

# Higher Order Correlation Attacks, XL Algorithm and Cryptanalysis of Toyocrypt

Nicolas T. Courtois

CP8 Crypto Lab, SchlumbergerSema,  
36-38 rue de la Princesse, BP 45, 78430 Louveciennes Cedex, France  
<http://www.nicolascourtois.net>  
[courtois@minrank.org](mailto:courtois@minrank.org)

**Abstract.** Many stream ciphers are built of a linear sequence generator and a non-linear output function  $f$ . There is abundant literature on (fast) correlation attacks, that use linear approximations of  $f$  to attack the cipher. In this paper we explore higher degree approximations, much less well known. We reduce the cryptanalysis of a stream cipher to solving a system of multivariate equations that is overdefined (much more equations than variables). We adapt the XL method, introduced at Eurocrypt 2000 for overdefined quadratic systems, to solving equations of higher degree. Though the exact complexity of XL remains an open problem, there is no doubt that it works perfectly well for such largely overdefined systems as ours, and we confirm this by computer simulations. Then we show that using XL, it is possible to break stream ciphers that were known to be immune to all previously known attacks. For example, we cryptanalyse the stream cipher Toyocrypt accepted to the second phase of the Japanese government Cryptrec program. Our best attack on Toyocrypt takes  $2^{92}$  CPU clocks for a 128-bit cipher. The interesting feature of our XL-based higher order correlation attacks is, their very loose requirements on the known keystream needed. For example they may work knowing ONLY that the ciphertext is in English.

*Key Words:* Multivariate cryptography, overdefined systems of multivariate equations, MQ problem, XL algorithm, Gröbner bases, stream ciphers, pseudo-random generators, nonlinear filtering, ciphertext-only attacks, Toyocrypt-HR1, Toyocrypt-HS1, Cryptrec.

## 1 Introduction

The security of most cryptographic schemes is usually based on impossibility to extract some secret information, given access to some encryption, signature oracles or other derived information. In most useful cases, there is no security in information-theoretic setting: the adversary has usually enough information to uniquely determine the secret (or the ability) he wants to acquire. Moreover the basic problem is always (in a sense) overdefined: the adversary is assumed to dispose of, for example, great

many plaintext and cipher text pairs, message and signature pairs, etc. He usually disposes of much more than the information needed to just determine the secret key. Thus, one might say, most cryptographic security relies on the hardness of largely overdefined problems. In public key cryptography, the problem is addressed by provable security, that will assure that each utilization of the cryptographic scheme does not leak useful information. The security is guaranteed by a hardness of a single difficult problem, and does not depend on how many times the scheme has been used. However unfortunately, there is yet very little provable security in secret key cryptography. It is also in secret key cryptography that the problems become most overdefined, due to the amounts of data that are usually encrypted with one single session key. This is especially true for stream ciphers: designed to be extremely fast in hardware, they can encrypt astronomic quantities of data, for example on an optical fiber.

In this paper we point out that many constructions of stream ciphers directly give an overdefined system of multivariate equations of low degree. The fact that solving such overdefined systems of equations, is much easier than expected, has been demonstrated at Eurocrypt 2000 by Courtois, Klimov, Patarin and Shamir. Later, the possibility to use multivariate polynomial equations to attack secret key cryptosystems such as AES has been proposed by Courtois and Pieprzyk [7]. Unfortunately, these attacks are, to say the least, heuristic. In this paper we apply similar techniques to stream ciphers. Unlike in the work of Courtois and Pieprzyk, our systems of equations will be much more overdefined. We show that in this case it is possible to predict the behaviour of the XL method with precision and confidence. We attack a large class of stream ciphers in which there is a linear part, producing a sequence with a large period, and a nonlinear part that produces the output, given the state of the linear part. The security of such stream ciphers have been studied by many authors. In [11], Golic gives a set of criteria that should be satisfied in order to resist to the known attacks on stream ciphers. For example, a stream cipher should resist to the fast correlation attack [14], the conditional correlation attack [1] and the inversion attack [11]. In this paper we show that correlation immunity of order one is not sufficient, and show that it is really possible to use (at least in theory) any correlation of any order to mount an attack. Moreover we demonstrate that such attacks can be much faster than exhaustive search for some real stream ciphers, for example for Toyocrypt.

The paper is organized as follows: In Section 2 and in Appendix A we study the XL algorithm from [23] for solving multivariate quadratic

equations, and extend it to equations of higher degree. In Section 3 we apply XL to the cryptanalysis of stream ciphers. In Section 4 we discuss the opportunity to use bent functions in stream ciphers. Then in Section 5 we apply our attack on Toyocrypt stream cipher. Finally we discuss various extensions of the attack and present our conclusions.

## 2 The XL Algorithm

In this paper we describe a rather obvious extension of the XL algorithm proposed by Courtois, Klimov, Patarin and Shamir at Eurocrypt 2000 [23]. Instead of solving a system of  $m$  multivariate quadratic equations with  $n$  variables of degree  $K = 2$  as in [23], we consider also higher degree equations, i.e. study the general case  $K \geq 2$ . Let  $D$  be the parameter of the XL algorithm. Let  $l_i = 0$  be the initial equations,  $i = 1 \dots m$ . The XL algorithm consists of multiplying both sides of these equations by products of variables:

**Definition 2.0.1 (The XL algorithm).** Execute the following steps:

1. **Multiply:** Generate all the products  $\prod_{j=1}^k x_{i_j} \cdot l_i$  with  $k \leq D - K$ , so that the total degree of these equations is  $\leq D$ .
2. **Linearize:** Consider each monomial in the  $x_i$  of degree  $\leq D$  as a new variable and perform Gaussian elimination on the equations obtained in 1. The ordering on the monomials must be such that all the terms containing one variable (say  $x_1$ ) are eliminated last.
3. **Solve:** Assume<sup>1</sup> that step 2 yields at least one univariate equation in the powers of  $x_1$ . Solve this equation over the finite field (e.g., with Berlekamp's algorithm).
4. **Repeat:** Simplify the equations and repeat the process. to find the values of the other variables.

It is usually assumed that once a few variables are fixed, the system becomes much easier to solve and therefore the complexity is essentially the complexity of the Gaussian reduction in the step 2.

### 2.1 The Necessary Condition for XL to Work

The XL algorithm consists of multiplying the initial  $m$  equations  $l_i$  by all possible monomials of degree up to  $D - K$ , so that the total degree of resulting equations is  $D$ . Let  $R$  be the number of equations generated in XL, and  $T$  be the number of all monomials. We have, (the first term is dominant):

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<sup>1</sup> Improved versions of the XL algorithm exist in which the system can still be solved even if this condition is not satisfied, see the FXL algorithm [23], and XL' and XL2 methods described in [6]. We do not need these improvements here.

$$R = m \cdot \left( \sum_{i=0}^{D-K} \binom{n}{i} \right) \approx m \cdot \binom{n}{D-K}, \quad T = \sum_{i=0}^D \binom{n}{i} \approx \binom{n}{D}$$

The main problem in the XL algorithm is that in practice not all the equations generated are independent. Let  $Free$  be the exact number of equations that are linearly independent in XL. We have  $Free \leq R$ . We also have necessarily  $Free \leq T$ .

The main heuristics behind XL is the following: it can be seen that for some  $D$  we have always  $R \geq T$ . Then we expect that  $Free \approx T$ , as obviously it cannot be bigger than  $T$ . More precisely, following [23], when  $Free \geq T - D$ , it is possible by Gaussian elimination, to obtain one equation in only one variable, and XL will work. Otherwise, we need a bigger  $D$ , or an improved algorithm<sup>2</sup>.

### The Saturation Problem in XL

Very little is known about the exact value of  $Free$  in XL. In [23] authors demonstrate that XL works with a series of computer simulations for  $K = 2$  and over  $GF(127)$ . In [6] authors show that it also works very well for  $K = 2$  and over  $GF(2)$ . Moreover they explain how to predict the exact number  $Free$  of linearly independent equations in XL. In this paper we extend these results for higher degree equations  $K > 2$  (still over  $GF(2)$ ), and also will give a formula that allows to compute  $Free$  (see Conjecture A.3.1). This will allow us to say that our (cryptanalytic) applications of XL should work **exactly** as predicted.

In order to XL algorithm to work, it is sufficient that for some  $D$ , the number  $Free$  of linearly independent equations satisfies  $Free \geq T - D$ . In [18], Moh states that "From the theory of Hilbert-Serre, we may deduce that the XL program will work for many interesting cases for  $D$  large enough". In Section 4 Moh shows a very special example on which the basic version<sup>2</sup> of XL always fails, for any  $D$  [18]. This example is very interesting, but here we consider random systems of quadratic (or degree  $K$ ) equations, i.e. we look at the behaviour of XL on most of the systems, and not on some very special systems.

**Remark:** In Section 3 the author presents an argument that is apparently wrong. He assumes  $D \gg n$  in a formula in which  $D = \mathcal{O}(\frac{n}{\sqrt{m}})$ . He shows that, apparently  $Free/R \approx \frac{(n+D)(n+D-1)}{D(D-1)m} = w$ , and it is obvious that  $w \rightarrow \frac{1}{m}$  when  $D \rightarrow \infty$ . However in XL,  $D$  is never as big as  $n$ , if we

<sup>2</sup> Improved versions of XL exist [23, 6], see the footnote 1 on the previous page.

assume that we have  $D \approx \frac{n}{\sqrt{m}}$  as in the previous section, we get  $w \approx 1$ . The conclusion of Moh is inappropriate, not to say incorrect.

We will see in this paper that in all interesting cases, not only  $Free \geq T - D$  is always achieved, but frequently end up by a complete saturation<sup>3</sup>, i.e.  $Free = T - 1$  is achieved.

## 2.2 Asymptotic Analysis of XL for Equations of Degree $K$

We assume that  $D \ll n$ . XL algorithm is expected to succeed when  $R \geq T$ , i.e. when

$$m \binom{n}{D-K} \geq \binom{n}{D} \Rightarrow m \geq \frac{(n-D+K) \cdots (n-D+1)}{D(D-1) \cdots (D-K+1)}$$

Thus (assuming that  $D \ll n$ ) we get:

$$D \approx \frac{n}{m^{1/K}}, \text{ and } T \approx \binom{n}{D}^\omega \approx \left( \frac{n}{m^{1/K}} \right)^\omega$$

Asymptotically this is expected to be a good evaluation, when  $m = \varepsilon n^K$  with a constant  $\varepsilon > 0$ .

## 2.3 The exact behaviour of XL for Interesting Cases $K \geq 2$ .

In this paper we don't use any of the above approximations. We study the exact behaviour of XL, and compute the exact values of  $Free$  for the interesting values of  $K$  and  $D$ . This part is in Appendix A.

## 3 Application of XL to Stream Ciphers

In this part we outline a general strategy to apply XL in cryptanalysis of a general class of stream ciphers. Later we will apply it to Toyocrypt.

### 3.1 The Stream Ciphers that May be Attacked

We consider only synchronous stream ciphers, in which each state is generated from the previous state independently of the plaintext, see for example [16]. We consider regularly clocked stream ciphers, and also (indifferently) stream ciphers that are clocked in a known way<sup>4</sup>.

For simplicity we restrict to binary stream ciphers in which the state and keystream are composed of a sequence of bits  $b_i$ , generating one bit at a time. Let  $L$  be the "connection function" that computes the next state. We restrict to the (very popular) case of cipher with linear

<sup>3</sup> It is impossible to have  $Free = T$  for a system that has a solution, this is because in our definition of  $T$  the constant term is counted and  $Free = T$  would imply, by elimination of all non-constant monomials, that  $1 = 0$ .

<sup>4</sup> This condition can sometimes be relaxed, see the attacks on LILI-128 in [5].

feedback, i.e. when  $L$  is linear over  $GF(2)$ . We assume that  $L$  is public, and only the state is secret. We also assume that the function  $f$  that computes the output bit from the state is public and does not depend on the secret key of the cipher. The only non-linear component of the cipher is  $f$  and this way of building stream ciphers is sometimes called "nonlinear filtering". It includes the very popular filter generator, in which the state of a single LFSR<sup>5</sup> is transformed by a boolean function, and also not less popular scenarios, in which outputs of several LFSR are combined by a boolean function (combinatorial function generators or nonlinear function generators).

The problem of cryptanalysis of such a stream cipher can be described as follows. Let  $(k_0, \dots, k_{n-1})$  be the initial state, then the output of the cipher (i.e. the keystream) is given by:

$$\begin{cases} b_0 = f(k_0, \dots, k_{n-1}) \\ b_1 = f(L(k_0, \dots, k_{n-1})) \\ b_2 = f(L^2(k_0, \dots, k_{n-1})) \\ \vdots \end{cases}$$

The problem we consider<sup>6</sup> is to recover  $(k_0, \dots, k_{n-1})$  given some  $b_i$ .

### 3.2 The Attack Scenario

We are going to design a partially known plaintext attack, i.e. we know some bits of the plaintext, and the corresponding ciphertext bits. These bits does not need to be consecutive. For example if the plaintext is written with latin alphabet and does not use too much special characters, it is very likely that all the characters have their most significant bit equal to 0. This will be enough for us, if the text is sufficiently long. In our later attacks we just assume that we have some  $m$  bits of the keystream at some known positions:  $\{(t_1, b_{t_1}), \dots, (t_m, b_{t_m})\}$ .

### 3.3 Criteria on the Function $f$

Let  $f$  be the boolean function<sup>7</sup> that is used to combine the bits of the linear part of a stream cipher (the entries of the function are for example some bits of the state of some LFSR's). There are many design criteria known on boolean functions. Some of them are obvious to justify, for example a function should be balanced in order to avoid statistical attacks.

<sup>5</sup> A Linear Feedback Shift Register, see for example [16]. It is also possible to use a Modular LFSR, i.e. a MLFSR, which is equivalent in theory, see, [17], but may be better in practice. A MLFSR is used in the Toyocrypt cipher that we study later.

<sup>6</sup> We do not consider attacks in which one can predict the future keystream, given some information on the current keystream, and without computing the key.

<sup>7</sup> We describe an attack with a single boolean function  $f$ , still it is easy to extend it to stream ciphers using several different boolean functions.

Some criteria are not well justified, no practical attacks are known when the function does not satisfy them, and they are used rather to prevent some new attacks.

It is obvious that for stream ciphers such as described above, the function  $f$  should be non-linear. The abundant literature on fast correlation attacks implies also that it should be highly non-linear<sup>8</sup> and also correlation immune at order 1. Similarly,  $f$  should have high order (i.e. an algebraic normal form of high degree), to prevent algebraic attacks and finally, a "good" boolean function should also be correlation immune at high order, as pointed out in [3, 12]. However up till now, no practical and non-trivial attacks on stream ciphers were published, when a function is of high degree, but not higher-order correlation immune. In this paper we design such a general attack based on the XL algorithm, and show that it can be successfully applied to Toyocrypt.

Our attack works in two cases:

- S1** When the boolean function  $f$  has a low algebraic degree  $K$ ,
- S2** and more importantly when  $f$  can be approximated<sup>9</sup> with good probability, by a function  $g$  that has a low algebraic degree  $K$ .

More precisely, we assume that:

$$f(s_0, \dots, s_{n-1}) = g(s_0, \dots, s_{n-1}) \text{ holds: } \begin{cases} 1. \text{ with probability } \geq 1 - \varepsilon \\ 2. \text{ and with } g \text{ of degree } K. \end{cases}$$

**Note:** In the first scenario S1, when  $f$  has just a low algebraic degree, it is known that the system can be easily broken given  $\binom{n}{K}$  keystream bits. A successful example of this attack is described for example in [2]. In this paper we show that, using XL, successful attacks can be mounted given much less keystream bits, and with much smaller complexities. More importantly we don't need that the function has a low algebraic degree (the second case above). For example in Toyocrypt the degree of  $f$  is 63, but in our attacks it will be approximated by a function of degree 2 or 4.

### 3.4 The Actual Attack

Given  $m$  bits of the keystream, we have the following  $m$  equations to solve:

$$\forall i = 1 \dots m, \quad b_{t_i} = f\left(L^{t_i}(k_0, \dots, k_{n-1})\right)$$

We recall that  $f$ , and all the  $L^{t_i}$  are public, and only the  $k_j$  are secret<sup>10</sup>. Each of the keystream bits gives one multivariate equation of degree  $K$ , with  $n$  variables  $(k_0, \dots, k_{n-1})$  and true with probability  $(1 - \varepsilon)$ :

<sup>8</sup> But maybe not perfectly non-linear, see Section 4.

<sup>9</sup> If such a (sufficiently good) approximation exists, there are efficient algorithms to find it. This problem is also known as "learning polynomials in the presence of noise", or as "decoding Reed-Muller codes". See for example [3, 12, 9].

<sup>10</sup> Important: If  $L$  is not public, as it may be the case in Toyocrypt, our later attacks will not work. Nevertheless they show that Toyocrypt is cryptographically weak.

$$\forall i = 1 \dots m, \quad b_{t_i} = g\left(L^{t_i}(k_0, \dots, k_{n-1})\right) \quad \text{with probability } \geq 1 - \varepsilon$$

If we choose  $m$  such that  $(1 - \varepsilon)^m \geq \frac{1}{2}$ , we may assume that all these equations are true and we have to find a solution to our system of  $m$  multivariate equations of degree  $K$  with  $n$  variables. More generally, even if  $(1 - \varepsilon)^m < \frac{1}{2}$ , the attack still works, if we repeat it about  $(1 - \varepsilon)^{-m}$  times, each time for a different subset of  $m$  keystream bits, and until it succeeds. The complexity of this attack will be the complexity of generalized XL obtained in Section 2.2, multiplied by the number of repetitions necessary to succeed:

$$WF = T^\omega (1 - \varepsilon)^{-m} \approx \binom{n}{n/m^{1/K}}^\omega (1 - \varepsilon)^{-m}$$

The above attack requires about  $m$  keystream bits, out of which we chose  $m$  at each iteration of the attack. We also need to chose  $m$  that minimizes the complexity given above. In practice, since the XL algorithm complexity varies by thresholds depending on the value of  $D$ , we will in fact chose  $D$  and determine a minimal  $m$  for which the attack works.

#### 4 Non-linear Filtering using Bent Functions

Due to numerous known fast correlation attacks, ciphers such as we described above (for example filter generators) should use a function  $f$  that is highly non-linear. For this, Meier and Staffelbach suggested at Eurocrypt'89 to use so called perfect non-linear functions, also known as "bent functions" [15, 21]. These functions achieve optimal resistance to the correlation attacks, because they have a minimum (possible) correlation to all affine functions, see Theorem 3.5. in [15]. It is therefore tempting to use a bent function as a combiner in a stream cipher. And indeed many cryptographic designs (e.g. Toyocrypt, and not only in stream ciphers) use such functions, or modified versions of such functions<sup>11</sup>.

Unfortunately optimality against one attack does not guarantee the security against other attacks. Following Anderson [1], any criteria on  $f$  itself cannot be sufficient. The author of [1] claims that "attacking a filter generator using a bent or almost bent function would be easy" and shows why on small examples. He considers "an augmented function" that consists of  $\alpha$  copies of the function  $f$  applied to consecutive windows of  $n$  consecutive bits, among the  $n + \alpha$  consecutive bits of an LFSR output stream. He

<sup>11</sup> In general the authors of [15] did not advocate to use pure bent functions, because it is known that these functions are not balanced and cannot have a very high degree. They advise to use modified bent functions, for which it is still possible to guarantee a high non-linearity, see [15].



shows explicit examples in which even if  $f : GF(2)^n \rightarrow GF(2)$  is a bent function, still the augmented function  $GF(2)^{n+\alpha} \rightarrow GF(2)^\alpha$  will have very poor statistic properties, and thus will be cryptographically weak.

For real ciphers, it is difficult to see if Anderson's remark is really dangerous. For example in Toyocrypt, an MLFSR is used instead of an LFSR, which greatly decreases the number of common bits between two consecutive states, and more importantly, only a carefully selected subset of state bits is used in each application of  $f$ . Thus it seems that Toyocrypt makes any version of the attacks described by Anderson in [1] completely impractical.

### Bent Function Used in Toyocrypt

The combining function  $f$  of Toyocrypt is built according to:

**Theorem 4.0.1 (Rothaus 1976 [21]).** Let  $g$  be any boolean function  $g : GF(2)^k \rightarrow GF(2)$ . All the functions  $f : GF(2)^{2k} \rightarrow GF(2)$  of the following form are bent:

$$f(x_1, x_2, \dots, x_{2k}) = x_1x_2 + x_3x_4 + \dots + x_{2k-1}x_{2k} + g(x_1, x_3, \dots, x_{2k-1})$$

**Remark:** More precisely, the function of Toyocrypt is a XOR of  $s_{127}$  and a function built according to the above theorem. We must say that using such a function as a non-linear filter is **not** a very good idea. It is easy to see that if we use a single LFSR or MLFSR, there will be always a "guess and find" attack on such a cipher. This is due to the fact that if we guess and fix  $k$  state bits, here it will be the odd-numbered bits, then the expression of the output becomes linear in the other state bits. This can be used to recover the whole state of the cipher given  $3k/2$  bits of it, i.e. the effective key length in such a scheme is only  $3k/2$  instead of  $2k$  bits. This attack is explained in details (on the example of Toyocrypt) in [17]. In this paper we do not use this property of  $f$ , and design a different attack, based on the low number of higher degree monomials, and thus being potentially able to break variants of Toyocrypt that are not based on the above theorem and for which there is no "guess and find" attacks.

## 5 Application of XL to the Cryptanalysis of Toyocrypt

In this section we present a general attack on Toyocrypt [17], a cipher that was, at the time of the design, believed to resist to all known attacks on stream ciphers. In Toyocrypt, we have one 128-bit LFSR, and thus  $n = 128$ . The boolean function is of the form:

$$f(s_0, \dots, s_{127}) = s_{127} + \sum_{i=0}^{62} s_i s_{\alpha_i} + s_{10} s_{23} s_{32} s_{42} + \dots + (\dots \text{a degree 17 monomial} \dots) \oplus (\dots \text{a degree 63 monomial} \dots).$$

with  $\{\alpha_0, \dots, \alpha_{62}\}$  being some permutation of the set  $\{63, \dots, 125\}$ . This system is quite vulnerable to the XL higher order correlation attack we described above: there are only a few higher-order monomials.

## A Quadratic Approximation

Most of the time, the system is quadratic. We put:  $g(s_0, \dots, s_{127}) = \sum_{i=0}^{62} s_i s_{\alpha_i}$ .

Then  $f(s) = g(s)$  holds with probability about  $1 - 2^{-4}$ . With the notations of the Section 3.4 we have  $K = 2$  and  $\varepsilon = 2^{-4}$ . This approximation does not allow efficient attacks.

## An Approximation of Degree $K = 4$

One can also see that if we put:

$$g(s_0, \dots, s_{127}) = \sum_{i=0}^{62} s_i s_{\alpha_i} + s_{10} s_{23} s_{32} s_{42}.$$

Then  $f(s) = g(s)$  holds with probability very close to  $1 - 2^{-17}$ . We have  $K = 4$  and we have approximately  $\varepsilon = 2^{-17}$ .

### 5.1 Our Higher Order Correlation Attack on Toyocrypt

The equation  $(1 - \varepsilon)^m \approx \frac{1}{2}$  gives  $m \approx 2^{16}$ . This is simply to say that if we consider some  $2^{16}$ , not necessarily consecutive bits of the keystream, the probability that for **all** of them we have  $f(s) = g(s)$  will be about  $1/2$ . A more precise evaluation shows that if we put  $m = 1.3 \cdot 2^{16}$ , we still have  $(1 - \varepsilon)^m = 0.52$ . This is the value we are going to use.

Thus, given some  $m$  keystream bits,  $m = 1.3 \cdot 2^{16}$ , one can write from Toyocrypt  $m$  equations of degree 4 and with 128 variables  $k_i$ . To this system of equations we apply generalized XL as described in Section 2. We have  $n = 128$  and let  $D \in \mathbb{N}$ . We multiply each of the  $m$  equations by all products of up to  $D-4$  variables  $k_i$ . The number of generated equations is:  $R = m \left( \sum_{i=0}^{D-4} \binom{n}{i} \right)$ . We also have  $T = \left( \sum_{i=0}^D \binom{n}{i} \right)$ . We observe that for  $D = 9$  we get  $R/T = 1.1401$ . Following our simulations and their analysis given in Section A.3, and since  $D < 3K$ , we expect that the exact number of linearly independent equations is  $Free = \min(T, R - \binom{m}{2} - m) - \epsilon$  with a very small  $\epsilon$ . This  $Free$  is sufficient: we have  $(R - \binom{m}{2} - m)/T = 1.13998$ , and thus  $R - \binom{m}{2} - m > T$  and  $R - \binom{m}{2} - m$  is not very close to  $T$ . From this, following Conjecture A.3.1 and our simulation results, we expect that  $Free = T - \epsilon$  with  $\epsilon = 1$ . XL works for  $D=9$ . The complexity of the attack is basically the complexity of solving a linear system  $T \times T$  (we don't need to take more than  $T$  equations). Though the best known algorithm for this problem is asymptotically in  $T^{2.376}$ , see [4], the best practical algorithm we know is Strassen's algorithm, see [24]. The complexity of Strassen's algorithm is about  $7 \cdot T^{\log_2 7}$ . In practice it gives about  $7 \cdot T^{\log_2 7} / 64$  CPU clocks, because the basic operations are multiplications and addition of bits modulo 2. All these operations can be implemented in a "bitslice" manner, thus transforming a 64 bit machine in a Single

Instruction Multiple Data (SIMD) machine with 64 1-bit processors. Thus the complexity of our attack is:

$$WF = \frac{7}{64} \cdot T^{\log_2 7} = 2^{122}.$$

We are able to break Toyocrypt faster than the exhaustive search of the key, using only about  $m$  bits of the keystream, i.e. about 9 kilobytes.

## 6 Improved XL Higher Correlation Attacks

We will now explore the tradeoff described in Section 3.4. The basic idea is that, if we diminish a little bit a success probability of the attack, we may use a higher  $m$ , the system will be more overdefined and we will be able to use a lower value of  $D$ . This in turn greatly diminishes the value of  $T$  that may compensate for the necessity to repeat the attack.

### Improved Attacks Exploring the Tradeoff

In the attack above we saw that  $Free = \min(T, R - \binom{m}{2} - m) - \epsilon$  and that we may in fact neglect  $\binom{m}{2} - m$ . Moreover if  $D$  becomes smaller, and when  $D < 2K = 8$ , following Section A.3 we expect to have  $Free = \min(T, R) - 1$ . Thus we may say that for  $D < 9$ , and  $R > 1.1 \cdot T$  the attack does certainly work. It gives the following condition on  $m$ :

$$m \binom{\sum_{i=0}^{D-4} \binom{n}{i}}{i} > 1.1 \cdot \binom{\sum_{i=0}^D \binom{n}{i}}{i}$$

From this, given  $D$ , we put  $m = 1.1 \binom{\sum_{i=0}^D \binom{n}{i}}{\binom{\sum_{i=0}^{D-4} \binom{n}{i}}{i}}$ . The probability that our approximation of degree 4 holds for all  $m$  equations is  $(1 - \frac{1}{2^{17}})^m$ . Finally, the complexity of the whole attack is:

$$WF = (1 - \frac{1}{2^{17}})^{-m} \cdot 7 \cdot T^{\log_2 7} / 64 = (1 - \frac{1}{2^{17}})^{-m} \cdot \frac{7}{64} \cdot \left( \sum_{i=0}^D \binom{n}{i} \right)^{\log_2 7}$$

The number of keystream bits required in the attack is about  $m$ , and the memory is  $T^2$  bits. In the following table we show possible tradeoffs:

$D$	4	5	6	7	8	9
Data	$2^{23}$	$2^{21}$	$2^{19}$	$2^{18}$	$2^{17}$	$2^{16}$
Memory	$2^{89}$	$2^{56}$	$2^{65}$	$2^{73}$	$2^{81}$	$2^{88}$
Complexity	$2^{200}$	$2^{102}$	<b><math>2^{96}</math></b>	$2^{102}$	$2^{112}$	$2^{122}$

Now, our best attack is in  $2^{96}$ , requires  $2^{65}$  bits of memory and only 82 kilobytes of keystream.

### Better Attacks with an Iterated Variant XL

It is possible to improve this attack slightly by iterating the XL algorithm. Here is one possible way to do this. We start with  $m = 1.6 \cdot 2^{18}$  keystream bits. The probability that all the corresponding  $m$  approximations of degree 4 are true is  $(1 - \frac{1}{2^{17}})^m \approx 2^{-4.62}$ . This means that the whole attack should be repeated on average  $2^{4.62}$  times. Now we apply the XL algorithm with  $D = 5$ , i.e. we multiply each equation by nothing or one of the variables. We have  $R = 129 \cdot 1.6 \cdot 2^{18}$ . The goal is however not to eliminate most of the terms, but only all the terms that contain one variable  $k_0$ . Let  $T'$  be the number of terms in  $T$  that does not contain the first variable  $k_0$ . We have  $T = \sum_{i=0}^D \binom{n}{i}$  and  $T' = \sum_{i=0}^D \binom{n-1}{i}$ . The number of remaining equations of degree  $K' = 5$  that contain only  $n' = 127$  variables is  $R - (T - T') = 129 \cdot 1.6 \cdot 2^{18} - \sum_{i=0}^5 \binom{128}{i} + \sum_{i=0}^5 \binom{127}{i} = 2^{25.37}$ . We have  $R'/(T - T') = 5.06$  and the elimination takes the time of  $7 \cdot T^{\log_2 7} / 64 = 2^{75.5}$ . Then we re-apply XL for  $K = 5$ ,  $n' = 127$ ,  $m' = R - (T - T') = 2^{25.37}$  and  $D = 6$ . We have  $R'/T' = 1.021$  and XL works with the complexity of  $2^{87.59}$ .

The complexity of the whole attack is:  $2^{4.62} (2^{75.5} + 2^{87.6}) = 2^{92.2}$  CPU clocks. Our best attack is now in  $2^{92}$ , it requires still  $2^{65}$  bits of memory, and now only 51 kilobytes of keystream.

## 7 Other Known Attacks on Toyocrypt

### Comparison with Previously Known Attacks

Our new attack is much better than the generic purpose time/memory/data tradeoff attack described by Shamir and Biryukov in [22], that given the same number of keystream bits, about  $2^{19}$ , will require about  $2^{109}$  computations (in pre-computation phase).

Our attack is sometimes better, and sometimes worse than the Mihaljevic and Imai attack from [17]. On one side, for example, in [17], given much more data, for example  $2^{48}$  bits, and in particular at least some 32 consecutive bits of the keystream, and given the same quantity of memory  $2^{64}$ , the key can be recovered with a pre-computation of  $2^{80}$  and processing time in  $2^{32}$ . On the other side, if only the keystream does not contain 32 consecutive bits, only our attack will work. Similarly, if only the keystream available is  $2^{19}$ , then the Mihaljevic and Imai attack from [17] requires a pre-computation of about  $2^{109}$ , exactly as in the generic tradeoff attack from [22].

### New Attack Scenarios S3 and S4

Since this paper has been written, there was substantial progress in alge-

braic attacks on stream ciphers. Generalizing the attack scenarios S1 and S2 described in this paper, two new attack scenarios S3 and S4 have been introduced by Courtois and Meier [5]. The principle of these new attacks is to generate new multivariate equations of substantially lower degree than the original ones. Thus, the authors are able to break Toyocrypt in  $2^{49}$  CPU clocks instead of  $2^{92}$ , and also present an attack in  $2^{57}$  for LILI-128.

## 8 Extensions and Generalizations

**Improved Elimination Methods.** One should expect that a careful implementation of our attack might be much faster. For this, one should use a more careful elimination algorithm, that generates the equations in a specific order and eliminates monomials progressively, so that they are not generated anymore. We also expect that fast Gröbner bases algorithms such as Faugère’s F5/2 [8] would improve our attack.

**Variants of Toyocrypt.** Our XL-based attacks can cryptanalyse not only Toyocrypt but also many variants of Toyocrypt that resist to all known attacks. For example, if in Toyocrypt we replace the bilinear part of  $f$  by a random quadratic form, such ”guess-and-find” attacks as in [17] are not possible anymore, still our XL-based higher degree correlation attack works all the same. The same is true when we leave the quadratic part unchanged and add to  $f$  some terms of degree 3 and 4 in variables  $x_2, x_4, \dots$ . It is also possible to see that, if the positions of the known bits of the keystream are sparsely distributed, and we do not have any known 32 consecutive bits, the attacks from [17] will not work anymore, and our attack still works.

**More Advanced Higher Order Correlation Attacks.** It seems that our XL-based higher order correlation attack, may be combined with almost any kind of fast correlation attack, another higher order correlation attack, or other attacks such as Anderson’s augmented function attacks from [1]. For example, let us assume that for Toyocrypt we know several linear approximations for all the  $s_i$  that appear in the term of degree 17 of  $f$ . Assume that all these approximations allow to predict when the corresponding  $s_i = 1$  with a probability only slightly smaller than  $1/2$ . Even if this deviation from  $1/2$  is small, the product of all the 17 will be equal to 1 with a probability somewhat smaller than  $2^{-17}$ . Thus, in our attack described above, we are able to select in the keystream, some  $m$  bits, for which  $K = 4$  and  $\varepsilon$  is smaller. This will strictly decrease the complexity.

## 9 Conclusion

In this paper we studied higher order correlation attacks on stream ciphers. Our approach is to reduce their cryptanalysis to the problem of solving an overdefined system of multivariate equations. In order to solve such systems of equations, we studied an extension of the XL algorithm proposed at Eurocrypt 2000 for the case of quadratic equations [23]. The problem about XL is that it is heuristic, not all equations that appear in XL are linearly independent, and thus it is somewhat difficult to say to what extent it works. In this paper we showed that we are always able to explain the origin of the linear dependencies that appear in XL and to predict the **exact** number of non-redundant equations in XL. We do not have a proof that this prediction is always correct, but there is no doubt that XL works exactly as predicted for largely overdefined systems.

From our results on XL, we presented a new attack on the Toyocrypt stream cipher, accepted to the second phase of the Japanese government Cryptrec call for cryptographic primitives. It is a 128-bit stream cipher, and at the time of submission of Toyocrypt, it was claimed to resist to all known attacks on stream ciphers. In our best version of the XL-based higher-order correlation attack we have  $K = 4$  and  $D < 2 * K$ . In this case, from our theory, confirmed by our simulations, we expect that all the equations generated in XL will be linearly independent, until the system achieves a complete saturation. Our faster attack on Toyocrypt requires  $2^{92}$  CPU clocks for a 128-bit cipher. This complexity can be achieved using only 51 kilobytes of the keystream, that does not have to be consecutive, and using  $2^{65}$  bits of memory. Other tradeoffs are possible and our new attack is in many cases the best attack known on Toyocrypt.

We conclude that higher order correlation immunity, should be taken more seriously than previously thought, in the design of stream ciphers.

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## A The exact behaviour of XL for $K \geq 2$ .

Let  $Free$  be the maximum number of equations that are linearly independent in XL algorithm. We will show how to compute  $Free$  exactly and compare the results with computer simulations.

In all the simulations that follow, we pick a random system of linearly independent equations  $y_i = f_i(x_0, \dots, x_{n-1})$  of degree  $\leq K$  (non-homogenous). Then we pick a random input  $x = (x_0, \dots, x_{n-1})$  and we modify the constants in the system in order to have a system that gives 0 in  $x$ , i.e. we write a system to solve as  $l_i(x_0, \dots, x_{n-1}) = 0$ , for  $i = 1, \dots, m$ .

### A.1 The behaviour of XL for $K = 2$ and $D = 3$ .

By definition,  $Free$  is smaller than  $R$  and cannot exceed  $T$ , see Section 2.1. Therefore:

$$Free \leq \min(T, R)$$

We have done various computer simulations with  $D = 3$  and in our simulations, for  $K = 2$  and  $D = 3$ , we have always<sup>12</sup>  $Free = \min(T, R) - \epsilon$  with  $\epsilon = 0, 1, 2$  or  $3$ . In the following table we fix  $n$  and try XL on a random system of  $m$  linearly independent equations with growing  $m$  and with a fixed  $D$ .

$K$	2	2	2	2	2	2	2	2	2	2	2	2	2
$n$	10	10	10	10	10	20	20	20	20	20	64	64	64
$m$	10	14	16	17	18	20	40	50	60	65	512	1024	1024
$D$	3	3	3	3	3	3	3	3	3	3	3	3	3
$R$	110	154	176	187	198	420	840	1050	1260	1365	33280	66560	66560
$T$	176	176	176	176	176	1351	1351	1351	1351	1351	43745	43745	43745
$Free$	110	154	174	175	175	420	840	1050	1260	1350	33280	43744	43744

Figure 1: XL simulations for  $K = 2$  and  $D = 3$ .

$n$  number of variables.

$m$  number of equations.

$D$  we generate equations of total degree  $\leq D$  in the  $x_i$ .

$R$  number of equations generated (independent or not).

$T$  number of monomials of degree  $\leq D$

$Free$  number of linearly independent equations among the  $R$  equations.

◇ XL will work when  $Free \geq T - D$ .

<sup>12</sup> Actually  $Free$  is almost always the minimum of the two functions, around the point where the two graphics meet, we sometimes observed a "smooth" transition, and in this case we observe that  $Free = \min(T, R) - \epsilon$  with  $\epsilon = 0, 1, 2$  or  $3$ . In our simulations the smooth transition is visible for  $n = 10$ ,  $m = 16$ ,  $D = 3$ .





**Conjecture A.3.1 (Behaviour of XL for  $D < 3K$ ).**

1. For  $D = K \dots 2K - 1$  there are no linear dependencies when  $R \geq T$  and we have  $Free = \min(T, R) - \epsilon$  with  $\epsilon = 0, 1, 2$  or  $3$ .
2. For  $D = 2K \dots 3K - 1$  there are linear dependencies and we have

$$Free = \min \left( T, R - \left( \sum_{i=0}^{D-2K} \binom{n}{i} \right) \left( \binom{m}{2} + m \right) \right) - \epsilon \text{ with } \epsilon = 0, 1, 2 \text{ or } 3.$$

The factor  $\left( \binom{m}{2} + m \right)$  is due to the linear dependencies of type  $l_i[l_j] = [l_i]l_j$  and  $l_i[l_i] = l_i$  as explained above. Moreover when  $D > 2K$  there are other linear dependencies that are products of these by monomials in  $x_i$  of degree up to  $D - 2K$ , and to count these we have multiplied their number by a factor  $\left( \sum_{i=0}^{D-2K} \binom{n}{i} \right)$ .

3. It is also possible to anticipate what happens for  $D \geq 3K$ . However, it is more complex, and in this paper we do not need to know this.

Here is a series of simulations with different  $K > 2$  and different values of  $D$  to see if our conjecture is verified in practice.

$K$	3	3	3	3	3	3	3	3	3	3	3	3	3	3
$n$	10	10	10	10	10	10	10	16	16	16	16	16	16	16
$m$	10	10	10	10	10	10	10	16	16	16	16	16	16	16
$D$	3	4	5	6	7	8		3	4	5	6	7		
$R$	10	110	560	1760	3860	6380		16	272	2192	11152	40272		
$T$	176	386	638	848	968	1013		697	2517	6885	14893	26333		
$Free$	10	110	560	846	966	1011		16	272	2192	11016	26330		

Figure 3: XL with  $K = 3$  (notations as on Fig. 1).

$K$	4	4	4	4	4	4	4	4	4	4	4	4	4	4
$n$	10	10	10	10	10	10	10	16	16	16	16	16	16	16
$m$	10	10	10	10	10	10	10	16	16	16	16	16	16	16
$D$	4	5	6	7	8	9	10	4	5	6	7	8		
$R$	10	110	560	1760	3860	6380	8480	16	272	2192	11152	40272		
$T$	386	638	848	968	1013	1023	1024	2517	6885	14893	26333	39202		
$Free$	10	110	560	966	1011	1021	1022	16	272	2192	11152	39200		

Figure 4: XL with  $K = 4$  (notations as on Fig. 1).

By inspection we see that these results, all our previous simulations, as well as those done in [6], always do confirm the Conjecture A.3.1.