

# An Information-Theoretic Model for Steganography

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## Abstract

An information-theoretic model for steganography with a passive adversary is proposed. The adversary's task of distinguishing between an innocent cover message  $C$  and a modified message  $S$  containing a hidden information is interpreted as a hypothesis testing problem. The security of a steganographic system is quantified in terms of the relative entropy (or discrimination) between  $P_C$  and  $P_S$ , which gives quantitative bounds on the detection capability of any adversary. It is shown that secure steganographic schemes exist in this model provided the coartext distribution satisfies certain conditions. A universal stegosystem is presented in this model that needs no knowledge of the coartext distribution, except that it is generated from independently repeated experiments.

## 1 Introduction

Steganography is the art and science of communicating in such a way that the presence of a message cannot be detected. It belongs to the field of information hiding, which has received considerable attention recently [10]. A survey of current information hiding is given by Petitcolas et al. [13]. One may distinguish two general directions in information hiding, determined by the power of an adversary: protection only against the detection of a message by a *passive* adversary and hiding a message such that not even an *active* adversary can remove it.

This paper views steganography as information hiding with a *passive* adversary. The model is perhaps best illustrated by Simmons' "Prisoners' Problem" [17]: Alice and Bob are in jail, locked up in separate cells far apart from each other, and wish to devise an escape plan. They are allowed to communicate by means of sending authenticated messages via trusted couriers, provided they do not deal with escape plans. The couriers are agents of the warden Eve (the adversary) and will leak all communication to her. If Eve detects any sign of conspiracy, she will thwart the escape plans by transferring both prisoners to high-security cells from which nobody has ever escaped. Alice and Bob are well aware of these facts, so that before getting locked up, they have shared a few secret codewords that they are now going to exploit for adding a hidden meaning to their seemingly innocent messages. Alice and Bob succeed if they can exchange information allowing them to coordinate their escape and Eve does not become suspicious.

Of course, Eve knows what a "legitimate" conversation among prisoners is like, and she also knows about the tricks that prisoners apply to embed a hidden meaning in a seemingly

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innocent message. Following the approach of information theory, we capture this knowledge by a *probabilistic model*, and view Eve’s task of detecting hidden messages as a problem of *hypothesis testing*.

**Our approach.** We consider only the scenario where Alice sends a message to Bob. Eve models an innocent message from Alice as a *coverttext*  $C$  with probability distribution  $P_C$ . A message with embedded hidden information is called *stegotext* and denoted by  $S$ .

In general, Eve might not know the process by which stegotext is generated; thus, Eve’s task would be to decide whether the observed message has been produced under the known coverttext distribution or under another distribution unknown to her. However, we adopt a stronger model and assume that Eve has complete knowledge of the embedding and extraction processes in a steganographic system, except for a short secret key  $K$  shared by Alice and Bob. This prudent tradition is adopted from in cryptology, where it is known as “Kerckhoffs’ principle.”

Upon observing the message sent by Alice, Eve has to decide whether it is coverttext or stegotext. This is the problem of choosing one of two different explanations for observed data, known as “hypothesis testing” in information theory [2, 3]. Recall that Eve knows the probability distributions of coverttext and stegotext, and draws her conclusion about the observed message only from this knowledge. However, Eve does not know if Alice produced the message according to  $P_C$  or  $P_S$ , nor is she willing to assign any a priori probabilities to these two explanations. (We note that it would be possible to assign such probabilities, but this would result in a different model.)

We define the security of the steganographic system used by Alice and Bob (or *stegosystem* for short) in terms of the *relative entropy*  $D(P_C||P_S)$  between  $P_C$  and  $P_S$ , which yields quantitative bounds on Eve’s detection performance. If the coverttext and the stegotext distributions are equal,  $D(P_C||P_S) = 0$  and we have a *perfectly secure* stegosystem; Eve has no information at all about the presence of an embedded message. This parallels Shannon’s notion of perfect secrecy for cryptosystems [16].

Note how our model differs from the scenario sometimes considered for steganography, where Alice uses a coverttext that is known to Eve and modifies it for embedding hidden information. Such schemes can only offer protection against adversaries with limited capability of comparing the modified stegotext to the coverttext. For instance, this applies to the popular use of steganography on visual images, where a stegoimage may be perceptually indistinguishable from the coverimage for humans, but not for an algorithm with access to the coverimage.

**Limitations.** The coverage of our information-theoretic model for real-world steganographic applications depends crucially on the assumption that there is a probabilistic model of the coverttext. Moreover, the users of a stegosystem need at least some knowledge about the coverttext distribution, as will become clear in the description of our stegosystems.

Probabilistic modeling of information is the subject of information theory, originating with Shannon’s pioneering work [15]. Information theory is today regarded as the “right” approach to quantifying information and to reasoning about the performance of communication channels. This confidence in the theory stems from many practical coding schemes that have been built according to the theory and perform well in real applications.

But the situation in steganography is more involved, since even a perfectly secure stegosystem requires that the users and the adversary share the same probabilistic model of the coverttext. For instance, *if* the coverttext distribution consists of uniformly random bits, *then* encrypt-

ing a message under a one-time pad results in a perfectly secure stegosystem according to our notion of security. But no reasonable warden would allow the prisoners to exchange random-looking messages in the Prisoners' Problem, since the use of encryption is clearly forbidden! Thus, the validity of a formal treatment of steganography is determined by the accuracy of a probabilistic model for the real data.

Assuming knowledge of the covertext distribution seems to render our model somewhat unrealistic for the practical purposes of steganography. But what are the alternatives? Should we rather study the perception and detection capabilities of the human cognition since most coverdata (images, text, and sound) is ultimately addressed to humans? Viewed in this way, steganography could fall entirely into the realms of image, language, and audio processing. However, it seems that an information-theoretic model, or any other formal approach, is more useful for deriving statements about the security of steganography schemes—and a formal security notion is one of the main reasons for introducing a mathematical model of steganography.

**Related work.** So far most formal models of information hiding address the case of *active adversaries*. This problem is different from the one considered here since the existence of a hidden message is typically known publicly, as for example in copyright protection schemes. Information hiding with active adversaries can be divided into watermarking and fingerprinting [14]. Watermarking supplies digital objects with an identification of origin; all objects are marked in the same way. Fingerprinting, conversely, attempts to identify individual copies of an object by means of embedding a unique marker in every copy that is distributed to a user. Cox et al. [4] propose a slightly different terminology and define watermarking in general as hiding covertext-dependent information, regardless of the adversary model.

As most objects to be protected by watermarking consist of audio, image, or video data, these domains have received the most attention so far. A large number of hiding techniques and domain-specific models have been developed for robust, imperceptible information hiding [4]. Ettinger [7] models active adversaries with game-theoretic techniques.

A general model for information hiding with active adversaries was formulated by Mittelholzer [11], but its hiding property also relies on the similarity of stegodata and coverdata in terms of a perceptually motivated distortion measure. Zöllner et al. [18] use information-theoretic methods to conclude that the embedding process in steganography must involve uncertainty. A discussion of these models with respect to ours is included in Section 6.

A large number of techniques for undetectable communication originate in the military domain, where they have found many applications. This includes radar, spread-spectrum communication, and covert channels. It is likely that our model is also applicable to those areas.

**Organization of the paper.** Section 2 contains the formal description of our model and the definition of security. After reviewing the theory of hypothesis testing, Section 3 presents the basic bounds on the detection performance for secure stegosystems. Section 4 provides some examples of unconditionally secure stegosystems; a universal stegosystem that requires no knowledge of the covertext statistics for the users is presented in Section 5. The paper concludes with a discussion in Section 6.

## 2 Model

**Preliminaries.** We define the basic properties of a stegosystem using the notions of entropy, mutual information, and relative entropy [2, 3].

The *entropy* of a probability distribution  $P_X$  over an alphabet  $\mathcal{X}$  is defined as

$$H(X) = - \sum_{x \in \mathcal{X}} P_X(x) \log P_X(x).$$

When  $X$  denotes a random variable with distribution  $P_X$ , the quantity  $H(X)$  is simply called the *entropy of the random variable  $X$*  (with the standard convention  $0 \log 0 = 0$  and logarithms to the base 2). Similarly, the *conditional entropy* of a random variable  $X$  given a random variable  $Y$  is

$$H(X|Y) = \sum_{y \in \mathcal{Y}} P_Y(y) H(X|Y = y),$$

where  $H(X|Y = y)$  denotes the entropy of the conditional probability distribution  $P_{X|Y=y}$ . The entropy of any distribution satisfies  $0 \leq H(X) \leq \log |\mathcal{X}|$ , where  $|\mathcal{X}|$  denotes the cardinality of  $\mathcal{X}$ .

The *mutual information* between  $X$  and  $Y$  is defined as the reduction of entropy that  $Y$  provides about  $X$ , i.e.,  $I(X;Y) = H(X) - H(X|Y)$ . It is symmetric in  $X$  and  $Y$ , i.e.,  $I(X;Y) = I(Y;X)$ , and always non-negative.

The *relative entropy* or *discrimination* between two probability distributions  $P_{Q_0}$  and  $P_{Q_1}$  is defined as

$$D(P_{Q_0} \| P_{Q_1}) = \sum_{q \in \mathcal{Q}} P_{Q_0}(q) \log \frac{P_{Q_0}(q)}{P_{Q_1}(q)}$$

(with  $0 \log \frac{0}{0} = 0$  and  $p \log \frac{p}{0} = \infty$  if  $p > 0$ ).

The *conditional relative entropy* between  $P_{Q_0}$  and  $P_{Q_1}$  given a random variable  $V$  defined in both probability spaces is

$$D(P_{Q_0|V} \| P_{Q_1|V}) = \sum_{v \in \mathcal{V}} P_V(v) \sum_{q \in \mathcal{Q}} P_{Q_0|V=v}(q) \log \frac{P_{Q_0|V=v}(q)}{P_{Q_1|V=v}(q)}.$$

The relative entropy between two distributions is non-negative and it is equal to 0 if and only if the distributions are equal. Although relative entropy is not a true distance measure in the mathematical sense, because it is not symmetric and does not satisfy the triangle inequality, it may be useful to think of it as a distance.

**Setting.** We adopt the standard terminology of information hiding [14] for our model of a stegosystem. There are two parties, Alice and Bob, who are the *users* of the stegosystem. Alice wishes to send an innocent-looking message with a hidden meaning over a public channel to Bob, such that the presence of hidden information goes unnoticed by a third party, the *adversary* Eve, who has perfect read-only access to the public channel.

Alice operates in one of two modes. In the first case, Alice is *inactive* and sends an innocent, legitimate message containing no hidden information, called *covert text* and denoted by  $C$ ; it is generated according to a distribution  $P_C$  known to Eve. One may imagine that the covert text is generated by a source to which only Alice has access. In the second case, Alice is *active* and sends a message containing *embedded information*  $E$  for Bob. Such a message is called *stegotext* and is denoted by  $S$ .

Alice may use a *private random source*  $R$  for the embedding operation. We assume that  $R$  is independent of  $E$  and  $C$ .

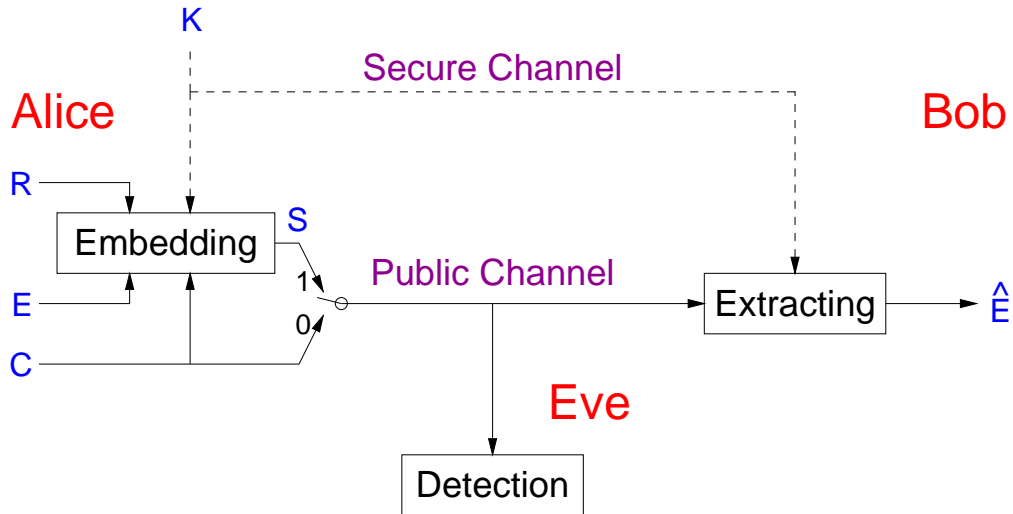


Figure 1: The model of a secret-key stegosystem. When Alice is active, the switch is in position 1 and she outputs stegotext  $S$ , which contains a hidden message  $E$  and was produced using knowledge of the secret key  $K$  shared by Alice and Bob. When Alice is inactive, the switch is in position 0, no embedding occurs, and Alice outputs covertext  $C$ .

The distributions of  $C$ ,  $E$ , and  $R$  are known to Eve. In general, it is not necessary for Alice or Bob to know the covertext and stegotext distributions.

In addition, Alice and Bob share a *secret key*  $K$ , which is unknown to Eve. The key has been chosen at random and communicated over a secure channel prior to the use of the stegosystem—in any case before the information  $E$  that Alice wants to communicate to Bob becomes known. Thus, we assume that  $K$  is independent of  $E$ ,  $R$ , and  $C$ .

Figure 1 shows the operation of a stegosystem in more detail. The switch at Alice's end of the public channel determines if Alice is active or not.

- In the first case (switch in position 0), Alice is inactive and sends only legitimate covertext  $C$  to Bob over the public channel. The covertext is generated by a covertext source, which is separate from the embedding process. The adversary Eve observes  $C$ .
- In the second case (switch in position 1), Alice is active and is given a message  $E$  that she embeds into the given covertext  $C$  using the shared key  $K$  to produce stegotext  $S$ . The embedding algorithm may be probabilistic, involving randomness from  $R$ . The stegotext is sent to Bob over the public channel. The adversary Eve and the receiver Bob observe  $S$ . Using his knowledge of  $K$ , Bob extracts a decision value  $\hat{E}$  from  $S$  in the hope that this gives him some information about  $E$ .

The embedding algorithm may exploit knowledge about the covertext distribution. We require, however, that the embedding function for a given covertext distribution is universal and works for any distribution of the message  $E$ . Thus, the embedding operation takes a concrete message as an input and must not depend on knowledge of the message distribution. This makes the stegosystem robust in the sense that the legitimate users do not have to worry about the adversary's knowledge of  $E$ . Such a precaution is often made in cryptographic contexts.

An alternative, but equivalent model would be to assume that the covertext originates with Alice herself, and not with a source external to her. Alice would use a probabilistic algorithm

involving her private random source  $R$  for generating  $C$ . Similarly, she would produce the stegotext without access to externally generated covertext. This formulation of the model emphasises the fact that neither Eve nor Bob has access to the covertext source.

The above discussion assumes that Bob has an oracle that tells him if Alice is active or not. This is a strong assumption, and we make it here in order to focus on the security properties of a stegosystem in general. Removing it does not hurt the security of a stegosystem, since if Bob was trying to extract an embedded message from the covertext when Alice is inactive, he would merely obtain garbage. As shown in Example 2 below, the oracle also does not open the way to trivial stegosystems. Later on in Example 3, we discuss a more practical class of stegosystems, in which this assumption is not necessary, because Bob may detect the presence of Alice’s message from redundancy in the embedded information.

**Secure stegosystems.** From the point of view of Eve, who does *not* know if Alice is active, the two cases above look similar: she observes data that is sent from Alice to Bob over the public channel. The data was generated either according to  $P_C$  or according to  $P_S$ ; these are the two explanations that Eve has for the observation. In the following, keep in mind that we do not assign probabilities to the events that Alice is active or not.

**Definition 1.** Let  $M$  denote the message received by Bob on the public channel, i.e.,  $M = S$  if Alice is active and  $M = C$  if Alice is inactive. A *stegosystem* consists of the components described above that satisfy the following conditions:

1.  $H(S|CEKR) = 0$ . The stegotext is determined uniquely by Alice’s probabilistic embedding algorithm from her inputs.
2.  $H(\hat{E}|MK) = 0$ . Bob’s output is determined uniquely by the information available to him.
3. If Alice is active and  $H(E) > 0$ , then  $I(\hat{E}; E) > 0$ . If Alice embeds a message that has non-zero entropy, then Bob’s output carries some information about the message.

Note that condition 3 implies that a stegosystem is “useful” in the sense that Bob obtains at least some information about  $E$ .

This model describes a stegosystem for *one-time use*, where Alice is always active or not. If Alice sends multiple dependent messages to Bob and at least one of them contains hidden information, she is considered to be active at all times and  $S$  consists of the concatenation of all her messages.

Eve, upon observing the message sent by Alice, has to decide whether it was covertext  $C$  or stegotext  $S$ , i.e., whether Alice is inactive or active. We quantify the security of a stegosystem in terms of the relative entropy between  $P_C$  and  $P_S$ .

**Definition 2.** A stegosystem is called *perfectly secure against passive adversaries* if

$$D(P_C \| P_S) = 0.$$

A stegosystem is called  *$\epsilon$ -secure against passive adversaries* if

$$D(P_C \| P_S) \leq \epsilon.$$

**Remarks.** It would be natural to require explicitly that a *perfectly* secure stegosystem provides also *perfect secrecy* for  $E$  in the sense of Shannon [16] by demanding that  $S$  and  $E$  are statistically independent (as for example in the definition of Mittelholzer [11]). However, this is not necessary since we required Alice’s embedding algorithm to work for *any* distribution  $P_E$ , without depending on the distribution itself. This guarantees perfect secrecy for  $E$  against Eve as follows. Fix a covert distribution and an embedding method. For all possible distributions of  $E$ , the embedding process must produce  $S$  with the same distribution as  $C$  (when given no further information). Since every concrete message value corresponds to a particular distribution of  $E$  but the distribution of  $S$  is the same for all values,  $S$  is statistically independent from  $E$ .

Analogously, we do not impose a secrecy constraint on  $E$  for stegosystems with  $\epsilon > 0$ . The implications on the secrecy of  $E$  are more involved and not investigated here; however, it is easy to construct stegosystems with perfect secrecy also in this case (see the second system in Section 4).

*Example 1.* In a world where pictures are stored on paper and not transmitted digitally, suppose Alice knows a “coverimage”  $m$  that she is going to mention to Bob. Eve has also seen  $m$ . Alice and Bob determine a set  $\mathcal{M} = \{m_0, \dots, m_{|\mathcal{M}|-1}\}$  of images that are perceptually indistinguishable from  $m$  for any viewer. Alice will not send  $m$  but a randomly chosen element of  $\mathcal{M}$ , which defines the probability space underlying the covert distribution  $C$ . Eve is aware of this as well. Furthermore, Alice and Bob share a uniformly random secret key  $K$  in  $\mathbb{Z}_{|\mathcal{M}|}$ . If Alice is active, she may embed a message  $E \in \mathbb{Z}_{|\mathcal{M}|}$  by sending  $S = m_{(K+E) \bmod |\mathcal{M}|}$ . Bob obtains  $E$  from  $\mathcal{M}$ ,  $S$ , and  $K$  easily. Since we assume the distribution of images sent by Alice to be uniform, covert and stegotext distributions are equal and this yields a perfectly secure stegosystem.

*Example 2.* In our definition of a stegosystem, Bob knows from an oracle if Alice is active or not. Hence, one might be tempted to construct the following “perfect” stegosystem that exploits this knowledge for transmitting hidden information without using a shared secret key. Consider an embedding algorithm consisting of an ideal source encoder that manages to compress some message  $E_1$  into stegotext  $S_1$ , which consists of independent and uniformly random bits. If the given  $C$  is a sequence of independent and uniformly random bits of the same length, the two distributions are the same and Eve cannot distinguish a compressed message from covert. In this case, Bob obtains  $E_1$  without any secret key. His advantage to distinguish stegotext from covert stems entirely from the oracle, and one might conclude that assuming such an oracle allows for trivial stegosystems.

However, this conclusion does not hold because the described stegosystem is not perfectly secure. Recall that the embedding method is fixed and required to work for *any* message distribution, so it must work also for some  $E_2$  with a smaller support and strictly less entropy than  $E_1$ —for instance, when Eve has partial or complete knowledge of the message. Let the embedding of  $E_2$  result in  $S_2$ . Then it is intuitively clear that the deterministic encoder will not output enough random bits and the distributions of  $C$  and  $S_2$  will differ.

Formally, this can be seen by expanding the mutual information between the message and the stegotext in two different ways. Since the encoder is deterministic *and* perfect, we have  $H(S_1) = H(E_1)$  from expanding  $I(E_1; S_1)$ . The same encoder applied to  $E_2$  also uniquely determines  $S_2$ , and therefore  $H(S_2) = H(E_2) - H(E_2|S_2) \leq H(E_2)$  from expanding  $I(E_2; S_2)$ . Hence,  $H(S_2) \leq H(E_2) < H(E_1) = H(S_1)$  by the assumption on  $E_2$ , which implies that the distributions of  $S_1$  and  $S_2$  are different and contradicts the assumption that the stegosystem is perfect.

**Average security.** It is often appropriate to model an information source as an infinite stochastic process. For example, the covertext may be generated from many independent repetitions of the same experiment. Although Eve observes the complete covertext sequence in our model above, it also makes sense to consider a restricted adversary who has only access to a small subset. This might apply in situations where the users of a stegosystem operate over a long period of time.

Let all random variables in our model above be extended to stochastic processes and let  $n$  denote the number of repetitions. Assume that the covertext is generated by a *stationary* information source. Hence, the *normalized* relative entropy between the covertext and stegotext processes determines the security in cases where Eve is restricted to see a finite part of the covertext sequence.

**Definition 3.** A stegosystem for stochastic processes with stationary covertext is called *perfectly secure on average against passive adversaries* whenever

$$\lim_{n \rightarrow \infty} \frac{1}{n} D(P_C \| P_S) = 0.$$

Analogously, a stegosystem for stochastic processes is called  *$\epsilon$ -secure on average against passive adversaries* whenever

$$\lim_{n \rightarrow \infty} \frac{1}{n} D(P_C \| P_S) \leq \epsilon.$$

Notice that Alice is still either active or inactive during the entire experiment, and the stegotext distribution will not be ergodic in general.

### 3 Detection Performance

This section analyzes Eve's capabilities of detecting an embedded message. Basic bounds on her performance are obtained from the theory of hypothesis testing. A brief review of hypothesis testing is given first, following the presentation by Blahut [2].

#### 3.1 Hypothesis Testing

Hypothesis testing is the task of deciding which one of two hypotheses  $H_0$  or  $H_1$  is the true explanation for an observed measurement  $Q$ . In other words, there are two plausible probability distributions, denoted by  $P_{Q_0}$  and  $P_{Q_1}$ , over the space  $\mathcal{Q}$  of possible measurements. If  $H_0$  is true, then  $Q$  was generated according to  $P_{Q_0}$ , and if  $H_1$  is true, then  $Q$  was generated according to  $P_{Q_1}$ . A *decision rule* is a binary partition of  $\mathcal{Q}$  that assigns one of the two hypotheses to each possible measurement  $q \in \mathcal{Q}$ . The two errors that can be made in a decision are called a *type I error* for accepting hypothesis  $H_1$  when  $H_0$  is actually true and a *type II error* for accepting  $H_0$  when  $H_1$  is true. The probability of a type I error is denoted by  $\alpha$ , the probability of a type II error by  $\beta$ .

A basic property in hypothesis testing is that *deterministic processing* cannot increase the relative entropy between two distributions. For any function  $f : \mathcal{Q} \rightarrow \mathcal{T}$ , if  $T_0 = f(Q_0)$  and  $T_1 = f(Q_1)$ , then

$$D(P_{T_0} \| P_{T_1}) \leq D(P_{Q_0} \| P_{Q_1}). \tag{1}$$



Let  $d(\alpha, \beta)$  denote the *binary relative entropy* of two distributions with parameters  $(\alpha, 1 - \alpha)$  and  $(1 - \beta, \beta)$ , respectively,

$$d(\alpha, \beta) = \alpha \log \frac{\alpha}{1 - \beta} + (1 - \alpha) \log \frac{1 - \alpha}{\beta}.$$

Because deciding between  $H_0$  and  $H_1$  is a special form of processing by a *binary* function, the type I and type II error probabilities  $\alpha$  and  $\beta$  satisfy

$$d(\alpha, \beta) \leq D(P_{Q_0} \| P_{Q_1}). \quad (2)$$

This bound is typically used as follows: Suppose that  $D(P_{Q_0} \| P_{Q_1}) < \infty$  and that an upper bound  $\alpha^*$  on the type I error probability is given. Then (2) yields a lower bound on the type II error probability  $\beta$ . For example,  $\alpha^* = 0$  implies that  $\beta \geq 2^{-D(P_{Q_0} \| P_{Q_1})}$ .

We note two properties of relative entropy that are useful in Section 5. The first one connects entropy, relative entropy, and the size of the alphabet for any random variable  $X \in \mathcal{X}$ : If  $P_U$  is the uniform distribution over  $\mathcal{X}$ , then

$$H(X) + D(P_X \| P_U) = \log |\mathcal{X}|. \quad (3)$$

The second property states that conditioning on derived information (side information, which has the same distribution in both cases) can only increase the discrimination: If there is a deterministic function  $f : \mathcal{Q} \rightarrow \mathcal{V}$  such that the random variables  $f(Q_0)$  and  $f(Q_1)$  have the same distribution  $P_V$ , then [2, Thm. 4.3.6]

$$D(P_{Q_0} \| P_{Q_1}) \leq D(P_{Q_0|V} \| P_{Q_1|V}). \quad (4)$$

### 3.2 Bounds for Secure Stegosystems

Consider Eve's decision process for detecting a hidden message in a stegosystem as a hypothesis testing problem. Any particular decision rule is a binary partition  $(\mathcal{C}_0, \mathcal{C}_1)$  of the set  $\mathcal{C}$  of possible covertexts. She decides that Alice is active if and only if the observed message  $c$  is contained in  $\mathcal{C}_1$ . Ideally, she would always detect a hidden message. (But this occurs only if Alice chooses an encoding such that valid covertexts and stegotexts are disjoint.) If Eve fails to detect that she observed stegotext  $S$ , she makes a type II error; its probability is denoted by  $\beta$ .

The opposite error, which usually receives less attention, is the type I error: Eve decides that Alice sent stegotext although it was a legitimate cover message  $C$ ; its probability is denoted by  $\alpha$ . An important special case is that Eve makes no type I error and never accuses Alice of sending hidden information when she is inactive ( $\alpha = 0$ ). Such a restriction might be imposed on Eve by external mechanisms, justified by the desire to protect innocent users.

The deterministic processing property (1) bounds the detection performance achievable by Eve. From (2) we obtain the following result.

**Theorem 1.** *In a stegosystem that is  $\epsilon$ -secure against passive adversaries, the probability  $\beta$  that the adversary does not detect the presence of the embedded message and the probability  $\alpha$  that the adversary falsely announces the presence of an embedded message satisfy*

$$d(\alpha, \beta) \leq \epsilon.$$

*In particular, if  $\alpha = 0$ , then*

$$\beta \geq 2^{-\epsilon}.$$

In a perfectly secure system, we have  $D(P_C \| P_S) = 0$  and therefore  $P_C = P_S$ ; thus, the observed message does not give Eve any information about whether Alice is active or not.

## 4 Secure Stegosystems

According to our model, we obtain a secure stegosystem whenever the stegotext distribution is close to the covertext distribution for an observer with no knowledge of the secret key. The embedding function depends crucially on the covertext distribution. We assume in this section that the covertext distribution is known to the users Alice and Bob, and describe how secure stegosystems can be constructed.

**One-time pad.** As already mentioned in the introduction, the *one-time pad* is a perfectly secure stegosystem whenever the covertext consists of uniformly random bits. Assuming such a covertext would be rather unrealistic, but in order to illustrate the model, we briefly describe this system formally.

*Example 3.* Assume the covertext  $C$  is a uniformly distributed  $n$ -bit string for some positive  $n$  and let Alice and Bob share an  $n$ -bit key  $K$  with uniform distribution. The embedding function (if Alice is active) consists of applying bitwise XOR to the  $n$ -bit message  $E$  and  $K$ , thus  $S = E \oplus K$ ; Bob can decode this by computing  $\hat{E} = S \oplus K$ . The resulting stegotext  $S$  is uniformly distributed in the set of  $n$ -bit strings and therefore  $D(P_C || P_S) = 0$ .

Using this stegosystem, we illustrate how to remove the assumption that Bob knows if Alice is active. Let the embedded message be  $k < n$  bits long and take a binary linear code with  $k$  information bits and block length  $n$ . Then Alice uses the message to select a codeword and embeds it in place of  $E$  using the one-time pad stegosystem. Bob checks if the vector extracted from the one-time pad is a codeword. If yes, he concludes that Alice is active and decodes it to obtain the embedded message.

Incidentally, the one-time pad stegosystem is equivalent to the basic scheme of visual cryptography [12]. This technique hides a monochrome picture by splitting it into two random layers of dots. When these are superimposed, the picture appears. Using a slight modification of the basic scheme, it is also possible to produce two innocent-looking pictures such that *both* of them together reveal a hidden embedded message that is perfectly secure against an observer who has only one picture.

**General distributions.** For arbitrary covertext distributions, we now describe a system that embeds a one-bit message in the stegotext. The extension to larger message spaces is straightforward, but might require even more detailed knowledge of the covertext distribution. Alice constructs the embedding function from a partition of the covertext space  $\mathcal{C}$  into two parts such that both parts are assigned approximately the same probability under  $P_C$ . In other words, let

$$\mathcal{C}_0 = \arg \min_{\mathcal{C}' \subseteq \mathcal{C}} \left| \sum_{c \in \mathcal{C}'} P_C(c) - \sum_{c \notin \mathcal{C}'} P_C(c) \right| \quad \text{and} \quad \mathcal{C}_1 = \mathcal{C} \setminus \mathcal{C}_0.$$

Alice and Bob share a one-bit secret key  $K$ . Define  $\mathcal{C}_0$  to be the random variable with alphabet  $\mathcal{C}_0$  and distribution  $P_{\mathcal{C}_0}$  equal to the conditional distribution  $P_{C|C \in \mathcal{C}_0}$  and define  $\mathcal{C}_1$  similarly on  $\mathcal{C}_1$ . Then Alice computes the stegotext to embed a message  $E \in \{0, 1\}$  as

$$S = C_{E \oplus K}.$$

Bob can decode the message because he knows that  $E = 0$  if and only if  $S \in \mathcal{C}_K$ . Note that the embedding provides perfect secrecy for  $E$ .

**Theorem 2.** *The one-bit stegosystem described above has security*

$$\frac{1}{\ln 2} (\Pr[C \in \mathcal{C}_0] - \Pr[C \in \mathcal{C}_1])^2$$

*against passive adversaries.*

*Proof.* Let  $\delta = \Pr[C \in \mathcal{C}_0] - \Pr[C \in \mathcal{C}_1]$ . We show only the case  $\delta > 0$ . It is straightforward but tedious to verify that

$$P_S(c) = \begin{cases} P_C(c)/(1 + \delta) & \text{if } c \in \mathcal{C}_0, \\ P_C(c)/(1 - \delta) & \text{if } c \in \mathcal{C}_1. \end{cases}$$

It follows that

$$\begin{aligned} D(P_C \| P_S) &= \sum_{c \in \mathcal{C}} P_C(c) \log \frac{P_C(c)}{P_S(c)} \\ &= \sum_{c \in \mathcal{C}_0} P_C(c) \log(1 + \delta) + \sum_{c \in \mathcal{C}_1} P_C(c) \log(1 - \delta) \\ &= \frac{1 + \delta}{2} \cdot \log(1 + \delta) + \frac{1 - \delta}{2} \cdot \log(1 - \delta) \\ &\leq \frac{1 + \delta}{2} \cdot \frac{\delta}{\ln 2} + \frac{1 - \delta}{2} \cdot \frac{-\delta}{\ln 2} \\ &= \delta^2 / \ln 2 \end{aligned}$$

using the fact that  $\log(1 + x) \leq x / \ln 2$ . □

In general, determining the optimal embedding function from a covertext distribution is an NP-hard combinatorial optimization problem. For instance, if we find an efficient embedding algorithm for the above one-bit stegosystem that achieves perfect security whenever possible, we have solved the NP-complete PARTITION problem [8], as can easily be verified.

## 5 Universal Stegosystems

Stegosystems of the kind described above require that the covertext distribution is known to the users Alice and Bob. This seems not realistic for many applications. In this section, we describe a method for obtaining a *universal* stegosystem where such knowledge is not needed. The idea is that Alice and Bob exploit a covertext signal that is produced by an infinite sequence of independent repetitions of the same experiment. Alice applies a *universal data compression* scheme to compute an approximation of the covertext distribution. She then produces stegotext with the approximate distribution of the covertext from *her own* randomness and embeds a message into the stegotext using the method of the one-time pad. Eve may have complete knowledge of the covertext distribution, but as long as she is restricted to observe only a finite part of the covertext sequence, this stegosystem achieves perfect average security asymptotically.

There are many practical universal data compression algorithms [1], and most encoding methods for perceptual data rely on them in some form. It is conceivable to combine them with our universal stegosystem for embedding messages in perceptual coverdata such as audio or video.

**The method of types.** One of the fundamental concepts of information theory is the *method of types* [6, 5]. It leads to simple proofs for the *asymptotic equipartition property (AEP)* and many other important results. The AEP states that the set of possible outcomes of  $n$  independent, identically distributed realizations of a random variable  $X$  can be divided into a typical set and a non-typical set, and that the probability of the typical set approaches 1 with  $n \rightarrow \infty$ . Furthermore, all typical sequences are almost equally likely and the probability of a typical sequence is close to  $2^{-nH(X)}$ .

Let  $x^n$  be a sequence of  $n$  symbols from  $\mathcal{X}$ . The *type* or *empirical probability distribution*  $U_{x^n}$  of  $x^n$  is the mapping that specifies the relative proportion of occurrences of each symbol  $x_0 \in \mathcal{X}$  in  $x^n$ , i.e.,  $U_{x^n}(x_0) = \frac{N_{x_0}(x^n)}{n}$ , where  $N_{x_0}(x^n)$  is the number of times that  $x_0$  occurs in the sequence  $x^n$ . The *set of types with denominator  $n$*  is denoted by  $\mathcal{U}_n$  and for  $U \in \mathcal{U}_n$ , the *type class*  $\{x^n \in \mathcal{X}^n : U_{x^n} = U\}$  is denoted by  $\mathcal{T}(U)$ .

The following standard result [6, 3] summarizes the basic properties of types.

**Lemma 3.** *Let  $X^n = X_1, \dots, X_n$  be a sequence of  $n$  independent and identically distributed random variables with distribution  $P_X$  and alphabet  $\mathcal{X}$  and let  $\mathcal{U}_n$  be the set of types. Then*

1. *The number of types with denominator  $n$  is at most polynomial in  $n$ , more particularly  $|\mathcal{U}_n| \leq (n+1)^{|\mathcal{X}|}$ .*
2. *The probability of a sequence  $x^n$  depends only on its type and is given by  $P_{X^n}(x^n) = 2^{-n(H(U_{x^n}) + D(U_{x^n} \| P_X))}$ .*
3. *For any  $U \in \mathcal{U}_n$ , the size of the type class  $\mathcal{T}(U)$  is on the order of  $2^{nH(U)}$ . More precisely,  $\frac{1}{(n+1)^{|\mathcal{X}|}} 2^{nH(U)} \leq |\mathcal{T}(U)| \leq 2^{nH(U)}$ .*
4. *For any  $U \in \mathcal{U}_n$ , the probability of the type class  $\mathcal{T}(U)$  is approximately  $2^{-nD(U \| P_X)}$ . More precisely,  $\frac{1}{(n+1)^{|\mathcal{X}|}} 2^{-nD(U \| P_X)} \leq \Pr[X^n \in \mathcal{T}(U)] \leq 2^{-nD(U \| P_X)}$ .*

**A universal data compression scheme.** A universal coding scheme  $(\mathcal{E}, \mathcal{D})$  for a memoryless source  $X$  works as follows. Fix a rate  $\rho < \log |\mathcal{X}|$  and let  $\rho_n = \rho - |\mathcal{X}| \frac{\log(n+1)}{n}$ . Define a set of sequences  $A_n = \{x^n \in \mathcal{X}^n : H(U_{x^n}) \leq \rho_n\}$ . The block code is given by an enumeration  $\mathcal{A} = \{1, \dots, |\mathcal{A}|\}$  of the elements of  $A_n$ . The encoder  $\mathcal{E}$  maps a sequence  $X^n$  to a codeword in  $\mathcal{A}$  if the entropy of the *type* of  $X^n$  does not exceed  $\rho_n$  and to a default value  $\Delta$  otherwise. Let  $Z$  denote the output of  $\mathcal{E}$ . Given a value  $S \in \mathcal{A} \cup \{\Delta\}$ , the decoder  $\mathcal{D}$  returns the appropriate sequence in  $A_n$  if  $S \neq \Delta$  or a default sequence  $x_0^n$  otherwise.

Using Lemma 3, it is easy to show that  $|A_n| \leq 2^{n\rho}$  and therefore  $\lceil n\rho \rceil$  bits are sufficient to encode all  $x^n \in A_n$  [6, 3]. Moreover, if  $H(X) < \rho$  then values outside  $A_n$  occur only with exponentially small probability and the error probability  $p_e^{(n)} = P_Z(\Delta)$  satisfies

$$p_e^{(n)} \leq (n+1)^{|\mathcal{X}|} 2^{-n \min_{U: H(U) > \rho_n} D(U \| P_X)}. \quad (5)$$

The following observation is needed below. Write

$$H(X^n) = H(X^n Z) \quad (6)$$

$$= P_Z(\Delta) H(X^n Z | Z = \Delta) + (1 - P_Z(\Delta)) H(X^n Z | Z \neq \Delta) \quad (7)$$

$$\leq P_Z(\Delta) H(X^n) + (1 - P_Z(\Delta)) (H(Z | Z \neq \Delta) + H(X^n | Z, Z \neq \Delta)) \quad (8)$$

$$\leq P_Z(\Delta) H(X^n) + H(Z | Z \neq \Delta), \quad (9)$$

where (6) follows because  $Z$  is determined uniquely by  $X^n$ , (7) follows from rewriting, (8) holds because  $Z$  is uniquely determined by  $X^n$  and by rewriting, and (9) follows because codewords  $Z \neq \Delta$  can be decoded uniquely. Rewriting this as

$$H(Z|Z \neq \Delta) \geq nH(X)(1 - p_e^{(n)}), \quad (10)$$

we see that the codeword  $Z$  carries almost all information of  $X^n$ .

**A universal stegosystem.** Suppose the coartext, which is given as input to Alice, consists of  $n$  independent realizations of a random variable  $X$ . Our universal stegosystem applies the above data compression scheme to the coartext. If Alice is active, she generates stegotext containing hidden information using the derived encoder and her private random source.

More precisely, given  $\rho > H(X)$  and  $n$ , Alice first maps the incoming coartext  $X^n$  to its encoding  $Z = \mathcal{E}(X^n)$ . W.l.o.g. assume the output of the encoder is a binary  $m$ -bit string for  $m = \lceil \log |\mathcal{A}| \rceil$  (or the special symbol  $\Delta$ ) and the shared key  $K$  is a uniformly random  $\ell$ -bit string with  $\ell \leq m$ ; furthermore, let the message  $E$  to be embedded be an  $\ell$ -bit string and let Alice's random source  $R$  generate uniformly random  $(m - \ell)$ -bit strings.

If Alice's encoder outputs  $Z = \Delta$ , she sends  $S = X^n$  and no message is embedded. Otherwise, she computes the  $m$ -bit string

$$T = (E \oplus K) \parallel R,$$

where  $\parallel$  denotes the concatenation of bit strings, and sends  $S = \mathcal{D}(T)$ .

Bob extracts the embedded message from the received stegotext  $S$  as follows. If  $\mathcal{E}(S) = \Delta$ , he declares a transmission failure and outputs a default value. Otherwise, he outputs

$$\hat{E} = \mathcal{E}(S)_{[1, \dots, \ell]} \oplus K,$$

where  $Z_{[1, \dots, \ell]}$  stands for the prefix of length  $\ell$  of a binary string  $Z$ .

Note that this stegosystem relies on Alice's private random source in a crucial way.

**Theorem 4.** *Let the coartext consist of a sequence  $(X_1, \dots, X_n)$  of  $n$  independently repeated random variables with the same distribution  $P_X$  for  $n \rightarrow \infty$ . Then given any  $\epsilon > 0$ , the algorithm above implements a universal stegosystem that is  $\epsilon$ -secure on average against passive adversaries and hides an  $\ell$ -bit message with  $\ell \leq nH(X)$ , for  $n$  sufficiently large.*

*Proof.* The first two conditions of a stegosystem are satisfied because the embedding and extraction algorithms are deterministic. For condition 3, it is easy to see from the given universal data compression scheme  $(\mathcal{E}, \mathcal{D})$  that

$$\hat{E} = \mathcal{E}(S)_{[1, \dots, \ell]} \oplus K = \mathcal{E}(\mathcal{D}(T))_{[1, \dots, \ell]} \oplus K = T_{[1, \dots, \ell]} \oplus K = E$$

whenever  $\mathcal{E}(S) \neq \Delta$ , which happens with overwhelming probability as shown below. It remains to show that the stegosystem is  $\epsilon$ -secure on average.

Let  $\rho = H(X) + \epsilon/2$ . Then

$$m = \lceil n\rho \rceil \leq \lceil nH(X) + n\epsilon/2 \rceil. \quad (11)$$

Define a binary random variable  $V$  as follows:

$$V = \begin{cases} 0 & \text{if } Z \neq \Delta, \\ 1 & \text{if } Z = \Delta. \end{cases}$$

We bound the relative entropy between coverttext and stegotext as

$$D(P_C \| P_S) \leq D(P_{C|V} \| P_{S|V}) \tag{12}$$

$$= P_V(0)D(P_{C|V=0} \| P_{S|V=0}) + P_V(1)D(P_{C|V=1} \| P_{S|V=1}) \tag{13}$$

$$\leq D(P_{C|V=0} \| P_{S|V=0}) \tag{14}$$

$$\leq D(P_{Z|V=0} \| P_T) \tag{15}$$

$$= m - H(Z|V = 0), \tag{16}$$

where (12) follows from the property (4) of relative entropy about conditioning on derived information and (13) from the definition of conditional relative entropy. The second term in (13) vanishes because the coverttext and stegotext distributions are the same whenever  $V = 1$ , and  $P_V(0) \leq 1$ , hence we obtain (14). Because  $C$  and  $S$  in the case  $V = 0$  are obtained from  $Z$  and  $T$  by deterministic processing, (15) follows from (1). Since  $T$  is uniformly distributed, the next step (16) follows using (3).

Using the fact that the events  $V = 0$  and  $Z \neq \Delta$  are the same, insert (10) and (11) into (16) to obtain

$$\begin{aligned} \frac{1}{n}D(P_C \| P_S) &\leq \frac{1}{n} \left( \lceil nH(X) + n\epsilon/2 \rceil - nH(X)(1 - p_e^{(n)}) \right) \\ &\leq \frac{1}{n} (p_e^{(n)}nH(X) + n\epsilon/2 + 1) \\ &= p_e^{(n)}H(X) + \epsilon/2 + \frac{1}{n}. \end{aligned}$$

Since  $\rho_n$  approaches  $\rho$  from below and  $\rho > H(X)$ , it follows that for all sufficiently large  $n$ , also  $\rho_n > H(X)$  and the value  $\min_{U: H(U) > \rho_n} D(U \| P_X)$  in the exponent in (5) is strictly positive. This implies that the last expression is smaller than  $\epsilon$  for all sufficiently large  $n$  and that the stegosystem is indeed  $\epsilon$ -secure on average.  $\square$

## 6 Discussion

The approach of this paper is to view steganography with a passive adversary as a problem of hypothesis testing because the adversary succeeds if he merely detects the presence of hidden information.

Other information-theoretic models for information hiding and steganography in the literature take a slightly different view:

- Zöllner et al. [18] recognize that breaking a steganographic system means detecting the use of steganography to embed a message. However, they formally require only that knowledge of the stegotext does not decrease the uncertainty about an embedded message, analogous to Shannon's notion of perfect secrecy for cryptosystems.
- Mittelholzer [11] considers steganography (with a passive adversary) and watermarking (with an active adversary). A stegosystem is required to provide perfect secrecy for the embedded message in sense of Shannon, and an encoder constraint is imposed in terms of a distortion measure between coverttext and stegotext. The expected mean squared error is proposed as a possible distortion measure.

Although these conditions may be necessary, they are not sufficient to guarantee undetectable communication, as can be seen from the following insecure stegosystem.

*Example 4.* Let the covertext consist of an  $m$ -bit string with *even* parity that is otherwise uniformly random ( $m \geq 2$ ). Let a ciphertext bit be computed as the XOR of a one-bit message and a one-bit random secret key; this is a random bit. Then the first bit of the covertext is replaced by the ciphertext bit and the last bit is adjusted such that the parity of the resulting stegotext is *odd*.

Clearly, the scheme provides perfect secrecy for the message. The squared error distortion between covertext and stegotext is  $1/m$  and vanishes as  $m \rightarrow \infty$ . Yet, an adversary can easily detect the presence of an embedded message *with certainty*. In the sense of Definition 2, such a scheme is completely insecure since the discrimination is infinite.

Simmons' original formulation of the prisoners' problem includes explicit authentication, that is, the secret key  $K$  shared by Alice and Bob is partially used for authenticating Alice's messages. The reason for this is that Alice and Bob want to protect themselves from the adversary and from malicious couriers (and they are allowed to do so), which may give rise to a subliminal channel in the authentication scheme. It would be interesting to extend our model for this scenario.

Another direction would be to model steganography with the security notions of modern cryptography [9], and to define a secure stegosystem such that the stegotext is computationally (or statistically) indistinguishable from the covertext.

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## References

- [1] T. C. Bell, J. G. Cleary, and I. H. Witten, *Text Compression*. Prentice Hall, 1990.
- [2] R. E. Blahut, *Principles and Practice of Information Theory*. Reading: Addison-Wesley, 1987.
- [3] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. Wiley, 1991.
- [4] I. J. Cox, M. L. Miller, and J. A. Bloom, *Digital Watermarking*. Morgan Kaufmann, 2002.
- [5] I. Csiszár, "The method of types," *IEEE Transactions on Information Theory*, vol. 44, pp. 2505–2523, Oct. 1998.
- [6] I. Csiszár and J. Körner, *Information Theory: Coding Theorems for Discrete Memoryless Systems*. New York: Academic Press, 1981.
- [7] M. Ettinger, "Steganalysis and game equilibria," in *Information Hiding, 2nd International Workshop* (D. Aucsmith, ed.), Lecture Notes in Computer Science, pp. 319–328, Springer, 1998.
- [8] M. R. Garey and D. S. Johnson, *Computers and Intractability — A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.
- [9] O. Goldreich, *Foundations of Cryptography: Basic Tools*. Cambridge University Press, 2001.

- [10] “Information hiding.” Series of International Workshops, 1996–2002. Proceedings in Lecture Notes in Computer Science, Springer.
- [11] T. Mittelholzer, “An information-theoretic approach to steganography and watermarking,” in *Information Hiding, 3rd International Workshop, IH’99* (A. Pfitzmann, ed.), vol. 1768 of *Lecture Notes in Computer Science*, pp. 1–16, Springer, 1999.
- [12] M. Naor and A. Shamir, “Visual cryptography,” in *Advances in Cryptology: EUROCRYPT ’94* (A. De Santis, ed.), vol. 950 of *Lecture Notes in Computer Science*, pp. 1–12, Springer, 1995.
- [13] F. A. Petitcolas, R. J. Anderson, and M. G. Kuhn, “Information hiding—a survey,” *Proceedings of the IEEE*, vol. 87, pp. 1062–1078, July 1999.
- [14] B. Pfitzmann, “Information hiding terminology,” in *Information Hiding, First International Workshop* (R. Anderson, ed.), vol. 1174 of *Lecture Notes in Computer Science*, Springer, 1996.
- [15] C. E. Shannon, “A mathematical theory of communication,” *Bell System Technical Journal*, vol. 27, pp. 379–423, 623–656, July, Oct. 1948.
- [16] C. E. Shannon, “Communication theory of secrecy systems,” *Bell System Technical Journal*, vol. 28, pp. 656–715, Oct. 1949.
- [17] G. J. Simmons, “The prisoners’ problem and the subliminal channel,” in *Advances in Cryptology: Proceedings of Crypto 83* (D. Chaum, ed.), pp. 51–67, Plenum Press, 1984.
- [18] J. Zöllner, H. Federrath, H. Klimant, A. Pfitzmann, R. Piotraschke, A. Westfeld, G. Wicke, and G. Wolf, “Modeling the security of steganographic systems,” in *Information Hiding, 2nd International Workshop* (D. Aucsmith, ed.), *Lecture Notes in Computer Science*, pp. 344–354, Springer, 1998.