Special Polynomial Families for Generating More Suitable Elliptic Curves for Pairing-Based Cryptosystems

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Abstract

Constructing non-supersingular elliptic curves for pairing-based cryptosystems have attracted much attention in recent years. The best previous technique builds curves with $\rho = lg(q)/lg(r) \approx 1$ ($k = 12$) and $\rho = lg(q)/lg(r) \approx 1.25$ ($k = 24$). When $k > 12$, most of the previous work address the question of representing $r(x)$ as a cyclotomic polynomial. In this paper, we propose a new method to find more pairing-friendly elliptic curves with arbitrary embedding degree k by certain special polynomial families. The new method generates curves with $lg(q)/lg(r) \approx 1$ ($k \geq 48$) with random forms of $r(x)$. Different representations of $r(x)$ allow us to obtain many new families of pairing-friend elliptic curves. In addition, we propose a equation to illustrate how to obtain small values of ρ by choosing appropriate forms of discriminant D and trace t. Numerous parameters of certain pairing-friendly elliptic curves are presented with support for the theoretical conclusions.

Keywords: *pairing-friendly elliptic curves, special polynomial families, cyclotomic polynomials*

1. Introduction

After the propositions of identity-based encryption scheme [12] and short signature scheme [13], paring-based cryptography has attracted significant attention in modern public-key cryptography. Over pairing-based cryptosystems, Elliptic Curve Discrete Logarithm Problem (ECDLP) on supersingular elliptic curves can be reduced to Discrete Logarithm Problem (DLP) over an extension field by Weil Pairing [10] or Tate Paring [15]. Although supersingular elliptic curves provides high efficiency for pairing-based cryptosystems [19, 20], since these curves only can be built when

embedding degree $k \leq 6$ [11], researchers have explored other form of curves, e.g. non-supersingular elliptic curves.

In 2001, Miyaji, Nakabayashi and Takano [8] first proposed a method to find suitable non-supersingular elliptic curves for pairing-based cryptosystems. They discussed the problem from the point of view of tract t. Scott and Barreto [1] extended the method of Miyaji *et al.* and found more suitable non-supersingular elliptic curves when $k \in [3, 4, 6]$. Gallbraith, Mckee and Valenca [3] summarized the method proposed by early researchers and presented some appropriate families of group orders of such elliptic curves when embedding degree $k \leq 6$. Duan, Cui and Chan [5] extended the work of Gallbraith *et al*. by proposing the idea of efficient polynomial families of pairing-friendly elliptic curves.

For larger values of k, Brezing and Weng proposed an alternative method to find these curves [7]. They used $t - 1$ as a kth root of unity modulo prime r. They generated the curves with best performance so far as $lg(q)/lg(r) = 1.25$ ($k = 24$). Dupont, Enge and Morain [16] proposed another method for finding the suitable non-supersingular elliptic curves. Most of the curves they found had $lg(q)/lg(r) \approx 2$. In their method, tract t was chosen large enough to make $4q - t^2$ small as to produce effective values of D. Barreto and Naehrig [17] generated non-supersingular elliptic curves with $lg(q)/lg(r) = 1$ when embedding degree $k = 12$. They presented the best curves with prime group order known so far. Their work was actually generated by a special polynomial family of $q(x)$, $t(x)$ and $r(x)$, where $4q(x) - t^2(x)$ can be factorized as one square polynomial multiplying with one constant number. In the most recent work, Murphy and Fitzpatrick [4] extended the work of Brezing *et al*. and generated pairing-friend elliptic curves over prime fields with discriminant $D > 4$ for arbitrary values of k.

In this paper we propose a new method for finding more pairing-friendly elliptic curves. Compared to the previous works, we explicitly present the special polynomial families for generating more elliptic curves with arbitrary values of embedding degree k. We also illustrate the relation to obtain small values of ρ by choosing appropriate forms of discriminant D, trace t and embedding degree k. In addition, our method allows to find more pairing-friendly elliptic curves with higher security and more efficiency by various representations of $r(x)$. In fact, the work of Brezing *et al*. [17] and Murphy *et al*. [4] can be represented as a special case of our new method, since they implement $r(x)$ only as cyclotomic polynomial in their work.

This paper is organized as follows. In Sections 2 we give a description of the mathematics background. In Section 3 we present the new method and discuss the difference compared with previous works. In Section 4 certain special polynomial families of pairing-friendly elliptic curves wit random forms of $r(x)$ and arbitrary embedding degree k are presented. In addition, we propose the idea to construct curves with $\rho = lg(q)/lg(r) \approx 1$ for large values of $k \geq 48$. We draw the conclusion in

Section 5. The parameters of some pairing-friendly elliptic curves, based on the proposed polynomial families, are presented in Appendix (a), (b), (c) and (d) over prime fields.

2. Mathematics Background

To find suitable elliptic curves for pairing-based cryptosystems, we need to solve certain equations. Assume the cofactor h is an integer, r is the order of a point as a big prime number and t is the trace of an elliptic curve, we want to find an elliptic curve over $\boldsymbol{F}_q$, where $q = p$ is a prime number (we only consider the prime fields in this paper). ECDLP on such elliptic curves can be reduced to DLP over $\boldsymbol{F}_q^k$, where k is the smallest integer satisfying certain conditions, defined as the embedding degree [1]. The following equations determine whether such an elliptic curve exists or not.

In a strict sense, to find the suitable elliptic curves for pairing-based cryptosystems [10], we need

$$r \mid q^k - 1 \tag{1}$$

However, under a mild condition [6], we just consider q as a kth root of unity modulo r [7]. Meanwhile, since k is the smallest integer satisfying $r \mid q^k - 1$, equation (1) should be presented as $r \mid q^k - 1$ and $q^i - 1$ is not divisible by r when $0 < i < k$. From [14] we have

$$dr = \Phi_k(q) \tag{2}$$

where d is an integer and $\Phi_k(q)$ is the cyclotomic polynomial of q with embedding degree k and

$$d'r \neq \Phi_i(q),\ 0 < i < k \tag{3}$$

Besides these conditions we need

$$hr = q + 1 - t \tag{4}$$

where h is an integer. Combining equation (2) and (4) together, we obtain

$$sr = \Phi_k(t - 1) \tag{5}$$

where s is also an integer[1]. Since k is the smallest integer, we have

$$s'r \neq \Phi_i(t - 1),\ 0 < i < k \tag{6}$$

By Hasse's bound we need

$$|t| \leq 2q^{1/2} \tag{7}$$

With all the above equations, we compute the elliptic curve by solving

$$DV^2 = 4q - t^2 \tag{8}$$

where D is chosen by certain conditions [2].

All the above equations aim for finding suitable elliptic curves for pairing-based cryptosystems in integer fields. But it is impossible to search the whole integer fields to obtain the suitable solutions. We should transfer the problem into polynomial fields. When analyzing in polynomial fields, we assume q, t, r as $q(x)$, $t(x)$ and $r(x)$; meanwhile h, d, s, D and V should be considered as $h(x)$, $d(x)$, $s(x)$, $D(x)$ and $V(x)$.

Duan *et al*. [5] have proposed a lemma which illustrates the fact that in polynomial fields, equation (2) and (5) are already both efficient and necessary conditions.
Thus for finding suitable elliptic curves for pairing-based cryptosystems in polynomial fields, the equations (2, 4, 5, 7, 8) are required and they can be rewritten as:

$$d(x)r(x) = \Phi_k(q(x)) \quad (9)$$

$$h(x)r(x) = q(x) + 1 - t(x) \quad (10)$$

$$s(x)r(x) = \Phi_k(t(x) - 1) \quad (11)$$

$$|t(x)| < 2q(x)^{1/2} \quad (12)$$

$$D(x)V(x)^2 = 4q(x) - t^2(x) \quad (13)$$

How to build pairing-friendly elliptic curves by finding the polynomial families satisfying equation (9) to (13) were presented in [1, 3, 5, 8, 17]. Most of the work concentrated on embedding degree $k \leq 6$. Only one special polynomial family whe $k = 12$ was found by Barreto *et al*. [17], which built the best curves with prime group order known so far.

For larger values of k, the polynomial families of $q(x)$, $t(x)$ and $r(x)$ will not satisy all the conditions from equations (9) to (13). Only some of the parameters will maintain the polynomial relations and the other ones will only be valid for certain x as x_0. Brezing and Weng [7] proposed a method to find these curves. In their method, in polynomial fields, $t(x)$ and $r(x)$ will satisfy equation (11) by representing $t(x) - 1$ as a kth root of unity modulo $r(x)$. The irreducible polynomial $r(x)$ is always set as a cyclotomic polynomial. q and DV^2 will not have the polynomial relations. They can not be represented as polynomials in the cyclotomic field. But all the parameters are satisfying equations (9) - (13) for a specific x_0. In the next section, we will present a new method for finding more pairing-friendly elliptic curves with arbitrary embeddin degree k by some special polynomial families. The new method allows $r(x)$ to be an irreducible polynomial with different forms.

3. A New Method for Producing More Pairing - Friendly Elliptic Curves with Special Polynomial Families

In this section the math evidence for the new method is provided. As proposed in [7], from equation (4) and (8), difference between $4q$ and t^2 can be obtained after knowing t and r:

$$DV^2 = 4q - t^2 = 4(hr + t - 1) - t^2 \quad (14)$$

Represented in polynomial fields, equation (16) can be rewritten as

$$DV^2(x) = 4h(x)r(x) - (t(x) - 2)^2 \quad (15)$$

where D is a square-free integer. This is the standard polynomial families as proposed in [1, 3, 5, 8, 17] of pairing-friendly elliptic curves with satisfaction to equation (9) to (13). In polynomial field, choose $r(x)$ and $t(x)$ as

$$s(x)r(x) = \Phi_k(t(x) - 1)$$

This can be viewed as

$$r(x) \mid \Phi_k(t(x) - 1) \tag{16}$$

Here we should mention that polynomial families are hard to find with satisfying equation (15) for large values of k, e.g. $k > 12$. To find more curves, we choose suitable $r(x)$ and $t(x)$ satisfying (16). Then for a specific x_0, assuming equation (15) is existing, this can be written as

$$DV^2(x_0) = 4h(x_0)r(x_0) - (t(x_0) - 2)^2 \tag{17}$$

Dividing D from both sides of equation (17), we have

$$V^2(x_0) = [4h(x_0)r(x_0) / D] - [t(x_0) - 2]^2 / D \tag{18}$$

where $4h(x_0)r(x_0)$ and $[t(x_0) - 2]^2$ divides D. Assuming $[t(x_0) - 2]^2 \mid 4Dh(x_0)$, equation (18) can be rewritten as

$$V^2(x_0) = \{[t(x_0) - 2]^2 / D^2\}\{[4Dh(x_0)r(x_0) / (t(x_0) - 2)^2] - D\} \tag{19}$$

Here we use a technique to consider $4Dh(x_0)/ (t(x_0) - 2)^2$ as a polynimail $h'(x)$ for certain x_0. This means for a specific x_0, $4Dh(x_0)$ divides $[t(x_0) - 2]^2$ and it can be represented in polynomial fields as a polynomial $h'(x)$. Then euqation (19) can be represented as

$$V^2(x_0) = \{[t(x_0) - 2]^2 / D^2\}\{h'(x)r(x) - D\} \tag{20}$$

Thus if $h'(x)r(x) - D$ can be viewed as a square polynoial $S^2(x)$, in equation (20) all parameters are represened in suqare forms. If $h'(x)r(x) - D = S^2(x)$, for any given x_0, equation (20) is satisfied and can be written as

$$V^2(x_0) = \{[t(x_0) - 2]^2 / D^2\}S^2(x) \tag{21}$$

This equation represents a modified polynomia family. For the specific x_0, we have

$$DV^2(x_0) = \{[t(x_0) - 2]^2 / D\}S^2(x_0) \tag{22}$$

Representing equation (14) into equation (22), we obtain

$$q = [t^2(x_0) + DV^2(x_0)] / 4 \tag{23}$$

We choose x_0 as to satisy that $r(x_0)$ is a prime integer. Then for the specific x_0, if $q = [t^2(x_0) + DV^2(x_0)] / 4$ is a prime integer, we find all the suitable parameters with satisfaction to a pairing-friendly elliptic curve.

The main idea can be presented as the following precedures. First we choose a speicific $r(x)$ and trace polynomial $t(x)$ with $r(x) \mid \Phi_k(t(x) - 1)$. Then aftern choosing a suitable discriminant D, we find polynomial families $h'(x)$ satisfying that $h'(x)r(x) - D = S^2(x)$ as a square polynomial. In the following step, we choose a suitable x_0 with $r(x_0)$ is a prime integer and test whether $q = \{t^2(x_0) + [(t(x_0) - 2)^2 / D]S(x0)\} / 4$ is a prime integer. Is q is according to the condition, we have found the suitable parameters of a pairing-friendly elliptic curves. In the procedure, $4Dh(x_0) / (t(x_0) - 2)^2 = h'(x)$, $D \mid 4h(x_0)$ and $D \mid [t(x_0) - 2]^2$ are three hiddern condtions since we represent $h'(x)$ as a polynomial.

The main step in the new method is to find a suitable discriminant D and speicial polynomia families with $h'(x)$, $r(x)$, $S(x)$ with $h'(x)r(x) - D = S^2(x)$. The other work to test prime $r(x_0)$ and $q(x_0)$ is trivial. Here we must mention that when $r(x)$ is taken as a standard cyclotomic polynomial, finding suitable $S(x)$ is equal to find the polynomial represention of $(-D)^{1/2}$ in cyclotomic field. This is because

$$S^2(x) + D \equiv 0 \bmod r(x) \quad (24)$$

In such circumstances, our method is same with the methods proposed by Brezing and Weng [7] since equation (24) can be rewritten as

$$DV^2(x) = \{[t(x) - 2]^2 / D\}S^2(x) \quad (25)$$

where $S(x)$ is the representation of $(-D)^{1/2}$ in cyclotomic field. This equation is same as the main relation proposed in [7]. Thus the work of Brezing *et al*. is a special case of our new method since in their method $r(x)$ is only taken as the standard cyclotomic polynomials. Our proposed method ignores the restriction imposed on the form of $r(x)$. By our method more pairing-friendly elliptic curves are found by vairous forms of $r(x)$.

Based on the above analysis, we propose a new algorithm for finding the suitable polynomial families of pairing-friendly elliptic curves.

Algorithm 1

Input: embedding degree k, $q^k \geq 2^{1024}$ and $r \geq 2^{160}$

Output: x_0, $q(x0)$, $t(x0)$, $r(x0)$, $DV^2(x0)$

1. Choose an irreducible polynomial $r(x)$.
2. Compute trace polynomial $t(x)$ by $\Phi_k(t(x) - 1) \equiv 0 \bmod r(x)$.
3. Choose a polynomial family $h'(x)$ and a suitable discriminant D with $h'(x)r(x) - D = S^2(x)$, where $S^2(x)$ is a square polynomial.
4. Find a specific x_0 with $r(x_0)$ is a prime integer and $q(x_0) = \{t^2(x_0) + S^2(x_0)\ [(t(x_0) - 2)^2 / D]\ \} / 4$ is also a prime integer.
5. Output x_0, $q(x_0)$, $t(x_0)$, $r(x_0)$, $DV^2(x_0)$
6. Establish the elliptic curve by CM method with the above parameters.
7. If no suitable parameters are found, repeat from step 1

Because the key procedure of the new method is to find peicial polynomia families with $h'(x)$, $r(x)$, $S^2(x)$, in the next section we will list some families with different embedding degree k, $\rho = lg(q)/lg(r)$ and $r(x)$.

4. Effective Polynomial Families for Producing More Pairing - Friendly Elliptic Curves

In this section we will present some special polynomial families obtained by the new method. These families can be used to generate more pairing-friendly elliptic curves

with different forms of $r(x)$, small values of ρ and arbitrary values of embedding degree k.

4.1 Special Polynomial Families with Small Values of ρ

By the new method, it can be found that the value of ρ is related to the choice of $r(x)$, $t(x)$ and $h'(x)$. It is because $\rho = lg(q)/lg(r) = degree(q(x)) / degree(r(x))$. Since $DV^2(x_0) = \{[t(x_0) - 2]^2 / D\}\{h'(x)r(x) - D\}$, we have $degree(q(x)) = degree(DV^2(x)) \approx 2degree(t(x)) + degree(h'(x)) + degree(r(x))$. Thus the value of $\rho = degree(q(x)) / degree(r(x)) = [2degree(t(x)) + degree(h'(x)) + degree(r(x))] / degree(r(x)) = 1 + [2degree(t(x)) + degree(h'(x))] / degree(r(x))$. It is to say that ρ will always be larger than 1. This can also be deduced from the condition used in the method. Since we assume $[t(x_0) - 2]^2 \mid 4Dh(x_0)$ in the algorithm, when $\rho = 1$, $h(x)$ will be a constant integer as h. Then $[t(x_0) - 2]^2 \mid 4Dh(x_0)$ will not be satisfied since $|t^2(x0)| > 4Dh$. When the degree of $h'(x)$ is 0 ($h'(x)$ is a constant number), ρ has the smallest values as $1 + 2degree(t(x)) / degree(r(x))$. Thus for finding ρ close to 1, $h'(x)$ should be chosen as a constant number and $t(x)$ should be chosen with smallest degree.

In the following paragraph, we construct a table with all the different special polynomial families for embedding degree $k \in [12, 14, 15, 16]$ over cyclotomic field ($k = 13$ does not contain such polynomial families). All the results are satisfying the nice representations of $(-D)^{1/2}$ in the work of Murphy and Fitzpatrick [4].

	h'(x)	r(x)	D	$S^2(x)$	t(x)	ρ
k = 12	$x^2 + 1$	$\Phi_{12}(x)$	1	x^6	x + 1	2
k = 12	4	$\Phi_{12}(x)$	3	$(2x^2 - 1)^2$	x + 1	1.5
k = 14	$4x^2 + 4x + 8$	$\Phi_{14}(x)$	7	$(2x^4 + 2x^2 - 2x + 1)^2$	x + 1	1.66
k = 15	$4x^2 + 4x + 4$	$\Phi_{15}(x)$	3	$(2x^5 + 1)^2$	x + 1	1.5
k = 15	$4x^6 + 4x^5 - 4x^4 + 8x^3 + 4x^2 + 24$	$\Phi_{15}(x)$	15	$(2x^7 - 2x^5 + 4x^4 - 2x^3 + 2x^2 + 4x - 3)$	x + 1	2

Table 1: Special polynomial families when $k = 12, 14, 15$

In Table 2 we tabulate all the possible special polynomial families when $k = 28$. When $r(x)$ is fixed as the cyclotomic polynomial with embedding degree k, the special polynomial families should be taken as $h'(x)$ and $t(x)$ with smallest degree to obtain the smallest values of ρ. Thus $t(x)$ always should be taken as $x + 1$ if $r(x) = \Phi_k(x)$. Although in cyclotomic field we can modulo $[t(x) - 2)]^2 / D$ by $\Phi_k(x)$ to a smaller integer; from the deductions of section 3 we can find that the best choice is still to set $t(x)$ as $x + 1$. Murphy *et al*. [4] implemented $t(x)$ as $x^3 + 1$ with standard cyclotomic polynomial $r(x)$ when $k = 28$. They found the elliptic curves with $\rho \approx 1.8$. We will present some pairing-friendly elliptic curves when $k = 15$ and 28 in Appendix (a). These curves have smaller values of ρ compared to the work of Murphy *et al*. [4] when representing $t(x)$ as $x + 1$.

	h'(x)	r(x)	D	$S^2(x)$	t(x)	ρ
k = 28	$x^2 + 1$	$\Phi_{28}(x)$	1	x^{14}	x + 1	1.3
k = 28	$4x^4 + 4x^2 + 8$	$\Phi_{28}(x)$	7	$(-2x^8 – 2x^4 + 2x^2 – 1)^2$	x + 1	1.5

Table 2: Special polynomial families when $k = 12, 14, 15$

Brezing *et al*. [7] and Murphy *et al*. [4] have implemented $r(x)$ as $\Phi_{ik}(x)$ for a given embedding degree k, where i is an integer. In fact, such techniques can not obtain elliptic curves with smaller values of ρ. The reason is when taken $r(x)$ as $\Phi_{ik}(x)$, $degree(r(x)) = degree(\Phi_{ik}(x)) = i \times degree(\Phi_k(x))$; meanwhile, $i \times degree(t(x))$ ($r(x)$ is $\Phi_k(x)$) = $degree(t(x))$ (r(x) is $\Phi_{ik}(x)$) and $i \times degree(h'(x))$ ($r(x)$ is $\Phi_k(x)$) = $degree(h'(x))$ ($r(x)$ is $\Phi_{ik}(x)$). Thus the value of ρ is not related to the choice of cyclotomic polynomials since the degrees of $r(x)$, $t(x)$ and $h'(x)$ are increased at same times. Brezing *et al*. [7] have found the best curves with $\rho \approx 1.25$, $k = 24$. They took $r(x)$ as $\Phi_{48}(x)$ and $t(x)$ as $x^2 + 1$. In Table 2, we will tabulate some polynomial families when $k = 24$ which illustrates that $\rho \approx 1.25$ is also obtained for $k = 24$ when $t(x) = x + 1$, $r(x) = \Phi_{24}(x)$. These polynomial families prove that the value of ρ has no relation with the choice of the degrees of $\Phi_k(x)$ and $t(x)$.

	h'(x)	r(x)	D	$S^2(x)$	t(x)	ρ
k = 24	4	$\Phi_{24}(x)$	3	$(2x^4 – 1)^2$	x + 1	1.25
k = 24	4	$\Phi_{48}(x)$	3	$(2x^8 – 1)^2$	$x^2 + 1$	1.25
k = 24	$x^2 + 2$	$\Phi_{24}(x)$	2	$(x^5 + x^3 – x)^2$	x + 1	1.5
k = 24	$x^4 + 2$	$\Phi_{48}(x)$	2	$(x^{10} + x^6 – x^2)^2$	$x^2 + 1$	1.5

Table 3: Special polynomial families when $k = 24$

4.2 Special Polynomial Families with Arbitrary Values of *k*

As finding the special polynomial families, we notice a fact which can be implemented to construct pairing-friendly elliptic curves with arbitrary embedding degree k. When k is an even integer, $\Phi_{sk}(x)$ will have the same form as $\Phi_k(x)$ where $s = 2^i$ (i is a positive integer). The only difference is that x in $\Phi_k(x)$ will be represented as x^s in $\Phi_{sk}(x)$. Meanwhile, $h'(x)$ will also have the same property. Thus if we find a special polynomial family for embedding degree k, we can easily obtain the families with the same forms for embedding degree sk ($s = 2^i$). The only work is to represent x as x^s, where s is a positive integer power of 2.

The beauty of this property is to find curves with larger embedding degree k with smaller ρ. It is because $\rho = 1 + [2degree(t(x)) + degree(h'(x))] / degree(r(x))$. When degree of $r(x)$ is increased with same multiples of $h'(x)$, if the degree of $t(x)$ is not chanced, the value of ρ will be decreased. When $h'(x)$ is a constant integer, the situations are more easily to analyze. Since if we always choose $t(x)$ as $x + 1$, the values of ρ equals $1 + [2 / degree(r(x))]$. By choosing larger values of embedding degree k, the degree of $r(x)$ will increase, then the value of ρ will be decreased (but it can not reach 1 for ever).

Koblitz and Menezes [18] have suggested to find pairing-friendly elliptic curves with $k = 2^i 3^j$ as large as possible. In Table 3, we tabulate the polynomial families when $k = 12, 24, 48, 96$. The value of ρ is decreased to 1 when taking larger values of k. When $k = 48$, we construct the parameters of certain paring-friendly elliptic curves with $\rho \approx 1.125$. When $k = 96$, we construct the parameters of certain paring-friendly elliptic curves with $\rho \approx 1.06$. These curves have better performance compared to previous works, since they have smaller values of ρ for larger values of k. The results are presented in Appendix (b).

	h'(x)	r(x)	D	$S^2(x)$	t(x)	ρ
k = 12	4	$\Phi_{12}(x)$	3	$(2x^2 - 1)^2$	x + 1	1.5
k = 24	4	$\Phi_{24}(x)$	3	$(2x^4 - 1)^2$	x + 1	1.25
k = 48	4	$\Phi_{48}(x)$	3	$(2x^8 - 1)^2$	x + 1	1.12
k = 96	4	$\Phi_{96}(x)$	3	$(2x^{16} - 1)^2$	x + 1	1.06

Table 4: More special polynomial families when $k = 12, 24, 48, 96$

With this technique we could find any pairing-friendly elliptic curves with arbitrary embedding degree k, e.g. finding curves with larger k than 96 is not a hard task. But since the suitable values of x become sparse when k is increased, finding parameters of elliptic curves with the essential security level (r is a 160 bits prime) [9] is more difficult with large k. This is the reason we find the parameters of an elliptic curve with r as a 416 bits prime integer when $k = 96$. . In addition, it is still not aware how large the embedding degree k should be for a pairing-friendly curve with best performance. (To find curves with prime r close to 160 bits integer for larger values of k, e.g. k = 96, we should use some other techniques. This question will be further discussed in section 4.4.)

4.3 Special Polynomial Families with Different Forms of *r(x)*

Brezing and Weng [7] had successfully found pairing-friendly elliptic curves with arbitrary embedding degree. But the limitation of their work was that $r(x)$ was only represented as the standard cyclotomic polynomial. It was because their method was to derive kth root of unity and polynomial representations of $(-D)^{1/2}$ in the cyclotomic field. They had not given any explanations to the circumstances when $r(x)$ was taken as an arbitrary irreducible polynomial. The work of Murphy and Fitzpatrick [4] was also based on the standard representation of $r(x)$ as $\Phi_k(x)$.

Our method ignores the limitation imposed on the form of $r(x)$ since we only need to find $h'(x)$ and D with $h'(x)r(x) - D = S^2(x)$. For this equation, $r(x)$ can be any irreducible polynomials satisfying $r(x) \mid \Phi_k(t(x) - 1)$. This point allows us to find much more ellitpc curves for various representations of $r(x)$. In Table 5 we will tabulate

more special polynomial families with different *r(x)* when *k = 12*, *14*. The trace *t(x)* is taken with degree as small as possible to obtain desired values of ρ. In Appendix (c) we generate certain parameters of pairing-friendly elliptic curves by some polynomial families in Table 5.

k	r(x)	t(x)
12	$36x^4 + 36x^3 + 18x^2 + 6x + 1$	$6x^2 + 1$
12	$4x^4 + 4x^3 + 2x^2 + 2x + 1$	$2x^2 + 1$
12	$2197x^4 – 1352x^3 + 299x^2 – 28x + 1$	$13x – 1$
12	$2197x^4 – 4056x^3 + 2795x^2 – 852x + 97$	$13x – 5$
14	$16807x^6 – 16807x^5 + 7203x^4 – 1715x^3 + 245x^2 – 21x + 1$	$7x$
14	$20511149x^6 – 30413083x^5 + 18803919x^4 – 6205739x^3 + 1153069x^2 – 114381x + 4733$	$29x – 6$

Table 5: More special polynomial families with different *r(x)*

Table 5 just tabulates some new polynomial families of different *r(x)* when *k = 12*, *14*. By our new method, more such families can be found for other values of *k*. These new polynomial families can be implemented for finding much more pairing-friendly elliptic curves.

4.4 More *r* with a small factor

In fact, when it is allowed that *r* contains a small factor *s* as *r = sn* (*n* is a prime larger than 2^{160}), much more suitable elliptic curves are found. The same technique has been used in [1]. When *r = sn*, *n* should be a large prime bigger than 2^{160} and cofactor *h* will be multiplied with the small factor *s*. Brezing *et al*. [7] and Murphy *et al*. [4] only implemented *r* as a large prime in their methods. By our find it is easy to find that the condition of prime *r(x)* can be loosed to *r = sn* without effecting the values of ρ much. Thus more elliptic curves different with their work are found. The value of ρ will not increase much if *s* is carefully chosen. In Appendix (d), we presented some examples with this technique when *k = 96*. By this technique, we effectively decrease the length of *r*.

5. Conclusion

In this paper, we propose a new method to find more pairing-friendly elliptic curves with arbitrary embedding degree *k* by certain special polynomial families. This method allows us to obtain new families of pairing-friend elliptic curves by representing *r(x)* with various forms. In addition, we propose a new technique to let prime *r* contain a small factor *s*. Numerous parameters of new pairing-friendly elliptic curves are found with the proposed method. These curves have higher security ($k \geq 48$)

and more efficiency ($\rho \leq 1.2$) compared to the previous work for paring-based cryptosystems.

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Appendix

(a) Special Polynomial Families with Small Values of ρ

(1) k = 15, $\rho \approx 1.5$
$S(x)^2 = 4x^{10} + 4x^5 + 1 = (2x^5 + 1)^2$, D = 3, $h'(x) = 4x^2 + 4x + 4$, $r(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$; $t = x + 1$

x = 962542
r = 736812625806875488411619464155914329720828004251 (160 bits)
t = 962543
q = 21082111624242378127392870715467849117003034307261239000753006319008248 1
DV^2 = 84328446496969512509571482861871396468012137229044956003011932627130307 5

x = 963691
r = 74387844410772769516212276726746144568521772216 1 (160 bits)
t = 963692
q = 21386094473663184255325672790155329822344906504518272856132597084927479 1
DV^2 = 85544377894652737021302691160621319289379626018073091424530295469482830 0

(2) k = 28, $\rho \approx 1.3$
$S(x)^2 = x^{14} = (x^7)^2$, D = 1, $h'(x) = x^2 + 1$, $r(x) = x^{12} - x^{10} + x^8 - x^6 + x^4 - x^2 + 1$

x = 9803
r = 787604206770071823093201411395167729501255763497 (160 bits)
t = 9804
q = 1818006543181202958475850993256242972686798649470526225526834373
DV^2 = 72720261727248118339034039730249718907471945978821049020112190 76

x = 11137
r = 364098284272844368011618772721554259182887991833 7 (162 bits)
t = 11138
q = 1400082181420459601283673155853812871854847437058818210822476889 7
DV^2 = 56003287256818384051346926234152514874193897482352728432775020544

(3) k = 28, $\rho \approx 1.5$
$S(x)^2 = 4x^{16} + 8x^{12} - 8x^{10} + 8x^8 - 8x^6 + 8x^4 - 4x^2 + 1 = (-2x^8 - 2x^4 + 2x^2 - 1)^2$, D = 7, $h'(x) = 4x^4 + 4x^2 + 8$, $r(x) = x^{12} - x^{10} + x^8 - x^6 + x^4 - x^2 + 1$; $t = x + 1$

x = 41707
r = 27701763920994110467659972445730100441267452808766711657 (160 bits)
t = 41708

q =
208277458642659286262143174064593219937166390246750999549996523200303792983736723 79
DV^2 =
833109834570637145048572696258372879748665560987003998199986092801215171917551322252

(b) Special Polynomial Families with Arbitrary Values of *k*

(1) k = 24, ρ ≈ 1.25
$S(x)^2 = 4x^8 - 4x^4 + 1 = (2x^4 - 1)^2$, D = 3, h'(x) = 4, $r(x) = x^8 - x^4 + 1$

x = 962833
r = 738597331833134695682651527040098006595721222721 (160 bits)
t = 962834
q = 228237767803155599843901956912501345675531789475531151146001
DV^2 = 91295107121262239937560782765000538270212715790119755527244 8

x = 965365
r = 754279653484785063609361644146986474789548990001 (160 bits)
t = 965366
q = 234311355599190690060865477933001341274428958628739903196197
DV^2 = 937245422396762760243461911732005365097715834514027681270832

(2) k = 48, ρ ≈ 1.125
$S(x)^2 = 4x^{16} - 4x^8 + 1 = (2x^8 - 1)^2$, D = 3, h'(x) = 4, $r(x) = x^{16} - x^8 + 1$; t = x + 1

x = 2470
r = 191933707364169721870043501834499777475161174390000000 1 (181 bits)
t = 2471
q = 390006798225797140633544085162179699332420329265212930203445 7
DV^2 = 156002719290318856253417634064871879732968131706085172020319 87

x = 5479
r = 659509122132996966757981555872075218495015674349333558784961 (199 bits)
t = 5480
q = 65969563131273617830018071305607584136680606478203955031538538751 87
DV^2 = 263878252525094471320072285222430336546722425912815820126153854703 48

(3) k = 96, ρ ≈ 1.06
$S(x)^2 = 4x^{32} - 4x^{16} + 1 = (2x^{16} - 1)^2$, D = 3, h'(x) = 4, $r(x) = x^{32} - x^{16} + 1$; t = x + 1

x = 8053
r = 97869274957462648053823048369599967807487149089726854198173277268671530882 7

46626642467570423271029692259277387159136085590721 (416 bits)
t = 8054
q = 21151084908539011258752644698068988370693194316787700109299072574747490593365787484542995859973106406242804468834519963349976870685 81
DV^2 = 84604339634156045035010578792275953482772777267150800437196290298989962373463149938171983439892425624971217875338079853399906834074 08

(c) Special Polynomial Families with Different Forms of *r(x)*

(1) k = 12
$r(x) = 2197x^4 - 1352x^3 + 299x^2 - 28x + 1$, $t(x) = 13x - 1$

x = 137438953782
r = 783915802287873738784769281395853185119086316917 (160 bits)
t = 1786706399165
q = 10844209581049657402029453962827548199750866489317427585866728276041510847
DV^2 =
433768383241986296081178158513101927990034659572665180237100959438533461 63 =
$3 \times 380249209178220512303502913837721631 1^2$
$\rho \approx 1.5$

(2) k = 14
$r(x) = 20511149x^6 - 30413083x^5 + 18803919x^4 - 6205739x^3 + 1153069x^2 - 114381x + 4733$, $t(x) = 29x - 6$

x = 5936652
r = 897923694407722064866709188398349353035596592369 (160 bits)
t = 172162902
q =
326811759720835564714732775797503236447913111267853657917916780522421188688368 2901
DV^2 =
130724703888334225885893110319001294579165244507141463167166712208672074827096 70000 =
$7 \times 432145319288933879814785107185576180841 00^2$
$\rho \approx 1.7$

(d) More *r* with a small factor

(1) k = 96

x = 790
r = 104347209516035823805625812181222021008778404957874578523 53 (193 bits)

t = 791
$DV^2 = 3 \times 12106516682165495677419277453752459999999999999737^2$
q =
109925809631663581112233601417794309888777824519815968844486928993958095044494654530000000000208297
ρ ≈ 1.67

x = 1075
r =
960244477302499018655886830888768384605031972092397401372 9 (193 bits)
h = 193 × 189697 × 5 548897 × 44511 743233 × 116 517189 305089
t = 1076
$DV^2 = 3 \times 22774478984574661338473130929283797740936279296871422^2$
q =
389007669764124673056208123236933160496464959539104119574614994745358704230572581291198730468750038556 7
ρ ≈ 1.75

x = 1462
r =
799856510600656312372792979921553204080181543003124919258645718098290466279718510 73 (276 bits)
h = 97 × 24464261226891361
t = 1463
$DV^2 = 3 \times 424342879248671780299130728442658508419523779578691097^2$
q =
135050159376789629563661162494941702923787864008293359224562874946890295339329479610347558831805883810832649
ρ ≈ 1.3

x = 1990
r =
499530549187789116902899835715635587618608075220280202334114331084455419325231174585117316929 93 (315 bits)
h = 769 × 95237953
t = 1991
$DV^2 = 3 \times 80203393950668375604161093376178805581259999999999999337^2$
q =
482443830090458143970598719142978471444730710196094627023340791823785775650918333739017784943693000000000 1320697
ρ ≈ 1.17

(5)
x = 2344

r =
3265344136080389756117061362948346136410185602877472164948887012210851476357245
639076892938770050488256 1 (344 bits)
h = 21121
t = 2345
$DV^2 = 3 \times 12971874761632176402622474504460802316195215015492158742912$
q =
1262021511236023750235671121066961149610353271316025640749876413335944371477169
742695240677028410003616420637439267
$\rho \approx 1.1$