

Deterministic Polynomial Time Equivalence of Computing the RSA Secret Key and Factoring

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Abstract. We address one of the most fundamental problems concerning the RSA cryptosystem: does the knowledge of the RSA public and secret key-pair (e, d) yield the factorization of $N = pq$ in polynomial time? It is well-known that there is a *probabilistic* polynomial time algorithm that on input (N, e, d) outputs the factors p and q . We present the first *deterministic* polynomial time algorithm that factors N provided that $e, d < \phi(N)$. Our approach is an application of Coppersmith's technique for finding small roots of univariate modular polynomials.

Keywords: RSA, Coppersmith's theorem.

1 Introduction

The most basic security requirement for a public key cryptosystem is that it should be hard to recover the secret key from the public key. To establish this property, one usually identifies a well-known hard problem P and shows that solving P is polynomial-time equivalent to recovering the secret key from the public key.

In this paper we consider the RSA cryptosystem [7]. We denote by $N = pq$ the modulus, product of two primes p and q of the same bit-size. Furthermore, we let e, d be integers such that $e \cdot d = 1 \pmod{\phi(N)}$, where $\phi(N) = (p-1) \cdot (q-1)$ is Euler's totient function. The public key is then (N, e) and the secret key is (N, d) .

It is well known that there exists a *probabilistic* polynomial time equivalence between computing d and factoring N . The proof is given in the original RSA paper by Rivest, Shamir and Adleman [7] and is based on a work by Miller [6].

In this paper, we show that the equivalence can actually be made deterministic, namely we present a *deterministic* polynomial-time algorithm that on input (N, e, d) outputs the factors p and q , provided that $e \cdot d \leq N^2$. Since for standard RSA, the exponents e and d are defined modulo $\phi(N)$, this gives $ed < \phi(N)^2 < N^2$ as required. Our technique is a variant of Coppersmith's

theorem for finding small roots of univariate polynomial equations [1], which is based on the LLL lattice reduction algorithm [4]. We also generalize our algorithm to the case of unbalanced prime factors p and q . We obtain that the more imbalanced the prime factors are, the larger is the required upper bound on ed . The paper is an extended version of the paper published by A. May at Crypto 2004 [5].

2 Background on Lattices

Let $u_1, \dots, u_\omega \in \mathbb{Z}^n$ be linearly independent vectors with $\omega \leq n$. The lattice L spanned by $\langle u_1, \dots, u_\omega \rangle$ consists of all integral linear combinations of u_1, \dots, u_ω , that is:

$$L = \left\{ \sum_{i=1}^{\omega} n_i \cdot u_i \mid n_i \in \mathbb{Z} \right\}$$

Such a set $\{u_1, \dots, u_\omega\}$ of vectors is called a lattice *basis*. All the bases have the same number of elements, called the *dimension* or *rank* of the lattice. We say that the lattice is full rank if $\omega = n$. Any two bases of the same lattice can be transformed into each other by a multiplication with some integral matrix of determinant ± 1 . Therefore, all the bases have the same Gramian determinant $\det_{1 \leq i, j \leq d} \langle u_i, u_j \rangle$. One defines the *determinant* of the lattice as the square root of the Gramian determinant. If the lattice is full rank, then the determinant of L is equal to the absolute value of the determinant of the $\omega \times \omega$ matrix whose rows are the basis vectors u_1, \dots, u_ω .

The LLL algorithm [4] computes a short vector in a lattice :

Theorem 1 (LLL). *Let L be a lattice spanned by (u_1, \dots, u_ω) . The LLL algorithm, given (u_1, \dots, u_ω) , finds in polynomial time a vector b_1 such that:*

$$\|b_1\| \leq 2^{(\omega-1)/4} \det(L)^{1/\omega}$$

3 The Case of Balanced p and q

In this section, we show the *deterministic* polynomial-time equivalence between recovering d and factoring N , when N is the product of two primes p and q of same bit-size; this is the standard RSA setting. We generalize to an $N = pq$ with unbalanced prime factors in the next section.

Theorem 2. *Let $N = p \cdot q$, where p and q are two prime integers of same bit-size. Let e, d be such that $e \cdot d = 1 \pmod{\phi(N)}$. Then assuming that $e \cdot d \leq N^2$, given (N, e, d) one can recover the factorization of N in deterministic polynomial time.*

Proof. Let $U = e \cdot d - 1$ and $s = p + q - 1$. Our goal is to recover s from N and U . Then given N and s it is straightforward to recover the factorization of N .

First, we assume that we are given the high-order bits s_0 of s . More precisely, we let X be some integer, and write $s = s_0 \cdot X + x_0$, where $0 \leq x_0 < X$. The integer s_0 will eventually be recovered by exhaustive search. Moreover, we denote $\phi = \phi(N)$. From $\phi = (p-1) \cdot (q-1) = N - s = N - s_0 \cdot X - x_0$ we obtain the following equations :

$$U = 0 \pmod{\phi} \tag{1}$$

$$x_0 - N + s_0 \cdot X = 0 \pmod{\phi} \tag{2}$$

Let m, k be integers. We consider the polynomials :

$$g_{ij}(x) = x^i \cdot (x - N + s_0 \cdot X)^j \cdot U^{m-j}$$

for $0 \leq j \leq m$ and $i = 0$, and for $j = m$ and $1 \leq i \leq k$. Then from equations (1) and (2), we have that for all previous (i, j) :

$$g_{ij}(x_0) = 0 \pmod{\phi^m}$$

Our goal is to find a non-zero integer linear combination $h(x)$ of the polynomials $g_{ij}(x)$, with small coefficients. Then $h(x_0) = 0 \pmod{\phi^m}$, and using the following lemma [3], if the coefficients of $h(x)$ are sufficiently small, then $h(x_0) = 0$ over the integers. Then x_0 can be recovered using any standard root-finding algorithm; eventually from x_0 one recovers the factorization of N . Given a polynomial $h(x) = \sum h_i x^i$, we denote by $\|h(x)\|$ the Euclidean norm of the vector of its coefficients h_i .

Lemma 1 (Howgrave-Graham). *Let $h(x) \in \mathbb{Z}[x]$ which is a sum of at most ω monomials. Suppose that $h(x_0) = 0 \pmod{\phi^m}$ where $|x_0| \leq X$ and $\|h(xX)\| < \phi^m / \sqrt{\omega}$. Then $h(x_0) = 0$ holds over the integers.*

Proof. We have :

$$\begin{aligned} |h(x_0)| &= \left| \sum h_i x_0^i \right| = \left| \sum h_i X^i \left(\frac{x_0}{X} \right)^i \right| \\ &\leq \sum \left| h_i X^i \left(\frac{x_0}{X} \right)^i \right| \leq \sum |h_i X^i| \\ &\leq \sqrt{\omega} \|h(xX)\| < \phi^m \end{aligned}$$

Since $h(x_0) = 0 \pmod{\phi^m}$, this gives $h(x_0) = 0$. □

We consider the lattice L spanned by the coefficient vectors of the polynomials $g_{ij}(xX)$. One can see that these coefficient vectors form a triangular basis of a full-rank lattice of dimension $\omega = m + k + 1$ (for an example, see Fig. 1). The determinant of the lattice is then the product of the diagonal entries, which gives :

$$\det L = X^{(m+k)(m+k+1)/2} U^{m(m+1)/2} \tag{3}$$

	1	x	x^2	x^3	x^4	x^5	x^6
$g_{00}(xX)$	U^3						
$g_{01}(xX)$	*	$U^2 X$					
$g_{02}(xX)$	*	*	UX^2				
$g_{03}(xX)$	*	*	*	X^3			
$g_{13}(xX)$		*	*	*	X^4		
$g_{23}(xX)$			*	*	*	X^5	
$g_{33}(xX)$				*	*	*	X^6

Fig. 1. The lattice L of the polynomials $g_{ij}(xX)$ for $k = m = 3$. The symbol '*' correspond to non-zero entries whose value is ignored.

Using LLL (theorem 1), one obtains a non-zero short vector b whose norm is guaranteed to satisfy :

$$\|b\| \leq 2^{(\omega-1)/4} \cdot (\det L)^{1/\omega}$$

The vector b is the coefficient vector of some polynomial $h(xX)$ with $\|h(xX)\| = \|b\|$. The polynomial $h(x)$ is then an integer linear combination of the polynomials $g_{ij}(x)$, which implies that $h(x_0) = 0 \pmod{\phi^m}$. In order to apply Lemma 1, it is therefore sufficient to have that :

$$2^{(\omega-1)/4} \cdot (\det L)^{1/\omega} < \frac{\phi^m}{\sqrt{\omega}}$$

Using the inequalities $\sqrt{\omega} \leq 2^{(\omega-1)/2}$, $\phi > N/2$ and $\omega - 1 = m + k \geq m$, we obtain the following sufficient condition :

$$\det L \leq N^{m \cdot \omega} \cdot 2^{-2 \cdot \omega \cdot (\omega-1)}$$

From equation (3) and inequality $U < N^2$, this gives :

$$X^{(m+k)(m+k+1)/2} \leq N^{m \cdot k} \cdot 2^{-2 \cdot \omega \cdot (\omega-1)}$$

which gives the following condition for X :

$$X \leq \frac{N^{\gamma(m,k)}}{16}, \quad \gamma(m,k) = \frac{2 \cdot m \cdot k}{(m+k) \cdot (m+k+1)}$$

Our goal is to maximize the bound X on x_0 , so that as few as possible bits will eventually have to be exhaustively searched. For a fixed m , the function $\gamma(m,k)$ is maximal for $k = m$. The corresponding bound for $k = m$ is then :

$$X \leq \frac{1}{16} \cdot N^{\frac{1}{2} - \frac{1}{4m+2}}. \quad (4)$$

In the following we denote by \log the logarithm to the base 2. For an X satisfying the previous inequality, the previous algorithm applies the LLL reduction algorithm on a lattice of dimension $2 \cdot m + 1$ and with entries bounded by $\mathcal{O}(N^{2m})$.

Since the running-time of LLL is polynomial in the dimension and in the size of the entries, given s_0 such that $s = s_0 \cdot X + x_0$ with $0 \leq x_0 < X$, the previous algorithm recovers the factorization of N in time polynomial in $(\log N, m)$.

Finally, taking the greatest integer X satisfying (4), and using $s = p + q - 1 \leq 3\sqrt{N}$, we obtain :

$$s_0 \leq \frac{s}{X} \leq 49 \cdot N^{1/(4m+2)}$$

Then, taking $m = \lfloor \log N \rfloor$, we obtain that s_0 is upper-bounded by a constant. The previous algorithm is then run for each possible value of s_0 , and the correct s_0 enables to recover the factorization of N , in time polynomial in $\log N$. \square

4 Generalization to Unbalanced Prime Factors

The previous algorithm fails when the prime factors p and q are unbalanced, because in this case we have that $s = p + q - 1 \gg \sqrt{N}$. This implies that s is much greater than the bound on X given by inequality (4).

In this section, we provide an algorithm which extends the result of the previous section to unbalanced prime-factors. We use a technique introduced by Durfee and Nguyen in [2], which consists in using two separate variables x and y for the primes p and q , and replacing each occurrence of $x \cdot y$ by N .

The following theorem shows that the factorization of N given (e, d) becomes easier when the prime factors are imbalanced. Namely, the condition on the product $e \cdot d$ becomes weaker. For example, we obtain that for $p < N^{1/4}$, the modulus N can be factored in polynomial time given (e, d) if $e \cdot d \leq N^{8/3}$ (instead of N^2 for prime factors of equal size).

Theorem 3. *Let β and $0 < \delta \leq 1/2$ be real values, such that $2\beta\delta(1 - \delta) \leq 1$. Let $N = p \cdot q$, where p and q are two prime integers such that $p < N^\delta$ and $q < 2 \cdot N^{1-\delta}$. Let e, d be such that $e \cdot d \equiv 1 \pmod{\phi(N)}$, and $0 < e \cdot d \leq N^\beta$. Then given (N, e, d) one can recover the factorization of N in deterministic polynomial time.*

Proof. Let $U = ed - 1$ as previously. Our goal is to recover p, q from N and U . We have the following equations :

$$U = 0 \pmod{\phi} \tag{5}$$

$$p + q - (N + 1) = 0 \pmod{\phi} \tag{6}$$

Let $m \geq 1$, $a \geq 1$ and $b \geq 0$ be integers. We define the following polynomials $g_{ijk}(x, y)$:

$$g_{ijk}(x, y) = x^i \cdot y^j \cdot U^{m-k} \cdot (x + y - (N + 1))^k$$

$$\begin{cases} i \in \{0, 1\}, & j = 0, & k = 0, \dots, m \\ 1 < i \leq a, & j = 0, & k = m \\ i = 0, & 1 \leq j \leq b, & k = m \end{cases}$$

In the definition of the polynomials $g_{ijk}(x, y)$, we replace each occurrence of $x \cdot y$ by N ; therefore, the polynomials $g_{ijk}(x, y)$ contain only monomials of the form x^r and y^r . From equations (5) and (6), we obtain that (p, q) is a root of $g_{ijk}(x, y)$ modulo ϕ^m , for all previous (i, j, k) :

$$g_{ijk}(p, q) = 0 \pmod{\phi^m}$$

Now, we assume that we are given the high-order bits p_0 of p and the high-order bits q_0 of q . More precisely, for some integers X and Y , we write $p = p_0 \cdot X + x_0$ and $q = q_0 \cdot Y + y_0$, with $0 \leq x_0 < X$ and $0 \leq y_0 < Y$. The integers p_0 and q_0 will eventually be recovered by exhaustive search.

We define the translated polynomials :

$$t_{ijk}(x, y) = g_{ijk}(p_0 \cdot X + x, q_0 \cdot Y + y)$$

It is easy to see that for all (i, j, k) , we have that (x_0, y_0) is a root of $t_{ijk}(x, y)$ modulo ϕ^m :

$$t_{ijk}(x_0, y_0) = 0 \pmod{\phi^m}$$

As in the previous algorithm, our goal is to find a non-zero integer linear combination $h(x, y)$ of the polynomials $t_{ijk}(x, y)$, with small coefficients. Then $h(x_0, y_0) = 0 \pmod{\phi^m}$, and using the following lemma, if the coefficients of $h(x, y)$ are sufficiently small, then $h(x_0, y_0) = 0$ over the integers. Then one can define the polynomial $h'(x) = (p_0 \cdot X + x)^{m+b} \cdot h(x, N/(p_0 \cdot X + x) - q_0 \cdot Y)$. Since $h(x, y)$ is not identically zero and $h(x, y)$ contains only x powers and y powers, the polynomial $h'(x)$ cannot be identically zero. Moreover $h'(x_0) = 0$, which enables to recover x_0 using any standard root-finding algorithm, and eventually the primes p and q . Given a polynomial $h(x, y) = \sum h_{ij} x^i y^j$, we denote by $\|h(x, y)\|$ the Euclidean norm of the vector of its coefficients h_{ij} .

Lemma 2 (Howgrave-Graham). *Let $h(x, y) \in \mathbb{Z}[x, y]$ which is a sum of at most ω monomials. Suppose that $h(x_0, y_0) = 0 \pmod{\phi^m}$ where $|x_0| \leq X$, $|y_0| \leq Y$ and $\|h(xX, yY)\| < \phi^m / \sqrt{\omega}$. Then $h(x_0, y_0) = 0$ holds over the integers.*

Proof. We have:

$$\begin{aligned} |h(x_0, y_0)| &= \left| \sum h_{ij} x_0^i y_0^j \right| = \left| \sum h_{ij} X^i Y^j \left(\frac{x_0}{X}\right)^i \left(\frac{y_0}{Y}\right)^j \right| \\ &\leq \sum \left| h_{ij} X^i Y^j \left(\frac{x_0}{X}\right)^i \left(\frac{y_0}{Y}\right)^j \right| \leq \sum |h_{ij} X^i Y^j| \\ &\leq \sqrt{\omega} \|h(xX, yY)\| < \phi^m \end{aligned}$$

Since $h(x_0, y_0) = 0 \pmod{\phi^m}$, this gives $h(x_0, y_0) = 0$. □

We consider the lattice L spanned by the coefficient vectors of the polynomials $t_{ijk}(xX, yY)$. One can see that these coefficient vectors form a triangular basis of a full-rank lattice of dimension $\omega = 2m + a + b + 1$ (for an example,

	1	x	y	x^2	y^2	x^3	y^3	x^4	x^5	y^4
$g_{000}(xX, yY)$	U^3									
$g_{100}(xX, yY)$	*	U^3X								
$g_{001}(xX, yY)$	*	*	U^2Y							
$g_{101}(xX, yY)$	*	*	*	U^2X^2						
$g_{002}(xX, yY)$	*	*	*	*	UY^2					
$g_{102}(xX, yY)$	*	*	*	*	*	UX^3				
$g_{003}(xX, yY)$	*	*	*	*	*	*	Y^3			
$g_{103}(xX, yY)$	*	*	*	*	*	*	*	X^4		
$g_{203}(xX, yY)$	*	*	*	*	*	*	*	*	X^5	
$g_{013}(xX, yY)$	*	*	*	*	*	*	*	*	*	Y^4

Fig. 2. The lattice L of the polynomials $g_{ijk}(xX, yY)$ for $m = 3$, $a = 2$ and $b = 1$. The symbol '*' correspond to non-zero entries whose value is ignored.

see Fig. 2). The determinant of the lattice is then the product of the diagonal entries, which gives :

$$\det L = X^{(m+a)(m+a+1)/2} Y^{(m+b)(m+b+1)/2} U^{m(m+1)} \quad (7)$$

As previously, using LLL, one obtains a non-zero polynomial $h(x, y)$ such that:

$$\|h(xX, yY)\| \leq 2^{(\omega-1)/4} \cdot (\det L)^{1/\omega}$$

In order to apply Lemma 2, it is therefore sufficient to have that :

$$2^{(\omega-1)/4} \cdot (\det L)^{1/\omega} < \phi^m / \sqrt{\omega}$$

As in the previous section, using $\sqrt{\omega} \leq 2^{(\omega-1)/2}$, $\phi > N/2$ and $\omega - 1 \geq m$, it is sufficient to have :

$$\det L \leq N^{m \cdot \omega} \cdot 2^{-2 \cdot \omega \cdot (\omega-1)} \quad (8)$$

Let $a = \lfloor (u-1) \cdot m - 1 \rfloor$ and $b = \lfloor (v-1) \cdot m - 1 \rfloor$ for some reals u, v . We obtain that $(m+a)(m+a+1) \leq m^2 u^2$ and $(m+b)(m+b+1) \leq m^2 v^2$. We denote $X = N^{\delta_x}$ and $Y = N^{\delta_y}$. From equation (7) we obtain that :

$$\frac{\log(\det L)}{\log N} \leq m^2 \cdot \left(\delta_x \cdot \frac{u^2}{2} + \delta_y \cdot \frac{v^2}{2} + \beta \right) + \beta \cdot m \quad (9)$$

Moreover, using $m(u+v) - 3 < \omega \leq m(u+v)$, we obtain :

$$\log \left(N^{m \cdot \omega} \cdot 2^{-2 \cdot \omega \cdot (\omega-1)} \right) \geq m(m(u+v) - 3) \log N - 2m^2(u+v)^2 \quad (10)$$

Therefore, we obtain from inequalities (8), (9) and (10) the following sufficient condition :

$$u + v - \delta_x \frac{u^2}{2} - \delta_y \frac{v^2}{2} - \beta \geq \frac{\beta + 3}{m} + \frac{2}{\log N} (u + v)^2$$

The function $u \rightarrow u - \delta_x \cdot u^2/2$ is maximal for $u = 1/\delta_x$, with a maximum equal to $1/(2\delta_x)$. The same holds for the function $v \rightarrow v - \delta_y \cdot v^2/2$. Therefore, taking $u = 1/\delta_x$ and $v = 1/\delta_y$, we obtain the sufficient condition :

$$\frac{1}{2\delta_x} + \frac{1}{2\delta_y} - \beta \geq \frac{\beta+3}{m} + \frac{2}{\log N} \left(\frac{1}{\delta_x} + \frac{1}{\delta_y} \right)^2 \quad (11)$$

For $X = N^{\delta_x}$ and $Y = N^{\delta_y}$ satisfying the previous condition, given p_0 and q_0 , the algorithm recovers x_0, y_0 and then p and q in time polynomial in $(m, \log N)$.

In the following, we show that p_0 and q_0 can actually be recovered by exhaustive search, while remaining polynomial-time in $\log N$.

Let ε be such that $0 < \varepsilon \leq \delta/2$. We have the following inequalities :

$$\frac{1}{\delta - \varepsilon} = \frac{1}{\delta(1 - \frac{\varepsilon}{\delta})} \geq \frac{1}{\delta} \left(1 + \frac{\varepsilon}{\delta} \right) \quad \text{and} \quad \frac{1}{1 - \delta - \varepsilon} \geq \frac{1}{1 - \delta} \left(1 + \frac{\varepsilon}{1 - \delta} \right)$$

From $2\beta\delta(1 - \delta) \leq 1$, we obtain :

$$2\beta \leq \frac{1}{\delta(1 - \delta)} = \frac{1}{\delta} + \frac{1}{1 - \delta}$$

which gives :

$$\frac{1}{\delta - \varepsilon} + \frac{1}{1 - \delta - \varepsilon} - 2\beta \geq \varepsilon \left(\frac{1}{\delta^2} + \frac{1}{(1 - \delta)^2} \right)$$

Therefore, taking $\delta_x = \delta - \varepsilon$ and $\delta_y = 1 - \delta - \varepsilon$, we obtain from (11) the following sufficient condition :

$$\frac{\delta}{2} \geq \varepsilon \geq 2 \cdot \left(\frac{\beta+3}{m} + \frac{2}{\log N} \left(\frac{1}{\delta} + \frac{1}{1 - \delta} \right)^2 \right) \left(\frac{1}{\delta^2} + \frac{1}{(1 - \delta)^2} \right)^{-1}$$

Taking $m = \lceil \log N \rceil$, this condition can always be satisfied for large enough $\log N$. Taking the corresponding lower-bound for ε , we obtain $\varepsilon = \mathcal{O}(1/\log N)$, which gives $N^\varepsilon \leq C$ for some constant C . Therefore, we obtain that p_0 and q_0 are upper-bounded by the constants C and $2C$:

$$p_0 \leq \frac{p}{X} \leq N^{\delta - \delta_x} \leq N^\varepsilon \leq C$$

$$q_0 \leq \frac{q}{Y} \leq 2N^{1 - \delta - \delta_y} \leq 2N^\varepsilon \leq 2C$$

This shows that p_0 and q_0 can be recovered by exhaustive search. The total running-time is still polynomial in $\log N$. \square

5 Practical Experiments

We have implemented the two algorithms of sections 3 and 4, using Shoup’s NTL library [8]. First, we describe in Table 1 the experiments with prime factors of equal bit-size, with $e \cdot d \simeq N^2$. We assume that we are given the ℓ high-order bits of s ; the observed running time for a single execution of LLL is denoted by t . The total running time for factoring N is then estimated as $T \simeq 2^\ell \cdot t$. We obtain that the factorization would take a few days for a 512-bit modulus, and a few years for a 1024-bit modulus.

N	bits given	dimension	t	T
512 bits	14 bits	21	70 s	13 days
512 bits	10 bits	29	7 min	5 days
512 bits	9 bits	33	16 min	5 days
1024 bits	26 bits	21	7 min	900 years
1024 bits	19 bits	29	40 min	40 years
1024 bits	17 bits	33	90 min	23 years

Table 1. Bit-size of N , number of bits to be exhaustively searched, lattice dimension, observed running-time for a single LLL-reduction t , and estimated total running-time T , with $e \cdot d \simeq N^2$. The experiments were performed on a 1.6 GHz PC running under Windows 2000/Cygwin.

The experiments with prime factors of unbalanced size with $e \cdot d \simeq N^2$ are summarized in Table 2. In this case, it was not necessary to know the high-order bits of p and q , and one recovers the factorization of N after a single application of LLL. The table shows that the factorization of N is easier when the prime factors are unbalanced.

N	δ	dimension	t
512 bits	0.25	16	2 s
512 bits	0.3	29	2 min
1024 bits	0.25	16	15 s
1024 bits	0.3	29	10 min

Table 2. Bit-size of the RSA modulus N such that $p < N^\delta$, lattice dimension, observed running-time for factoring N , with $e \cdot d \simeq N^2$. The experiments were performed on a 1.6 GHz PC running under Windows 2000/Cygwin.

6 Conclusion

We have shown the first *deterministic* polynomial time algorithm that factors an RSA modulus N given the pair of public and secret exponents e and d , provided that $e \cdot d < N^2$. The algorithm is a variant of Coppersmith's technique for finding small roots of univariate modular polynomial equations. We have also generalized our algorithm to the case of unbalanced prime factors.

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