Pairing-Friendly Elliptic Curves of Prime Order

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Abstract. Previously known techniques to construct pairing-friendly curves of prime or near-prime order are restricted to embedding degree $k \leqslant 6$. More general methods produce curves over $\mathbb{F}_p$ where the bit length of p is often twice as large as that of the order r of the subgroup with embedding degree k; the best published results achieve $\rho \equiv \log(p)/\log(r) \sim 5/4$. In this paper we make the first step towards surpassing these limitations by describing a method to construct elliptic curves of prime order and embedding degree $k = 12$. We also discuss the role of large CM discriminants D to minimize ρ; in particular, for embedding degree $k = 2q$ where q is prime we show that the ability to handle $\log(D)/\log(r) \sim (q-3)/(q-1)$ enables building curves with $\rho \sim q/(q-1)$.

Keywords: elliptic curves, pairing-based cryptosystems.

1 Introduction

A non-supersingular elliptic curve over $\mathbb{F}_p$ is called pairing-friendly if it contains a subgroup of order r whose embedding degree k is not too large, so that computations in the field $\mathbb{F}_{p^k}$ are feasible. The optimal case occurs when the entire curve has prime order and the desired embedding degree.

Pairing-friendly curves of prime or near-prime order are absolutely essential in certain pairing-based schemes like short signatures with longer useful life. For instance, the length of BLS signatures [5] is the size of the base field p; at the 128-bit security level the scheme should be defined on a group of 256-bit order r and be mapped to a finite field of roughly 3072-bit size p^k. In the optimal case, the embedding degree should be $k = 12$. Of course, other systems would benefit as well, since the space requirements for all quantities involved in cryptographic protocols except pairing values would be kept to a minimum (pairing compression techniques [15, 10] would help reducing the bandwidth for pairing values as well).

The pioneering work of Miyaji, Nakabayashi, and Takano [13] describes how to construct elliptic curves of prime order and embedding degree $k \in \{3, 4, 6\}$. Such curves are now dubbed MNT curves, and satisfy $p \sim r$ by the Hasse bound. Extensions of the original MNT construction to curves of near-prime order were investigated by Scott and Barreto [16], and more recently by Galbraith, McKee, and Valença [9][3]. Unfortunately, those methods are restricted to $k \leqslant 6$ and hence only allow for a tradeoff: one has to choose between increasing the base field to a 512-bit prime p (thus doubling the signature size, which ceases to be "short") or contenting oneself with the lower security level of a 1536-bit finite field $\mathbb{F}_{p^6}$.

Let $\rho \equiv \log(p)/\log(r)$ be the ratio between the bit lengths of the finite field and the order of the subgroup with embedding degree k. Several methods have been proposed to construct curves with arbitrary k, including a "folklore" algorithm [4, chapter 9] credited to Cocks and Pinch [7] and related methods due to Barreto, Lynn, and Scott [1] and to Dupont, Enge, and Morain [8]. In general these algorithms only achieve $\rho \sim 2$.

Algebraic methods may produce curves with ρ closer to unity for certain values of k. Such techniques include the families of curves described by Barreto, Lynn, and Scott [3], and by Brezing and Weng [7]. The latter presents the best known results, achieving $\rho \sim 5/4$ for families of curves with $k = 8$ or $k = 24$, and $\rho \sim (k + 2)/(k - 1)$ for prime k (hence $\rho \leqslant 5/4$ for prime $k \geqslant 13$). Such ratios are already useful under many circumstances. Still, for most embedding degrees the value of ρ is larger than this; for instance, the best published value for $k = 12$ is $\rho \sim 3/2$. Besides, the use of prime k precludes many optimizations that are only possible for even k [2], making the computation of pairings much less efficient.

In spite of all these efforts, constructing pairing-friendly curves with prime order has remained an elusive open problem since it was posed by Boneh *et al.* [5, section 3.5] (see also [6, section 4.5]).

Our main contribution in this paper (section 2) is a surprisingly simple algorithm to construct curves of prime order and embedding degree $k = 12$. The resulting security enhancement is even better than the lower bound of $k = 10$ required by Boneh *et al.*. Using the proposed method, even a 160-bit signature maps to 1920-bit field, where the best algorithms to compute discrete logarithms are worse than Pollard-rho in the elliptic group itself.

We also show (section 3) that the ability to handle large complex multiplication (CM) discriminants may have a positive influence on the minimization of ρ. In particular, for embedding degree $k = 2q$ where q is prime we describe how to build curves with $\rho \sim q/(q - 1)$ and $\log(D)/\log(r) \sim (q - 3)/(q - 1)$. We finally discuss the possibility of building curves of nearly-prime order over extension fields.

[3] Interestingly, the latter work also considers the construction of hyperelliptic curves of genus $g = 2$ analogous to MNT elliptic curves, for which the range of embedding degrees is $k \in \{5, 8, 10, 12\}$, but the security ratio k/g is still bound by 6.

2 The proposed method for $k = 12$

Theorem 1. *There exists an efficient algorithm to construct elliptic curves of prime order (of nearly arbitrary bitlength) and embedding degree $k = 12$ over a prime field.*

Proof. We follow the strategy of parametrising $p(x)$, $n(x)$ and $t(x)$, and using the property that $n \mid \Phi_k(t-1)$ as in [1]. Since Φ_{12} is quartic and we know from the Hasse bound that $n \sim p \sim t^2$, we must take $t(x)$ to be a quadratic polynomial such that $n(x)$ is a quartic factor of $\Phi_{12}(t-1)$.

Galbraith *et al.* showed [9] that the only quadratic polynomials $u(x)$ such that $\Phi_{12}(u(x))$ splits into two irreducible quartic factors are $u(x) = 2x^2$ and $u(x) = 6x^2$. Setting the trace of the Frobenius to be $t(x) = 6x^2 + 1$, we obtain

$$\Phi_{12}(t(x) - 1) = n(x)n(-x),$$

where $n(x) = 36x^4 + 36x^3 + 18x^2 + 6x + 1$. From the relation $n = p + 1 - t$ we get the irreducible polynomial $p(x) = n(x) + t(x) - 1 = 36x^4 + 36x^3 + 24x^2 + 6x + 1$. The CM norm equation becomes

$$DV^2 = 4p - t^2 = 3(6x^2 + 4x + 1)^2.$$

Assuming that, for some x_0, both $n = n(x_0)$ and $p = p(x_0)$ evaluate to prime numbers, the CM method for discriminant $D = 3$ [12, 14] produces curves of form $E(\mathbb{F}_p) : y^2 = x^3 + b$, with $b \neq 0$.

Finding b is actually very simple: take the smallest $b \neq 0$ such that $b + 1$ is a quadratic residue mod$p$ and the point $G = (1, \sqrt{b+1} \bmod p)$, which is clearly on the curve[4], satisfies $nG = \infty$. This method is a simplification of the technique described in [11, section A.14.4] and quickly converges to a suitable b.

We see that the bitlength m of the curve order can be easily tailored by a suitable choice of x_0, namely, start with the smallest $x \sim 2^{m/4}$ such that $n(x)$ has bitlength m and increase it until finding some x_0 such that $n(x_0)$ and $p(x_0)$ are prime. □

In summary, we have the following parametrisation, where x may take either positive or negative values:

$$\begin{aligned}
t &= 6x^2 + 1, \\
n &= 36x^4 + 36x^3 + 18x^2 + 6x + 1, \\
p &= 36x^4 + 36x^3 + 24x^2 + 6x + 1, \\
DV^2 &= 108x^4 + 144x^3 + 84x^2 + 24x + 3 = 3(6x^2 + 4x + 1)^2.
\end{aligned}$$

Algorithm 1 shows how the CM method simplifies in our setting. The algorithm takes as input value the desired bitlength of the primes p and n, and

[4] Since the curve order $n(x)$ is a large prime, there is no point of form $(0, y)$, which would necessarily have order 3.

returns instances of these primes computed as indicated in the proof of theorem 1, plus a parameter $b \in \mathbb{F}_p$ such that the curve $E(\mathbb{F}_p) : y^2 = x^3 + b$ has order n over the field $\mathbb{F}_p$, and the coordinate y of a sample generator $G = (1, y)$. Appendix A gives a few examples of cryptographic interest. Algorithm 1 tends to produce the smallest p and n of the desired bitlength, but it is straightforward to modify it so that the output parameters meet other simple criteria (for instance, the examples in appendix A were selected to maximize p and n while keeping $b = 3$).

Algorithm 1 Constructing a curve of prime order with $k = 12$

INPUT: the approximate desired size m of the curve order (in bits).

OUTPUT: parameters p, n, b, y such that the curve $y^2 = x^3 + b$ has order n over $\mathbb{F}_p$ and the point $G = (1, y)$ is a generator of the curve.

Let $P(x) \equiv 36x^4 + 36x^3 + 24x^2 + 6x + 1$
Compute the smallest $x \approx 2^{m/4}$ such that $\lceil \log_2 P(-x) \rceil = m$.
loop
    $t \leftarrow 6x^2 + 1$
    $p \leftarrow P(-x), \quad n \leftarrow p + 1 - t$
    if p and n are prime **then**
        exit loop
    end if
    $p \leftarrow P(x), \quad n \leftarrow p + 1 - t$
    if p and n are prime **then**
        exit loop
    end if
    $x \leftarrow x + 1$.
end loop
$b \leftarrow 0$
repeat
    repeat
        $b \leftarrow b + 1$
    until $b + 1$ is a quadratic residue mod p
    Compute y such that $y^2 = b + 1 \bmod p$
    $G \leftarrow (1, y)$ on the curve $E : y^2 = x^3 + b$
until $nG = \infty$
return p, n, b, y.

Contrary to the case $k = 12$, finding parametrisations when $\varphi(k) > 4$ (but keeping k reasonably small) seems a rather difficult problem. The method suggested in [9] to find polynomials $u(x)$ such that $\Phi_k(u(x))$ splits, when extended to those cases, implies finding integer or rational points on curves of higher genus. We leave it as an open problem the task of buidling curves of prime order and $\varphi(k) > 4$.

3 Considerations on composite order

Under some circumstances, a reasonably small cofactor may be quite acceptable. For instance, if 256-bit prime fields do not have a substantial impact on bandwidth occupation, the Brezing-Weng family of curves for $k = 8$ and $\rho \sim 5/4$ could provide roughly 200-bit group orders and map the discrete logarithm on the curve to a 2048-bit finite field. Besides, as we already pointed out even values of k are advantageous from the point of view of efficient implementation of the pairing algorithm. It is thus interesting to investigate ways to produce more curves that meet the conditions that k be even and $\rho > 1$ be as small as possible (say, $\rho \leqslant 5/4$).

A naive approach to solving the norm equation $DV^2 = 4h\Phi_k(t-1) - (t-2)^2$, namely, by choosing t and hoping to be able to handle the resulting D, is in general bound to failure since $D \sim t^{\varphi(k)}$, where $\varphi(k)$ is Euler's totient function. For instance, for $k = 2q$ where q is an odd prime we expect to find $D \sim t^{q-1}$.

However, it would be quite simple to obtain curves with $k = 2q$ if we could handle a CM discriminant D as large as t^{q-3}, attaining $\rho \equiv \log(p)/\log(r) \sim q/(q-1)$ as the following reasoning reveals. Let the trace of Frobenius have the form $t = -4u^2 + 2$ for some u (notice that t is negative), and let $x = t - 1$. Assume that $\Phi_k(x)$ takes a prime value. Then set:

$$
\begin{aligned}
h &= -(x-1)/4, \\
r &= \Phi_k(x) \\
&= x^{q-1} - x^{q-2} + x^{q-3} - x^{q-4} + x^{q-5} - \cdots - x + 1 \\
&= x^{q-1} - x^{q-3}(x-1) - x^{q-5}(x-1) - \cdots - x^2(x-1) - (x-1), \\
p &= hr + t - 1, \\
DV^2 &= 4hr - (t-2)^2 \\
&= -(x-1)x^{q-1} + x^{q-3}(x-1)^2 + x^{q-5}(x-1)^2 + \cdots + (x-1)^2 - (x-1)^2 \\
&= -(x-1)x^2[x^{q-3} - (x-1)(x^{q-5} + x^{q-7} + \cdots + 1)].
\end{aligned}
$$

By construction, the $-(x-1)x^2$ factor is a square, so D is the square-free part of $z = x^{q-3} - (x-1)(x^{q-5} + x^{q-7} + \cdots + 1)$. Since $p = hr + t - 1$, it is also clear that $\rho \sim q/(q-1)$. For instance, if $k = 10$ (i.e. $\rho \sim 5/4$) we get $z = x^2 - x + 1$, and a simple search produces parameters like these:

$$
\begin{aligned}
t &= -931556989582 : 40 \text{ bits} \\
r &= 753074106157227719531468778253698105623799226081 : 160 \text{ bits} \\
p &= 175382861816372173247473133505975362972517516867279787545493 : 197 \text{ bits} \\
\rho &\sim 1.237425 \\
D &= 867798424841873127503473 : 80 \text{ bits}
\end{aligned}
$$

Another example, now for $k = 14$ (i.e. $\rho \sim 7/6$) where $z = x^4 - x^3 + x^2 - x + 1$:

$$
\begin{aligned}
t &= -82011134 : 27 \text{ bits} \\
r &= 304254450525046050085067914513460261078757135361 : 158 \text{ bits} \\
p &= 6238063280153705754947329076599940825481364534683333889 : 183 \text{ bits} \\
\rho &\sim 1.153987 \\
D &= 45236739484946456935793243535361 : 106 \text{ bits}
\end{aligned}
$$

Unfortunately, with currently available CM technology the only case where this construction is tractable occurs for $k = 6$, where we get $D = 1$ but also $\rho \sim 3/2$, much worse than plain MNT curves that attain $\rho \sim 1$.

3.1 Curves over extension fields

Another interesting observation is that, while none of the currently known methods to construct pairing-friendly curves for arbitrary k is able to produce curves over an extension field $\mathbb{F}_{p^m}$, it may be possible to fill this gap if sufficiently large D can be handled. As Galbraith *et al.* point out [9], parametrising $t = 5x^2 + 1$ causes $\Phi_5(t-1)$ to split as $\Phi_5(t-1) = r(x)r(-x)$, where $r(x) = 25x^4 + 25x^3 + 15x^2 + 5x + 1$. We observe that with cofactor $h = 4$, this gives $hr + t - 1 = (10x^2 + 5x + 2)^2$, a perfect square. This means that finding an odd value $x \in \mathbb{Z}$ such that r and $p = 10x^2 + 5x + 2$ are both prime enables constructing an elliptic curve over a finite field $\mathbb{F}_{p^2}$ with near-prime order $n = 4r$.

The CM equation here has the form $DV^2 = 5(5x^2 \pm 2x + 1)(15x^2 \pm 10x + 3)$. Solving a Pell-like equation can make one but not both of the factors $5x^2 \pm 2x + 1$ or $15x^2 \pm 10x + 3$ to assume the form dy^2 for small d and some y. One might hope that techniques like Hensel lifting could reduce the square-free part of the other factor to $O(x)$, but it is not clear how to harmonise such techniques to solutions of the Pell-like equation. As a consequence, we expect that $D \sim p \sim r^{1/2}$; practical values of p would need $D \sim 2^{100}$ at least.

Nevertheless, such a parametrisation hints that algebraic methods to build ordinary pairing-friendly curves over extension fields might exist for other embedding degrees, and deserved further research.

4 Conclusion

We have presented a very simple algorithm to construct pairing-friendly curves of prime order and embedding degree $k = 12$. This closes (and actually exceeds) the open problem proposed by Boneh *et al.* [5, section 3.5], and enhances the security level of most pairing-based cryptosystems without increasing bandwidth requirements other than those of the pairing values themselves. We leave it as an open problem the task of extending the method for higher values of k.

We have also discussed ways to produce curves of composite order and reasonably small cofactor as long as large discriminants fall within reach of CM methods, and pointed out the possibility of closing yet another problem, namely, building pairing-friendly curves of nearly-prime order over extension fields. Further exploration of such possibilities is left for future research.

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A Some curves of prime order and $k = 12$

All of the following curves satisfy the equation $E(\mathbb{F}_p) : y^2 = x^3 + 3$, with prime order n and trace of the Frobenius t. A sample generator for any of them is $G = (1, 2)$. In all cases $p \equiv 3 \pmod 4$, and the bitlengths of p and n are equal.

160 bits:

$$p = 1461488496467164787840903033753814258625438681423$$
$$n = 1461488496467164787840901824833429588724731730809$$
$$t = 1208920384669900706950615$$

192 bits:

$$p = 6277097970187419498128716739946879326177323005360194305891$$
$$n = 6277097970187419498128716739867651187424809866926505892541$$
$$t = 79228138752513138433688413351$$

224 bits:

$$p = 26959946583411299945503036141273401768802840431793499004315814874803$$
$$n = 26959946583411299945503036141273396576505989960770895419537263820397$$
$$t = 5192296850471022603584778551054407$$

256 bits:

$$p = 115792089236900232079058937854845988624251390961024547321803220395971\backslash$$
$$200794623$$
$$n = 115792089236900232079058937854845988623911108594104220061519282602749\backslash$$
$$935632649$$
$$t = 340282366920327260283937793221265161975$$