

# New Proofs for NMAC and HMAC: Security without Collision-Resistance

MIHIR BELLARE\*

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## Abstract

HMAC was proved in [2] to be a PRF assuming that (1) the underlying compression function is a PRF, and (2) the iterated hash function is weakly collision-resistant. However, recent attacks show that assumption (2) is false for MD5 and SHA-1, removing the proof-based support for HMAC in these cases. This paper proves that HMAC is a PRF under the sole assumption that the compression function is a PRF. This recovers a proof based guarantee since no known attacks compromise the pseudorandomness of the compression function, and it also helps explain the resistance-to-attack that HMAC has shown even when implemented with hash functions whose (hidden-key) collision resistance is compromised. We also show that an even weaker-than-PRF condition on the compression function, namely that it is a privacy-preserving MAC, suffices to establish HMAC is a MAC as long as the hash function meets the very weak requirement of being computationally almost universal, where again the value lies in the fact that known attacks do not invalidate the assumptions made.

**Keywords:** Message authentication, hash functions, Pseudorandom Functions, Carter-Wegman.

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\*Dept. of Computer Science & Engineering 0404, University of California San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0404, USA. Email: [mihir@cs.ucsd.edu](mailto:mihir@cs.ucsd.edu). URL: <http://www-cse.ucsd.edu/users/mihir>. Supported in part by NSF grants ANR-0129617, CCR-0208842 and CNS-0524765 and by an IBM Faculty Partnership Development Award.

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# 1 Introduction

HMAC [2] is a popular cryptographic-hash-function-based MAC. The basic construct is actually NMAC, of which HMAC can be viewed as a derivative.

THE CONSTRUCTIONS. Succinctly:

$$\begin{aligned}\text{NMAC}(K_{\text{out}}\|K_{\text{in}}, M) &= H^*(K_{\text{out}}, H^*(K_{\text{in}}, M)) \\ \text{HMAC}(K_{\text{out}}\|K_{\text{in}}, M) &= H(K_{\text{out}}\|H(K_{\text{in}}\|M)).\end{aligned}$$

Here  $H$  is a cryptographic hash function, eg. MD5 [27], SHA-1 [25], or RIPEMD-160 [15]. Let  $h: \{0, 1\}^c \times \{0, 1\}^b \rightarrow \{0, 1\}^c$  denote the underlying compression function. (Here  $b = 512$  while  $c$  is 128 or 160.) Let  $h^*$  be the iterated compression function which on input  $K \in \{0, 1\}^c$  and a message  $x = x[1] \dots x[n]$  consisting of  $b$ -bit blocks, lets  $a[0] = K$  and  $a[i] = h(a[i-1], x[i])$  for  $i = 1, \dots, n$ , and finally returns  $a[n]$ . Then  $H^*(K, M) = h^*(K, M^*)$  and  $H(M) = H^*(IV, M)$ , where  $M^*$  denotes  $M$  padded appropriately to a length that is a positive multiple of  $b$  and  $IV$  is a public  $c$ -bit initial vector that is fixed as part of the description of  $H$ . Both NMAC and HMAC use two keys, which in the case of NMAC are of length  $c$  bits each, and in the case of HMAC of length  $b$  bits each and derived from a single  $b$ -bit key.

USAGE. HMAC is standardized via an IETF RFC [20], a NIST FIPS [23] and ANSI X9.71 [1], and implemented in SSL, SSH, IPSEC and TLS amongst other places. It is often used as a PRF (pseudorandom function [17]) rather than merely as a MAC. In particular this is the case when it is used for key-derivation, as in TLS [14], IKE (the Internet Key Exchange protocol of IPSEC) [18] and NIST's key establishment draft standard [24]. HMAC is also used as a PRF in a proposed standard for one-time passwords [22].

WHAT'S KNOWN. The results are for NMAC but can be lifted to HMAC. It is shown in [2] that NMAC is a secure PRF if (1) the underlying compression function  $h$  is a secure PRF, and also (2) that the hash function  $H$  is *weakly collision resistant* (WCR). The latter, introduced in [2], is a relaxation of collision resistance (CR) that asks that it be computationally infeasible for an adversary, given an oracle for  $H^*(K, \cdot)$  under a hidden key  $K$ , to find a collision, meaning distinct inputs  $M_1, M_2$  such that  $H^*(K, M_1) = H^*(K, M_2)$ .

THE PROBLEM. HMAC is almost always implemented with MD5 or SHA-1. But, due to recent attacks [31, 30], these functions are *not* WCR. Thus the assumption on which the proof of [2] is based is not true. This does not reflect any actual weaknesses in the NMAC or HMAC constructs, on which no attacks are known. (Being iterated MACs, the generic birthday based forgery attacks of [26] always break NMAC and HMAC in time  $2^{c/2}$ , but we mean no better-than-birthday attacks are known.) But it means that we have lost the proof-based guarantees. We are interested in recovering them.

LOSS OF WCR. First we pause to expand on the claim above that our main hash functions are not WCR. Although WCR appears to be a weaker requirement than CR due to the hidden key, in fact, for iterated hash functions, it ends up not usually being so. The reason is that collision-finding attacks such as those on MD5 [31] and SHA-1 [30] extend to find collisions in  $H^*(IV, \cdot)$  for an arbitrary but given  $IV$ , and, any such attack, via a further extension attack, can be used to compromise WCR, meaning to find a collision in  $H^*(K, \cdot)$ , given an oracle for this function, even with  $K$  hidden. This was pointed out in [2, 19], and, for the curious, we recall the attack in Appendix A.

MAIN RESULT. We show (Theorem 3.3) that NMAC is a PRF under the *sole* assumption that the underlying compression function  $h$  is itself a PRF. In other words, the additional assumption that

the hash function is WCR is dropped. (And, in particular, as long as  $h$  is a PRF, the conclusion is true even if  $H$  is *not* WCR, let alone CR.)

The main advantage of our result is that it is based on an assumption that is not refuted by any known attacks. (There are to date no attacks that compromise the pseudorandomness of the compression functions of MD5 or SHA-1.) Another feature of our result is that it is the first proof for NMAC that is based solely on an assumption about the compression function rather than also assuming something about the entire iterated hash function.

TECHNIQUES. We show (Lemma 3.1) that if a compression function  $h$  is a PRF then the iterated compression function  $h^*$  is *computationally almost universal* (cAU), a relaxation of the standard notion of almost-universality (AU) of [12, 32, 29]. We then conclude with Lemma 3.2 which says that the composition of a PRF and a cAU function is a PRF. (This can be viewed as a computational relaxation of the Carter-Wegman paradigm [12, 32].)

RELATED WORK. If  $h^*$  were a PRF, it would imply it is cAU, but  $h^*$  is not a PRF due to the extension attack. It is however shown by [3] that if  $h$  is a PRF then  $h^*$  (which they call the cascade) is a “pf-PRF” (prefix-free PRF), meaning a PRF as long as no query of the adversary is a prefix of another query. It was pointed out to us by Shoup after seeing an early draft of this paper that it is possible to apply this in a black-box way to show that  $h^*$  is cAU. However Lemma 3.1 is a somewhat stronger result and bound whose proof distills and strengthens the ideas of [3] and also involves some new ones. For comparison, we do present the indirect proof in Appendix B.

Dodis, Gennaro, Håstad, Krawczyk and Rabin show [16, Lemma 4] that the cascade over a family of *random* functions is AU as long as the two messages whose collision probability one considers have the same length. (In this model,  $h(K, \cdot)$  is a random function for each  $K \in \{0, 1\}^c$ . That is, it is like Shannon’s ideal cipher model, except the component maps are functions not permutations.) This does not imply Lemma 3.1 (showing the cascade  $h^*$  is cAU if  $h$  is a PRF), because we need to allow the two messages to have different lengths, and also because it is not clear what implication their result has for the case when  $h$  is a PRF. (A PRF does *not* permit one to securely instantiate a *family* of random functions.) A second result [16, Lemma 5] in the same paper says that if  $h^*(K, M)$  is close to uniformly distributed then so is  $h^*(K, M \| X)$ . (Here  $M$  is chosen from some distribution,  $K$  is a random but known key, and  $X$  is a fixed block.) This result only assumes  $h$  is a PRF, but again we are not able to discern any implications for the problems we consider, because in our case the last block of the input is not fixed, we are interested in the cAU property rather than randomness, and our inputs are not drawn from a distribution.

ANOTHER RESULT. The fact that compression functions are underlain by block ciphers, together with the fact that no known attacks compromise the pseudorandomness of the compression functions of MD5, SHA-1, may give us some confidence that it is ok to assume these are PRFs, but it still behoves us to be cautious. What can we prove about NMAC without assuming the compression function is a PRF? We would not expect to be able to prove it is a PRF, but what about just a secure MAC? (Any PRF is a secure MAC [5, 7], so our main result implies NMAC is a secure MAC, but we are interested in seeing whether this can be proved under weaker assumptions.) We show (Theorem 4.3) that NMAC is a secure MAC if  $h$  is a *privacy-preserving MAC* (PP-MAC) and  $h^*$  (equivalently,  $H^*$ ) is cAU. A PP-MAC, defined in Section 4, is stronger than a MAC but weaker than a PRF. This result reverts to the paradigm of [2] of making assumptions both about the compression function and its iteration, but the point is that cAU is a very weak assumption compared to WCR and PP-MAC is a weaker assumption than PRF.

FROM NMAC TO HMAC. The formal results (both previous and new) we have discussed so far pertain to NMAC. However, discussions (above and in the literature) tend to identify NMAC and

HMAC security-wise. This is explained by an observation of [2] which says that HMAC inherits the security of NMAC as long as the compression function is a PRF when keyed via the data input. (So far when we have talked of it being a PRF, it is keyed via the chaining variable.) In our case this means that HMAC is a PRF if the compression function is a “dual-PRF,” meaning a PRF when keyed by either of its two inputs.

However, the analysis above assumes that the two keys  $K_{\text{out}}, K_{\text{in}}$  of HMAC are chosen *independently* at random, while in truth they are equal to  $K \oplus \text{opad}$  and  $K \oplus \text{ipad}$  respectively, where  $K$  is a random  $b$ -bit key and  $\text{opad}, \text{ipad}$  are fixed, distinct constants. We apply the theory of PRFs under related-key attacks [6] to extend the observation of [2] to this single-key version of HMAC, showing it inherits the security of NMAC as long as the data-input-keyed compression function is a PRF under an appropriate (and small) class of related key attacks. Assuming additionally that the compression function is a PRF in the usual sense, we obtain a (in fact, the first) security proof of the single-key version of HMAC. These results are in Section 5.

## 2 Definitions

NOTATION. We denote by  $s_1 \| s_2$  the concatenation of strings  $s_1, s_2$ , and by  $|s|$  the length of string  $s$ . Let  $b$  be a positive integer representing a block length, and let  $B = \{0, 1\}^b$ . Let  $B^+$  denote the set of all strings of length a positive multiple of  $b$  bits. Whenever we speak of blocks we mean  $b$ -bit ones. If  $M \in B^+$  then  $\|M\|_b = |M|/b$  is the number of blocks in  $M$ , and  $M[i]$  denotes its  $i$ -th  $b$ -bit block, meaning  $M = M[1] \dots M[n]$  where  $n = \|M\|_b$ . If  $M_1, M_2 \in B^+$ , then  $M_1$  is a prefix of  $M_2$ , written  $M_1 \subseteq M_2$ , if  $M_2 = M_1 \| A$  for some  $A \in B^*$ . If  $S$  is a set then  $s \stackrel{\$}{\leftarrow} S$  denotes the operation of selecting  $s$  uniformly at random from  $S$ . An adversary is a (possibly randomized) algorithm that may have access to one or more oracles. We let

$$A^{\mathcal{O}_1, \dots}(x_1, \dots) \Rightarrow 1 \quad \text{and} \quad y \stackrel{\$}{\leftarrow} A^{\mathcal{O}_1, \dots}(x_1, \dots)$$

denote, respectively, the event that  $A$  with the indicated oracles and inputs outputs 1, and the experiment of running  $A$  with the indicated oracles and inputs and letting  $y$  be the value returned. (This value is a random variable depending on the random choices made by  $A$  and its oracles.) Either the oracles or the inputs (or both) may be absent, and often will be.

A family of functions is a two-argument map  $f: \text{Keys} \times \text{Dom} \rightarrow \text{Rng}$  whose first argument is regarded as a key. We fix one such family  $h: \{0, 1\}^c \times B \rightarrow \{0, 1\}^c$  to model a compression function that we regard as being keyed via its  $c$ -bit chaining variable. Typical values are  $b = 512$  and  $c = 128$  or  $160$ . The *iteration* of family  $h: \{0, 1\}^c \times B \rightarrow \{0, 1\}^c$  is the family of functions  $h^*: \{0, 1\}^c \times B^+ \rightarrow \{0, 1\}^c$  defined via:

```
Function  $h^*(K, M)$  //  $K \in \{0, 1\}^c, M \in B^+$ 
   $n \leftarrow \|M\|_b$ ;  $a[0] \leftarrow K$ 
  For  $i = 1, \dots, n$  do  $a[i] \leftarrow h(a[i-1], M[i])$ 
  Return  $a[n]$ 
```

This represents the Merkle-Damgård [21, 13] iteration method used in all the popular hash functions but without the “strengthening,” meaning that there is no  $|M|$ -based message padding.

PRFs. A prf-adversary  $A$  against a family of functions  $f: \text{Keys} \times \text{Dom} \rightarrow \text{Rng}$  takes as oracle a function  $g: \text{Dom} \rightarrow \text{Rng}$  and returns a bit. The prf-advantage of  $A$  against  $f$  is the difference between the probability that it outputs 1 when its oracle is  $g = f(K, \cdot)$  for a random key  $K \stackrel{\$}{\leftarrow} \text{Keys}$ , and the probability that it outputs 1 when its oracle  $g$  is chosen at random from the set

$\text{Maps}(Dom, Rng)$  of all functions mapping  $Dom$  to  $Rng$ , succinctly written as

$$\mathbf{Adv}_f^{\text{prf}}(A) = \Pr \left[ A^{f(K, \cdot)} \Rightarrow 1 \right] - \Pr \left[ A^{\$} \Rightarrow 1 \right]. \quad (1)$$

In both cases the probability is over the choice of oracle and the coins of  $A$ .

CAU AND COLLISION-PROBABILITY. Let  $F: \{0, 1\}^k \times Dom \rightarrow Rng$  be a family of functions. cAU is measured by considering an almost-universal (au) adversary  $A$  against  $F$ . It (takes no inputs and) returns a pair of messages in  $Dom$ . Its au-advantage is

$$\mathbf{Adv}_F^{\text{au}}(A) = \Pr \left[ F(K, M_1) = F(K, M_2) \wedge M_1 \neq M_2 : (M_1, M_2) \stackrel{\$}{\leftarrow} A; K \stackrel{\$}{\leftarrow} \text{Keys} \right].$$

Note this corresponds to a weaker notion than WCR because in the latter the adversary has an oracle for  $F(K, \cdot)$ . For  $M_1, M_2 \in Dom$  it is also useful to let  $\text{Coll}_F(M_1, M_2) = \Pr[F(K, M_1) = F(K, M_2)]$ , the probability being over  $K \stackrel{\$}{\leftarrow} \{0, 1\}^k$ .

SECURITY AND RESOURCES. As usual with the concrete security approach [5], there is no formal notion of security of a primitive (eg. a PRF) but informally, when we talk of, say,  $f$  being a PRF, we mean that the prf-advantage of any (no input) prf-adversary of “practical” resources is “low.” Similarly  $H$  is computationally au (cAU) if the au-advantage of any adversary of “practical” resources is “low”. Formal results state the concrete security of reductions, bounding the advantage and resources of one adversary as a function of those of others.

The following conventions will be adopted in measuring resource usage. The time-complexity of an adversary is defined as the total execution time of an overlying experiment (meaning, includes not only the running time of the adversary but the time to compute replies to oracle queries and the time to perform any initializations or to test whether the adversary was successful) plus the size of the code of the adversary, in some fixed model of computation. (In cases like PRFs, where equation (1) defining the advantage involves two experiments, we consider the maximum of the two execution times, with the convention that the picking of a random function is done by building an on-line table while responding to oracle queries. More details on these conventions will appear when they are used.) When we say that a resource measure (such as the time-complexity, number of oracle queries, or their lengths) is at most a certain value, we mean this holds for all coin tosses of the adversary and regardless of how its oracle queries are answered. With regard to time-complexity we will permit ourselves the use of big-oh notation, even though there are no asymptotics here, with the intent that it hides some constant depending only on the model of computation.

MACs. Any PRF is a MAC. (This was established for the basic notion of MAC security in [5], but holds even for the most stringent notions and with tight security reductions [7].) Accordingly, we do not explicitly discuss MACs until Section 4.1 where we consider PP-MACs.

### 3 Security of NMAC

Let  $h: \{0, 1\}^c \times \{0, 1\}^b \rightarrow \{0, 1\}^c$  be a family of functions that represents the compression function, here assumed to be a PRF. Let  $\text{pad}$  denote a padding function such that  $s^* = s \parallel \text{pad}(|s|) \in B^+$  for any string  $s$ . (Such padding functions are part of the description of current hash functions. Note the pad depends only on the length of  $s$ .) Then the family NMAC:  $\{0, 1\}^{2c} \times D \rightarrow \{0, 1\}^c$  is defined by  $\text{NMAC}(K_{\text{out}} \parallel K_{\text{in}}, M) = h(K_{\text{out}}, h^*(K_{\text{in}}, M^*) \parallel \text{fpad})$  where  $\text{fpad} = \text{pad}(c) \in \{0, 1\}^{b-c}$  and  $h^*$  is the iterated compression function as defined in Section 2. The domain  $D$  is the set of all strings up to some maximum length, which is  $2^{64}$  for current hash functions.

It turns out that our security proof for NMAC does not rely on any properties of  $\text{pad}$  beyond the fact that  $M^* = M \parallel \text{pad}(|M|) \in B^+$ . (In particular, the Merkle-Damgård strengthening, namely

inclusion of the message length in the padding, that is used in current hash functions and is crucial to collision resistance of the hash function, is not important to the security of NMAC.) Accordingly, we will actually prove the security of a more general construct that we call generalized NMAC. The family **GNMAC**:  $\{0, 1\}^{2c} \times B^+ \rightarrow \{0, 1\}^c$  is defined by  $\text{GNMAC}(K_{\text{out}} \| K_{\text{in}}, M) = h(K_{\text{out}}, h^*(K_{\text{in}}, M) \| \text{fpad})$  where **fpad** is any (fixed)  $b-c$  bit string. Note the domain is  $B^+$ , meaning inputs have to have a length that is a positive multiple of  $b$  bits. (But can be of any length.) NMAC is nonetheless a special case of GNMAC via  $\text{NMAC}(K_{\text{out}} \| K_{\text{in}}, M) = \text{GNMAC}(K_{\text{out}} \| K_{\text{in}}, M^*)$  and thus the security of NMAC is implied by that of GNMAC. (Security as a PRF or a MAC, respectively, for both.)

### 3.1 The results

**MAIN LEMMA.** The following says that if  $h$  is a PRF then its iteration  $h^*$  is cAU.

**Lemma 3.1** *Let  $B = \{0, 1\}^b$ . Let  $h: \{0, 1\}^c \times B \rightarrow \{0, 1\}^c$  be a family of functions, and let  $A^*$  be an au-adversary against  $h^*$ . Assume that the two messages output by  $A^*$  are at most  $n_1, n_2 \geq 1$  blocks long, respectively. Then there exists a prf-adversary  $A$  against  $h$  such that*

$$\mathbf{Adv}_{h^*}^{\text{au}}(A^*) \leq (n_1 + n_2) \cdot \mathbf{Adv}_h^{\text{prf}}(A) + \frac{1}{2^c}. \quad (2)$$

*Furthermore,  $A$  has time-complexity at most  $O((n_1+n_2)T_h)$ , where  $T_h$  is the time for one evaluation of  $h$ , and makes at most 2 oracle queries. ■*

The proof is in Section 3.3. The quality of the reduction is good because the time-complexity of the constructed adversary  $A$  is small and in particular independent of the time-complexity of  $A^*$  (the proof shows how this is possible) and furthermore  $A$  makes only two oracle queries.

**THE PRF(CAU)=PRF LEMMA.** The *composition* of families  $h: \{0, 1\}^c \times \{0, 1\}^b \rightarrow \{0, 1\}^c$  and  $F: \{0, 1\}^k \times D \rightarrow \{0, 1\}^b$  is the family  $hF: \{0, 1\}^{c+k} \times D \rightarrow \{0, 1\}^c$  defined by  $hF(K_{\text{out}} \| K_{\text{in}}, M) = h(K_{\text{out}}, F(K_{\text{in}}, M))$ . The following lemma says that if  $h$  is a PRF and  $F$  is cAU then  $hF$  is a PRF.

**Lemma 3.2** *Let  $B = \{0, 1\}^b$ . Let  $h: \{0, 1\}^c \times B \rightarrow \{0, 1\}^c$  and  $F: \{0, 1\}^k \times D \rightarrow B$  be families of functions, and let  $hF: \{0, 1\}^{c+k} \times D \rightarrow \{0, 1\}^c$  be defined by*

$$hF(K_{\text{out}} \| K_{\text{in}}, M) = h(K_{\text{out}}, F(K_{\text{in}}, M))$$

*for all  $K_{\text{out}} \in \{0, 1\}^c, K_{\text{in}} \in \{0, 1\}^k$  and  $M \in D$ . Let  $A_{hF}$  be a prf-adversary against  $hF$  that makes at most  $q \geq 2$  oracle queries, each of length at most  $n$ , and has time-complexity at most  $t$ . Then there exists a prf-adversary  $A_h$  against  $h$  and an au-adversary  $A_F$  against  $F$  such that*

$$\mathbf{Adv}_{hF}^{\text{prf}}(A_{hF}) \leq \mathbf{Adv}_h^{\text{prf}}(A_h) + \binom{q}{2} \cdot \mathbf{Adv}_F^{\text{au}}(A_F). \quad (3)$$

*Furthermore,  $A_h$  has time-complexity at most  $t$  and makes at most  $q$  oracle queries, while  $A_F$  has time-complexity  $O(T_F(n))$  and the two messages it outputs have length at most  $n$ , where  $T_F(n)$  is the time to compute  $F$  on an  $n$ -bit input. ■*

This extends the analogous PRF(AU)=PRF lemma by relaxing the condition on  $F$  from AU to cAU. (The PRF(AU)=PRF lemma is alluded to in [3, 10], and variants are in [10, 11].) A simple proof of Lemma 3.2, using games [9, 28], is in Section 3.4.

The reduction of Lemma 3.2 may look loose due to the  $\binom{q}{2}$  factor. (And in some settings this is indeed the case and has led to the use of alternative constructs [10].) However in our case this factor turns out only to reflect the existing birthday attack on GNMAC [26] and thus does not reflect

a loose reduction. Note that the time-complexity of  $A_F$  is small and in particular independent of the time-complexity of  $A_{hF}$ .

GNMAC IS A PRF. We now combine the two lemmas above to conclude that if  $h$  is a PRF then so is GNMAC.

**Theorem 3.3** *Assume  $b \geq c$  and let  $B = \{0, 1\}^b$ . Let  $h: \{0, 1\}^c \times B \rightarrow \{0, 1\}^c$  be a family of functions and let  $\text{fpad} \in \{0, 1\}^{b-c}$  be a fixed padding string. Let GNMAC:  $\{0, 1\}^{2c} \times B^+ \rightarrow \{0, 1\}^c$  be defined by*

$$\text{GNMAC}(K_{\text{out}} \| K_{\text{in}}, M) = h(K_{\text{out}}, h^*(K_{\text{in}}, M) \| \text{fpad})$$

for all  $K_{\text{out}}, K_{\text{in}} \in \{0, 1\}^c$  and  $M \in B^+$ . Let  $A_{\text{GNMAC}}$  be a prf-adversary against GNMAC that makes at most  $q$  oracle queries, each of at most  $m$  blocks, and has time-complexity at most  $t$ . Then there exist prf-adversaries  $A_1, A_2$  against  $h$  such that

$$\text{Adv}_{\text{GNMAC}}^{\text{prf}}(A_{\text{GNMAC}}) \leq \text{Adv}_h^{\text{prf}}(A_1) + \binom{q}{2} \left[ 2m \cdot \text{Adv}_h^{\text{prf}}(A_2) + \frac{1}{2^c} \right]. \quad (4)$$

Furthermore,  $A_1$  has time-complexity at most  $t$  and makes at most  $q$  oracle queries, while  $A_2$  has time-complexity at most  $O(mT_h)$  and makes at most 2 oracle queries, where  $T_h$  is the time for one computation of  $h$ . ■

**Proof of Theorem 3.3:** Define  $F: \{0, 1\}^c \times B^+ \rightarrow \{0, 1\}^b$  by  $F(K_{\text{in}}, M) = h^*(K_{\text{in}}, M) \| \text{fpad}$ . Then  $\text{GNMAC} = hF$ . Apply Lemma 3.2 (with  $k = c$ ,  $D = B^+$  and  $A_{hF} = A_{\text{GNMAC}}$ ) to get prf-adversary  $A_1$  and au-adversary  $A_F$  with the properties stated in the lemma. Note that  $\text{Adv}_F^{\text{au}}(A_F) = \text{Adv}_{h^*}^{\text{au}}(A_F)$ . (Because a pair of messages is a collision for  $h^*(K_{\text{in}}, \cdot) \| \text{fpad}$  iff it is a collision for  $h^*(K_{\text{in}}, \cdot)$ .) Now apply Lemma 3.1 to  $A^* = A_F$  to get prf-adversary  $A_2$ . ■

### 3.2 Tightness of bound

If  $t$  is a time-complexity then let  $\bar{t} = t/T_h$ . The best known attacks on GNMAC are either the birthday one of [26] or exhaustive key search. This implies that with  $q$  queries and time-complexity  $t$ , one can achieve a prf-advantage of about  $\alpha(t, q) = (\bar{t} + q^2)2^{-c}$  against GNMAC. This hits 1 when  $\bar{t} = q \approx 2^{c/2}$ , leading to an estimate of about  $2^{c/2}$  queries (and time) to break NMAC.

To assess how close to  $\alpha(t, q)$  is the upper bound of Theorem 3.3, assume that the best attack against  $h$  as a PRF is exhaustive key search. (Birthday attacks do not apply since  $h$  is not a family of permutations.) This means that  $\text{Adv}_h^{\text{prf}}(A) \leq \bar{t} \cdot 2^{-c}$  for any prf-adversary  $A$  of time complexity  $t$  making  $q \leq \bar{t}$  queries. Plugging this into (4) and simplifying, the upper bound on the prf-advantage of any adversary against GNMAC who has time-complexity  $t$  and makes at most  $q$  queries is  $\beta(t, q, m) = O(\bar{t} + m^2 q^2 T_F) \cdot 2^{-c}$ . Roughly, this matches  $\alpha(t, q)$  up to factors that depend on the number  $m$  of blocks in the message. In particular, if we ignore the  $m^2$  and  $T_F$  terms, then this hits 1 when  $\bar{t} = q \approx 2^{c/2}$ . This means that the bound justifies NMAC up to roughly  $2^{c/2}$  queries in the case that the queried messages are short. (We note that the messages in the attack of [26] are short.)

We do not expect to be able to entirely remove the dependence on  $m$  from the upper bound. (Rather, we imagine it possible that the attacks could be extended so that the lower bound grows as some function of  $m$ .) But we imagine that it might be possible to reduce it from the current quadratic to a smaller function of  $m$ , perhaps via techniques from [16, 8].



<p><b>Game</b> <math>G_1(M_1, M_2, l) \ // \ 0 \leq l \leq \ M_1\ _b</math></p> <p><math>m_1 \leftarrow \ M_1\ _b ; m_2 \leftarrow \ M_2\ _b</math></p> <p><math>a[l] \xleftarrow{\\$} \{0, 1\}^c</math></p> <p>For <math>i = l + 1</math> to <math>m_2</math> do</p> <p style="padding-left: 20px;"><math>a[i] \leftarrow h(a[i - 1], M_2[i])</math></p> <p>If <math>a[m_1] = a[m_2]</math> then return 1</p> <p>else return 0</p> <p><b>Adversary</b> <math>A_3^g(M_1, M_2)</math></p> <p><math>m_1 \leftarrow \ M_1\ _b ; m_2 \leftarrow \ M_2\ _b</math></p> <p><math>l \xleftarrow{\\$} \{1, \dots, m_1 + 1\}</math></p> <p>If <math>l = m_1 + 1</math> then return <math>A_2^g(M_1, M_2)</math></p> <p>Else return <math>A_1^g(M_1, M_2, l)</math></p>	<p><b>Adversary</b> <math>A_1^g(M_1, M_2, l) \ // \ 1 \leq l \leq \ M_1\ _b</math></p> <p><math>m_1 \leftarrow \ M_1\ _b ; m_2 \leftarrow \ M_2\ _b</math></p> <p><math>a[l] \leftarrow g(M_2[l])</math></p> <p>For <math>i = l + 1</math> to <math>m_2</math> do</p> <p style="padding-left: 20px;"><math>a[i] \leftarrow h(a[i - 1], M_2[i])</math></p> <p>If <math>a[m_1] = a[m_2]</math> then return 1</p> <p>else return 0</p> <p><b>Adversary</b> <math>A_2^g(M_1, M_2)</math></p> <p><math>m_1 \leftarrow \ M_1\ _b ; m_2 \leftarrow \ M_2\ _b</math></p> <p><math>a[m_1 + 1] \leftarrow g(M_2[m_1 + 1])</math></p> <p>For <math>i = m_1 + 2</math> to <math>m_2</math> do</p> <p style="padding-left: 20px;"><math>a[i] \leftarrow h(a[i - 1], M_2[i])</math></p> <p><math>y \xleftarrow{\\$} B \setminus \{M_2[m_1 + 1]\}</math></p> <p>If <math>h(a[m_2], y) = g(y)</math> then return 1</p> <p>else return 0</p>
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Figure 1: Games and adversaries taking input distinct messages  $M_1, M_2$  such that  $M_1 \subseteq M_2$ . The adversaries take an oracle  $g: \{0, 1\}^b \rightarrow \{0, 1\}^c$ .

### 3.3 Proof of Lemma 3.1

SOME DEFINITIONS. In this proof it will be convenient to consider prf-adversaries that take inputs. The advantage of  $A$  against  $h$  on inputs  $x_1, \dots$  is defined as

$$\text{Adv}_h^{\text{prf}}(A(x_1, \dots)) = \Pr \left[ A^{h(K, \cdot)}(x_1, \dots) \Rightarrow 1 \right] - \Pr \left[ A^{\$}(x_1, \dots) \Rightarrow 1 \right],$$

where in the first case  $K \xleftarrow{\$} \{0, 1\}^c$  and in the second case the notation means that  $A$  is given as oracle a map chosen at random from  $\text{Maps}(\{0, 1\}^b, \{0, 1\}^c)$ .

OVERVIEW. To start with, we ignore  $A_{hF}$  and upper bound  $\text{Coll}_F(M_1, M_2)$  as some appropriate function of the prf-advantage of a prf-adversary against  $h$  that takes  $M_1, M_2$  as input. We consider first the case that  $M_1 \subseteq M_2$  ( $M_1$  is a prefix of  $M_2$ ) and then the case that  $M_1 \not\subseteq M_2$ , building in each case a different adversary.

THE CASE  $M_1 \subseteq M_2$ . We begin with some high-level intuition. Suppose  $M_1 \subseteq M_2$  with  $m_2 = \|M_2\|_b \geq 1 + m_1$ , where  $m_1 = \|M_1\|_b$ . The argument to upper bound  $\text{Coll}_{h^*}(M_1, M_2)$  has two parts. First, a hybrid argument is used to show that  $a[m_1] = h^*(K, M_1)$  is computationally close to random when  $K$  is drawn at random. Next, we imagine a game in which  $a[m_1]$  functions as a key to  $h$ . Let  $a[m_1 + 1] = h(a[m_1], M_2[m_1 + 1])$  and  $a[m_2] = h^*(a[m_1 + 1], M_2[m_1 + 2] \dots M_2[m_2])$ . Now, if  $a[m_2] = a[m_1]$  then we effectively have a way to recover the “key”  $a[m_1]$  given  $a[m_1 + 1]$ , amounting to a key-recovery attack on  $h(a[m_1], \cdot)$  based on one input-output example of this function. But being a PRF,  $h$  is also secure against key-recovery.

In the full proof that follows, we use the games and adversaries specified in Figure 1. Adversaries  $A_1, A_2$  represent, respectively, the first and second parts of the argument outlined above, while  $A_3$  integrates the two.

**Claim 3.4** Let  $M_1, M_2 \in B^+$  with  $M_1 \subseteq M_2$  and  $1 + \|M_1\|_b \leq \|M_2\|_b$ . Suppose  $1 \leq l \leq \|M_1\|_b$ .

Then

$$\begin{aligned}\Pr \left[ A_1^{\$}(M_1, M_2, l) \Rightarrow 1 \right] &= \Pr [ G_1(M_1, M_2, l) \Rightarrow 1 ] \\ \Pr \left[ A_1^{h(K, \cdot)}(M_1, M_2, l) \Rightarrow 1 \right] &= \Pr [ G_1(M_1, M_2, l-1) \Rightarrow 1 ] . \blacksquare\end{aligned}$$

Recall the notation means that in the first case  $A_1$  gets as oracle  $g \stackrel{\$}{\leftarrow} \text{Maps}(\{0, 1\}^b, \{0, 1\}^c)$  and in the second case  $K \stackrel{\$}{\leftarrow} \{0, 1\}^c$ .

**Proof of Claim 3.4:**  $A_1^g(M_1, M_2, l)$  sets  $a[l] = g(M_2[l])$ . If  $g$  is chosen at random then this is equivalent to the  $a[l] \stackrel{\$}{\leftarrow} \{0, 1\}^c$  assignment in  $G_1(M_1, M_2, l)$ . On the other hand if  $g = h(K, \cdot)$  for a random  $K$ , then  $K$  plays the role of  $a[l-1]$  in  $G_1(M_1, M_2, l-1)$ .  $\blacksquare$

**Claim 3.5** Let  $M_1, M_2 \in B^+$  with  $M_1 \subseteq M_2$  and  $1 + \|M_1\|_b \leq \|M_2\|_b$ . Then

$$\begin{aligned}\Pr \left[ A_2^{\$}(M_1, M_2) \Rightarrow 1 \right] &= 2^{-c} \\ \Pr \left[ A_2^{h(K, \cdot)}(M_1, M_2) \Rightarrow 1 \right] &\geq \Pr [ G_1(M_1, M_2, m_1) \Rightarrow 1 ] . \blacksquare\end{aligned}$$

**Proof of Claim 3.5:** Suppose  $g$  is chosen at random. Since  $y \neq M_2[m_1 + 1]$ , the quantity  $g(y)$  is not defined until the last line of the code of  $A_2$ , at which point  $h(a[m_2], y)$  is fixed, and thus the probability that the two are equal is  $2^{-c}$  due to the randomness of  $g(y)$ . Now suppose  $g = h(K, \cdot)$  for a random  $K$ . Think of  $K$  as playing the role of  $a[m_1]$  in  $G_1(M_1, M_2, m_1)$ . Then  $a[m_2] = K$  in  $A_2^g(M_1, M_2)$  exactly when  $a[m_1] = a[m_2]$  in  $G_1(M_1, M_2, m_1)$ , meaning exactly when the latter game returns 1. But if  $a[m_2] = K$  then certainly  $h(a[m_2], y) = h(K, y)$ , and the latter is  $g(y)$ , so  $A_1^g(M_1, M_2)$  will return 1. (However, it could be that  $h(a[m_2], y) = h(K, y)$  even if  $a[m_2] \neq K$ , which is why we have an inequality rather than an equality in the claim.)  $\blacksquare$

**Claim 3.6** Let  $M_1, M_2 \in B^+$  with  $M_1 \subseteq M_2$  and  $1 + \|M_1\|_b \leq \|M_2\|_b$ . Let  $m_1 = \|M_1\|_b$ . Then

$$\text{Adv}_h^{\text{prf}}(A_3(M_1, M_2)) \geq \frac{1}{m_1 + 1} (\text{Coll}_{h^*}(M_1, M_2) - 2^{-c}) . \blacksquare$$

**Proof of Claim 3.6:** From the description of  $A_3$ , whether  $g = \$$  or  $g = h(K, \cdot)$ ,

$$\Pr [ A_3^g(M_1, M_2) \Rightarrow 1 ] = \frac{1}{m_1 + 1} \left( \Pr [ A_2^g(M_1, M_2) \Rightarrow 1 ] + \sum_{l=1}^{m_1} \Pr [ A_1^g(M_1, M_2, l) \Rightarrow 1 ] \right) .$$

Now Claims 3.5 and 3.4 imply that  $\Pr [ A_3^{h(K, \cdot)}(M_1, M_2) \Rightarrow 1 ]$  is

$$\begin{aligned}&\geq \frac{1}{m_1 + 1} \left( \Pr [ G_1(M_1, M_2, m_1) \Rightarrow 1 ] + \sum_{l=1}^{m_1} \Pr [ G_1(M_1, M_2, l-1) \Rightarrow 1 ] \right) \\ &= \frac{1}{m_1 + 1} \cdot \sum_{l=0}^{m_1} \Pr [ G_1(M_1, M_2, l) \Rightarrow 1 ] .\end{aligned}\tag{5}$$

On the other hand, Claims 3.5 and 3.4 also imply that  $\Pr [ A_3^{\$}(M_1, M_2) \Rightarrow 1 ]$  is

$$= \frac{1}{m_1 + 1} \left( 2^{-c} + \sum_{l=1}^{m_1} \Pr [ G_1(M_1, M_2, l) \Rightarrow 1 ] \right) .\tag{6}$$

<p><b>Game</b> <math>G_2(M_1, M_2, l)</math> // <math>0 \leq l \leq \ M_1\ _b + \ M_2\ _b - \text{LCP}(M_1, M_2)</math></p> <p>200 <math>m_1 \leftarrow \ M_1\ _b</math>; <math>m_2 \leftarrow \ M_2\ _b</math>; <math>p \leftarrow \text{LCP}(M_1, M_2)</math></p> <p>210 If <math>0 \leq l \leq m_1</math> then</p> <p>220 <math>a_1[l] \stackrel{\\$}{\leftarrow} \{0, 1\}^c</math>; For <math>i = l + 1</math> to <math>m_1</math> do <math>a_1[i] \leftarrow h(a_1[i - 1], M_1[i])</math></p> <p>230 If <math>m_1 + 1 \leq l \leq m_1 + m_2 - p</math> then <math>a_1[m_1] \stackrel{\\$}{\leftarrow} \{0, 1\}^c</math></p> <p>240 If <math>0 \leq l \leq p</math> then <math>n \leftarrow l</math>; <math>a_2[n] \leftarrow a_1[n]</math></p> <p>250 If <math>p + 1 \leq l \leq m_1</math> then <math>n \leftarrow p + 1</math>; <math>a_2[n] \stackrel{\\$}{\leftarrow} \{0, 1\}^c</math></p> <p>260 If <math>m_1 + 1 \leq l \leq m_1 + m_2 - p</math> then <math>n \leftarrow l - m_1 + p + 1</math>; <math>a_2[n] \stackrel{\\$}{\leftarrow} \{0, 1\}^c</math></p> <p>270 For <math>i = n + 1</math> to <math>m_2</math> do <math>a_2[i] \leftarrow h(a_2[i - 1], M_2[i])</math></p> <p>280 If <math>a_1[m_1] = a_2[m_2]</math> then return 1 else return 0</p> <p><b>Adversary</b> <math>A_4^g(M_1, M_2, l)</math> // <math>1 \leq l \leq \ M_1\ _b + \ M_2\ _b - \text{LCP}(M_1, M_2)</math></p> <p>a00 <math>m_1 \leftarrow \ M_1\ _b</math>; <math>m_2 \leftarrow \ M_2\ _b</math>; <math>p \leftarrow \text{LCP}(M_1, M_2)</math></p> <p>a10 If <math>1 \leq l \leq m_1</math> then</p> <p>a20 <math>a_1[l] \leftarrow g(M_1[l])</math>; For <math>i = l + 1</math> to <math>m_1</math> do <math>a_1[i] \leftarrow h(a_1[i - 1], M_1[i])</math></p> <p>a30 If <math>m_1 + 1 \leq l \leq m_1 + m_2 - p</math> then <math>a_1[m_1] \stackrel{\\$}{\leftarrow} \{0, 1\}^c</math></p> <p>a40 If <math>1 \leq l \leq p</math> then <math>n \leftarrow l</math>; <math>a_2[n] \leftarrow a_1[n]</math></p> <p>a50 If <math>l = p + 1</math> then <math>n \leftarrow p + 1</math>; <math>a_2[n] \leftarrow g(M_2[n])</math></p> <p>a51 If <math>p + 2 \leq l \leq m_1</math> then <math>n \leftarrow p + 1</math>; <math>a_2[n] \stackrel{\\$}{\leftarrow} \{0, 1\}^c</math></p> <p>a60 If <math>m_1 + 1 \leq l \leq m_1 + m_2 - p</math> then <math>n \leftarrow l - m_1 + p + 1</math>; <math>a_2[n] \leftarrow g(M_2[n])</math></p> <p>a70 For <math>i = n + 1</math> to <math>m_2</math> do <math>a_2[i] \leftarrow h(a_2[i - 1], M_2[i])</math></p> <p>a80 If <math>a_1[m_1] = a_2[m_2]</math> then return 1 else return 0</p>
--

Figure 2: Games and adversaries taking input distinct messages  $M_1, M_2 \in B^+$  such that  $M_1 \not\subseteq M_2$  and  $\|M_1\|_b \leq \|M_2\|_b$ .

Subtracting (6) from (5) and exploiting the cancellation of terms, we get

$$\text{Adv}_h^{\text{prf}}(A_3(M_1, M_2)) \geq \frac{1}{m_1 + 1} (\Pr[G_1(M_1, M_2, 0) \Rightarrow 1] - 2^{-c}).$$

Now examination of Game  $G_1(M_1, M_2, 0)$  shows that that in this game,  $a[m_1] = h^*(a[0], M_1)$ ,  $a[m_2] = h^*(a[0], M_2)$ , and  $a[0]$  is selected at random. Since the game returns 1 iff  $a[m_1] = a[m_2]$ , the probability that it returns 1 is exactly  $\text{Coll}_{h^*}(M_1, M_2)$ . ■

THE CASE  $M_1 \not\subseteq M_2$ . For  $M_1, M_2 \in B^+$  with  $\|M_1\|_b \leq \|M_2\|_b$  and  $M_1 \not\subseteq M_2$ , we let  $\text{LCP}(M_1, M_2)$  denote the length of the longest common blockwise prefix of  $M_1, M_2$ , meaning the largest integer  $p$  such that  $M_1[1] \dots M_1[p] = M_2[1] \dots M_2[p]$  but  $M_1[p + 1] \neq M_2[p + 1]$ . We consider the games and adversaries of Figure 2.

**Claim 3.7** Let  $M_1, M_2 \in B^+$  with  $M_1 \not\subseteq M_2$ , and  $\|M_1\|_b \leq \|M_2\|_b$ . Suppose  $1 \leq l \leq \|M_1\|_b + \|M_2\|_b - \text{LCP}(M_1, M_2)$ . Then

$$\begin{aligned} \Pr[A_4^{\$}(M_1, M_2, l) \Rightarrow 1] &= \Pr[G_2(M_1, M_2, l) \Rightarrow 1] \\ \Pr[A_4^{h(K, \cdot)}(M_1, M_2, l) \Rightarrow 1] &= \Pr[G_2(M_1, M_2, l - 1) \Rightarrow 1]. \quad \blacksquare \end{aligned}$$

**Proof of Claim 3.7:** The first equality, namely the one where  $g \stackrel{\$}{\leftarrow} \text{Maps}(\{0, 1\}^b, \{0, 1\}^c)$ , is quite easy to see. Let us compare the code of  $G_2(M_1, M_2, l)$  and  $A_4^g(M_1, M_2, l)$  and look at places where

```

Game  $G_3(M_1, M_2, l)$  //  $1 \leq l \leq \|M_1\|_b + \|M_2\|_b - \text{LCP}(M_1, M_2)$ 
300  $m_1 \leftarrow \|M_1\|_b$ ;  $m_2 \leftarrow \|M_2\|_b$ ;  $p \leftarrow \text{LCP}(M_1, M_2)$ 
310 If  $0 \leq l - 1 \leq m_1$  then
320    $a_1[l - 1] \stackrel{\$}{\leftarrow} \{0, 1\}^c$ ; For  $i = l$  to  $m_1$  do  $a_1[i] \leftarrow h(a_1[i - 1], M_1[i])$ 
330 If  $m_1 + 1 \leq l - 1 \leq m_1 + m_2 - p$  then  $a_1[m_1] \stackrel{\$}{\leftarrow} \{0, 1\}^c$ 
340 If  $0 \leq l - 1 \leq p$  then  $n \leftarrow l - 1$ ;  $a_2[n] \leftarrow a_1[n]$ 
350 If  $p + 1 \leq l - 1 \leq m_1$  then  $n \leftarrow p + 1$ ;  $a_2[n] \stackrel{\$}{\leftarrow} \{0, 1\}^c$ 
360 If  $m_1 + 1 \leq l - 1 \leq m_1 + m_2 - p$  then  $n \leftarrow l - m_1 + p$ ;  $a_2[n] \stackrel{\$}{\leftarrow} \{0, 1\}^c$ 
370 For  $i = n + 1$  to  $m_2$  do  $a_2[i] \leftarrow h(a_2[i - 1], M_2[i])$ 
380 If  $a_1[m_1] = a_2[m_2]$  then return 1 else return 0

Game  $G_4(M_1, M_2, l)$  //  $1 \leq l \leq \|M_1\|_b + \|M_2\|_b - \text{LCP}(M_1, M_2)$ 
400  $m_1 \leftarrow \|M_1\|_b$ ;  $m_2 \leftarrow \|M_2\|_b$ ;  $p \leftarrow \text{LCP}(M_1, M_2)$ 
410 If  $1 \leq l \leq m_1$  then
420    $a_1[l - 1] \stackrel{\$}{\leftarrow} \{0, 1\}^c$ ; For  $i = l$  to  $m_1$  do  $a_1[i] \leftarrow h(a_1[i - 1], M_1[i])$ 
430 If  $m_1 + 1 \leq l \leq m_1 + m_2 - p$  then  $a_1[m_1] \stackrel{\$}{\leftarrow} \{0, 1\}^c$ 
440 If  $1 \leq l \leq p$  then  $a_2[l - 1] \leftarrow a_1[l - 1]$ ;  $n \leftarrow l$ ;  $a_2[n] \leftarrow h(a_2[n - 1], M_2[n])$ 
441 If  $l = p + 1$  then  $a_2[p] \leftarrow a_1[p]$ ;  $n \leftarrow p + 1$ ;  $a_2[n] \leftarrow h(a_2[n - 1], M_2[n])$ 
450 If  $p + 2 \leq l \leq m_1 + 1$  then  $n \leftarrow p + 1$ ;  $a_2[n] \stackrel{\$}{\leftarrow} \{0, 1\}^c$ 
460 If  $m_1 + 2 \leq l \leq m_1 + m_2 - p$  then
461    $n \leftarrow l - m_1 + p + 1$ ;  $a_2[n - 1] \stackrel{\$}{\leftarrow} \{0, 1\}^c$ ;  $a_2[n] \leftarrow h(a_2[n - 1], M_2[n])$ 
470 For  $i = n + 1$  to  $m_2$  do  $a_2[i] \leftarrow h(a_2[i - 1], M_2[i])$ 
480 If  $a_1[m_1] = a_2[m_2]$  then return 1 else return 0

Game  $G_5(M_1, M_2, l)$  //  $1 \leq l \leq \|M_1\|_b + \|M_2\|_b - \text{LCP}(M_1, M_2)$ 
500  $m_1 \leftarrow \|M_1\|_b$ ;  $m_2 \leftarrow \|M_2\|_b$ ;  $p \leftarrow \text{LCP}(M_1, M_2)$ 
501  $K \stackrel{\$}{\leftarrow} \{0, 1\}^c$ 
510 If  $1 \leq l \leq m_1$  then
520    $a_1[l - 1] \leftarrow K$ ; For  $i = l$  to  $m_1$  do  $a_1[i] \leftarrow h(a_1[i - 1], M_1[i])$ 
530 If  $m_1 + 1 \leq l \leq m_1 + m_2 - p$  then  $a_1[m_1] \stackrel{\$}{\leftarrow} \{0, 1\}^c$ 
540 If  $1 \leq l \leq p$  then  $n \leftarrow l$ ;  $a_2[n] \leftarrow a_1[n]$ 
541 If  $l = p + 1$  then  $n \leftarrow p + 1$ ;  $a_2[n] \leftarrow h(K, M_2[n])$ 
550 If  $p + 2 \leq l \leq m_1$  then  $n \leftarrow p + 1$ ;  $a_2[n] \stackrel{\$}{\leftarrow} \{0, 1\}^c$ 
560 If  $m_1 + 1 \leq l \leq m_1 + m_2 - p$  then
561    $n \leftarrow l - m_1 + p + 1$ ;  $a_2[n - 1] \leftarrow K$ ;  $a_2[n] \leftarrow h(a_2[n - 1], M_2[n])$ 
570 For  $i = n + 1$  to  $m_2$  do  $a_2[i] \leftarrow h(a_2[i - 1], M_2[i])$ 
580 If  $a_1[m_1] = a_2[m_2]$  then return 1 else return 0

```

Figure 3: Games equivalent to  $G_2(M_1, M_2, l - 1)$ , where  $M_1, M_2 \in B^+$  are such that  $M_1 \not\subseteq M_2$  and  $\|M_1\|_b \leq \|M_2\|_b$ .

they differ. The first such place is line a20, but since  $g$  is random, this is equivalent to line 220. Then we get to line a50. This is equivalent to what line 250 does for  $l = p + 1$  because  $g(M_2[n])$  is distributed uniformly and independently of anything else. (This is true because  $g$  is random and we know that  $M_2[p + 1] \neq M_1[p + 1]$  because  $p = \text{LCP}(M_1, M_2)$  and  $M_1$  is not a prefix of  $M_2$ .) Next, line a60 is equivalent to line 260 because  $g$  is random and, in this case, has not previously

been invoked.

The second equality, namely the one where  $K$  is selected at random from  $\{0, 1\}^c$  and  $g = h(K, \cdot)$ , needs more work. We will justify it by considering the games of Figure 3. We claim that

$$\Pr [G_2(M_1, M_2, l-1) \Rightarrow 1] = \Pr [G_3(M_1, M_2, l) \Rightarrow 1] \quad (7)$$

$$= \Pr [G_4(M_1, M_2, l) \Rightarrow 1] \quad (8)$$

$$= \Pr [G_5(M_1, M_2, l) \Rightarrow 1] \quad (9)$$

$$= \Pr [A_4^{h(K, \cdot)}(M_1, M_2, l) \Rightarrow 1] \quad (10)$$

We justify the above relations in turn by arguing that the games are equivalent.

Game  $G_3(M_1, M_2, l)$  is just a copy of  $G_2(M_1, M_2, l-1)$ , so (7) is clear.

Now we argue that  $G_3(M_1, M_2, l)$  is equivalent to  $G_4(M_1, M_2, l)$ . The changes to 310, 330 resulting in 410, 430 have simply moved the case  $l = m_1 + 1$ , covered by 310, into 430. This is right because when  $l = m_1 + 1$ , line 320 results in  $a_1[m_1]$  being chosen at random. Line 440 has been formed by pulling one iteration of 370 into 340 for the case  $1 \leq l \leq p$ , while the case  $l = p + 1$  of 340 is covered similarly by 441. Line 450 is clearly equivalent to 350. Lines 460, 461 have been formed by pulling one iteration of 370 into 360. Line 370 results in the same value of  $a_2[m_2]$  as 470 because, although we have in some cases incremented  $n$  by one in  $G_4(M_1, M_2, l)$  as compared to its value in  $G_3(M_1, M_2, l)$ , we have in these cases pulled the first iteration of the loop of 370 into some previous statement. This justifies (8).

Next we argue that  $G_4(M_1, M_2, l)$  is equivalent to  $G_5(M_1, M_2, l)$ . Line 520 is equivalent to 420 because  $K$  was chosen at random at 501. To obtain line 540 from 440, we first use the fact that  $a_1[l-1] = K$  from 520. Now since  $1 \leq l = n \leq p < m_1$  we have  $M_2[n] = M_1[n]$ , and thus from 440 we have  $h(a_2[n-1], M_2[n]) = h(a_1[n-1], M_1[n])$ . But this equals  $a_1[n]$  due to 520. Line 541 follows from 441 because  $a_1[p] = K$  due to 520. (Here  $l = p + 1$ .) Line 550 mimics 450 but excludes the case  $l = m_1 + 1$  of the latter. This case is now covered by 560, 561. (In the case  $l = m_1 + 1$ , these lines set  $n = p + 2$  and  $a_2[p + 1] = K$ . The latter is equivalent to the random choice of  $a_2[p + 1]$  in 450 because if  $l = m_1 + 1$  then  $K$  has not been used before in  $G_5(M_1, M_2, l)$ . Additionally, one iteration of 470 is pulled into 561, with  $n$  incremented accordingly.) On the other hand 560, 561 mimic 460, 461 in the case  $m_1 + 2 \leq l \leq m_1 + m_2 - p$  because in this case  $K$  has not been used before in  $G_5(M_1, M_2, l)$  and thus  $a_2[n-1] \leftarrow K$  is equivalent to choosing it at random as in 461. Finally, 570 mimics 470. This justifies (9).

Finally, (10) is easy to see: just identify the key  $K$  in the  $h(K, \cdot)$  oracle given to  $A_4$  with the one chosen at 501. ■

We now define prf adversary  $A_5^g(M_1, M_2)$  against  $h$  as follows. It picks  $l \xleftarrow{\$} \{1, \dots, \|M_1\|_b + \|M_2\|_b - \text{LCP}(M_1, M_2)\}$  and returns  $A_4^g(M_1, M_2, l)$ .

**Claim 3.8** Let  $M_1, M_2 \in B^+$  with  $M_1 \not\subseteq M_2$  and  $\|M_1\|_b \leq \|M_2\|_b$ . Let  $m = \|M_1\|_b + \|M_2\|_b - \text{LCP}(M_1, M_2)$ . Then

$$\mathbf{Adv}_h^{\text{prf}}(A_5) \geq \frac{1}{m} \cdot (\text{Coll}_{h^*}(M_1, M_2) - 2^{-c}) . \blacksquare$$

**Proof of Claim 3.8:** By Claim 3.7,

$$\mathbf{Adv}_h^{\text{prf}}(A_5) = \frac{1}{m} \cdot \sum_{l=1}^m \Pr [A_4^{h(K, \cdot)}(M_1, M_2, l) \Rightarrow 1] - \frac{1}{m} \cdot \sum_{l=1}^m \Pr [A_4^{\$}(M_1, M_2, l) \Rightarrow 1]$$

$$\begin{aligned}
&= \frac{1}{m} \cdot \sum_{l=1}^m \Pr [G_2(M_1, M_2, l-1) \Rightarrow 1] - \frac{1}{m} \cdot \sum_{l=1}^m \Pr [G_2(M_1, M_2, l) \Rightarrow 1] \\
&= \frac{1}{m} \cdot (\Pr [G_2(M_1, M_2, 0) \Rightarrow 1] - \Pr [G_2(M_1, M_2, m) \Rightarrow 1]) .
\end{aligned}$$

Let  $m_1 = \|M_1\|_b$  and  $m_2 = \|M_2\|_b$ . Examination of Game  $G_2(M_1, M_2, 0)$  shows that that, in this game,  $a_1[m_1] = h^*(a_1[0], M_1)$ ,  $a_2[m_2] = h^*(a_2[0], M_2)$ , and  $a_1[0] = a_2[0]$  is selected at random. Since the game returns 1 iff  $a_1[m_1] = a_2[m_2]$ , the probability that it returns 1 is exactly  $\text{Coll}_{h^*}(M_1, M_2)$ . On the other hand, since  $m_2 \geq m_1 \geq p+1$ , the values  $a_1[m_1]$  and  $a_2[m_2]$  in  $G_2(M_1, M_2, m)$  end up being chosen independently at random, and so the probability that they are equal, which is the probability this game returns 1, is  $2^{-c}$ .  $\blacksquare$

PUTTING IT TOGETHER. The final step to construct the prf-adversary  $A$  against  $h$ , claimed in the lemma, is to appropriately combine  $A_3, A_5$ . We assume wlog that the two messages  $M_1, M_2$  output by  $A^*$  are always distinct, in  $B^+$ , and satisfy  $\|M_1\|_b \leq \|M_2\|_b$ . We first define

$$\begin{array}{l|l}
\text{Adversary } A_6^g(M_1, M_2) & \text{Adversary } A_7^g \\
\text{If } M_1 \subseteq M_2 \text{ then return } A_3^g(M_1, M_2) & (M_1, M_2) \stackrel{\$}{\leftarrow} A^* \\
\text{Else return } A_5^g(M_1, M_2) & \text{Return } A_6^g(M_1, M_2)
\end{array}$$

**Claim 3.9**  $\text{Adv}_{h^*}^{\text{au}}(A^*) \leq 2^{-c} + (n_1 + n_2) \cdot \text{Adv}_h^{\text{prf}}(A_7)$   $\blacksquare$

**Proof of Claim 3.9:** We note that

$$\text{Adv}_{h^*}^{\text{au}}(A^*) = \sum_{M_1 \subseteq M_2} \text{Coll}_{h^*}(M_1, M_2) \cdot \Pr [M_1, M_2] + \sum_{M_1 \not\subseteq M_2} \text{Coll}_{h^*}(M_1, M_2) \cdot \Pr [M_1, M_2]$$

where  $\Pr[M_1, M_2]$  denotes the probability that  $A^*$  outputs  $(M_1, M_2)$ . Now use Claims 3.6 and 3.8, and also the assumptions  $\|M_1\|_b \leq n_1$ ,  $\|M_2\|_b \leq n_2$  and  $n_2 \geq 1$  from the lemma statement, to get

$$\begin{aligned}
\text{Adv}_{h^*}^{\text{au}}(A^*) &\leq \sum_{M_1 \subseteq M_2} \left[ (n_1 + n_2) \cdot \text{Adv}_h^{\text{prf}}(A_3(M_1, M_2)) + 2^{-c} \right] \cdot \Pr [M_1, M_2] \\
&\quad + \sum_{M_1 \not\subseteq M_2} \left[ (n_1 + n_2) \cdot \text{Adv}_h^{\text{prf}}(A_5(M_1, M_2)) + 2^{-c} \right] \cdot \Pr [M_1, M_2] \\
&= 2^{-c} + (n_1 + n_2) \cdot \text{Adv}_h^{\text{prf}}(A_7) ,
\end{aligned}$$

where in the last line we used the definition of prf-adversary  $A_7$ .  $\blacksquare$

Prf-Adversary  $A_7$  achieves the prf-advantage we seek, but has time-complexity that of  $A^*$  since it runs the latter. We now use a standard ‘‘coin-fixing’’ argument to reduce this time-complexity. Note that

$$\text{Adv}_h^{\text{prf}}(A_7) = \mathbf{E}_{M_1, M_2} \left[ \text{Adv}_h^{\text{prf}}(A_6(M_1, M_2)) \right]$$

where the expectation is over  $(M_1, M_2) \stackrel{\$}{\leftarrow} A^*$ . Thus there must exist distinct  $M_1, M_2 \in B^+$  ( $\|M_1\|_b \leq \|M_2\|_b$ ) such that  $\text{Adv}_h^{\text{prf}}(A_6(M_1, M_2)) \geq \text{Adv}_h^{\text{prf}}(A_7)$ . Let  $A$  be the prf-adversary that has  $M_1, M_2$  hardwired as part of its code and, given oracle  $g$ , runs  $A_6^g(M_1, M_2)$ . Since the latter has time complexity  $O(mT_h)$ , the proof of Lemma 3.1 is complete.

<p><b>Game <math>G_0</math></b></p> $K_{\text{in}} \xleftarrow{\$} \{0, 1\}^k$ $K_{\text{out}} \xleftarrow{\$} \{0, 1\}^c$ <b>On query <math>M</math>:</b> Reply $h(K_{\text{out}}, F(K_{\text{in}}, M))$	<p><b>Game <math>G_1</math></b></p> $K_{\text{in}} \xleftarrow{\$} \{0, 1\}^k$ $f \xleftarrow{\$} \text{Maps}(\{0, 1\}^b, \{0, 1\}^c)$ <b>On query <math>M</math>:</b> Reply $f(F(K_{\text{in}}, M))$	<p><b>Game <math>G_2</math></b></p> $g \xleftarrow{\$} \text{Maps}(D, \{0, 1\}^c)$ <b>On query <math>M</math>:</b> Reply $g(M)$
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Figure 4: Games  $G_0, G_1, G_2$  for the proof of Lemma 3.2.

### 3.4 Proof of Lemma 3.2

Game  $G_0$  of Figure 4 implements an oracle for  $hF(K_{\text{out}}\|K_{\text{in}}, \cdot)$  with the keys chosen at random, while Game  $G_2$  implements an oracle for a random function. So

$$\begin{aligned}
& \mathbf{Adv}_{hF}^{\text{prf}}(A_{hF}) \\
&= \Pr \left[ A_{hF}^{G_0} \Rightarrow 1 \right] - \Pr \left[ A_{hF}^{G_2} \Rightarrow 1 \right] \\
&= \left( \Pr \left[ A_{hF}^{G_0} \Rightarrow 1 \right] - \Pr \left[ A_{hF}^{G_1} \Rightarrow 1 \right] \right) + \left( \Pr \left[ A_{hF}^{G_1} \Rightarrow 1 \right] - \Pr \left[ A_{hF}^{G_2} \Rightarrow 1 \right] \right) \quad (11)
\end{aligned}$$

where in the last step we simply subtracted and then added back in the value  $\Pr \left[ A_{hF}^{G_1} \Rightarrow 1 \right]$ .

It is easy to construct a prf-adversary  $A_h$  against  $h$  such that

$$\mathbf{Adv}_h^{\text{prf}}(A_h) = \Pr \left[ A_{hF}^{G_0} \Rightarrow 1 \right] - \Pr \left[ A_{hF}^{G_1} \Rightarrow 1 \right]. \quad (12)$$

(Namely,  $A_h$ , given an oracle for a function  $f: B \rightarrow \{0, 1\}^c$ , picks  $K_{\text{in}} \xleftarrow{\$} \{0, 1\}^k$ . It then runs  $A_{hF}$ , replying to oracle query  $M$  by  $f(F(K_{\text{in}}, M))$ , and returns whatever output  $A_{hF}$  returns. We omit the simple analysis that establishes (12).) The main part of the proof is to construct au-adversary  $A'_F$  against  $F$  such that

$$\Pr \left[ A_{hF}^{G_1} \Rightarrow 1 \right] - \Pr \left[ A_{hF}^{G_2} \Rightarrow 1 \right] \leq \binom{q}{2} \cdot \mathbf{Adv}_F^{\text{au}}(A'_F). \quad (13)$$

This adversary, however, will have time-complexity  $t$  (and output messages of at most  $n$  bits). A standard ‘‘coin-fixing’’ argument will then be used to derive from  $A'_F$  an au-adversary  $A_F$  that has time-complexity  $O(T_F(n))$  (and also outputs messages of  $n$  bits) such that

$$\mathbf{Adv}_F^{\text{au}}(A'_F) \leq \mathbf{Adv}_F^{\text{au}}(A_F). \quad (14)$$

Combining (11), (12), (13) and (14) we get (3), completing the proof of the lemma.

Towards constructing  $A'_F$ , consider Games  $G_3, G_4, G_5$  of Figure 5. (Game  $G_3$  is defined by the code on the left of the Figure. Game  $G_4$  is the same except that the boxed code-statement is omitted.) We will assume now that a prf-adversary never repeats an oracle query. This is wlog, and is used below without explicit mention.

**Claim 3.10** Game  $G_4$  is equivalent to Game  $G_2$  while Game  $G_3$  is equivalent to Game  $G_1$ .  $\blacksquare$

**Proof of Claim 3.10:** In Game  $G_4$ , the ‘‘If’’ statement does nothing beyond setting the bad flag, and the reply to query  $M_s$  is always the random value  $Z_s$ . Thus, Game  $G_4$  implements a random function just like Game  $G_2$ . Game  $G_3$  returns random values except that it also ensures that if

<p><b>Games <math>G3, G4</math></b></p> <p><math>K_{\text{in}} \stackrel{\\$}{\leftarrow} \{0, 1\}^k; s \leftarrow 0</math></p> <p><math>Z_1, \dots, Z_q \stackrel{\\$}{\leftarrow} \{0, 1\}^c</math></p> <p><b>On query <math>M</math>:</b></p> <p><math>s \leftarrow s + 1; M_s \leftarrow M</math></p> <p><math>Y_s \leftarrow F(K_{\text{in}}, M_s)</math></p> <p>If <math>(\exists r &lt; s : Y_r = Y_s)</math> then</p> <p style="padding-left: 20px;"><math>\text{bad} \leftarrow \text{true}; \boxed{Z_s \leftarrow Z_r}</math></p> <p>Reply <math>Z_s</math></p>	<p><b>Game <math>G5</math></b></p> <p><math>s \leftarrow 0</math></p> <p><math>Z_1, \dots, Z_q \stackrel{\\$}{\leftarrow} \{0, 1\}^c</math></p> <p><b>On query <math>M</math>:</b></p> <p><math>s \leftarrow s + 1; M_s \leftarrow M</math></p> <p>Reply <math>Z_s</math></p>
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Figure 5: Games  $G3, G4$  are defined by the code on the left, where Game  $G3$  includes the boxed statement while Game  $G4$  does not.

$F(K_{\text{in}}, M_i) = F(K_{\text{in}}, M_j)$  then the answers to queries  $M_i, M_j$  are the same. Thus, it is equivalent to Game  $G1$ . ■

Now we have:

$$\Pr [A_{hF}^{G1} \Rightarrow 1] - \Pr [A_{hF}^{G2} \Rightarrow 1] = \Pr [A_{hF}^{G3} \Rightarrow 1] - \Pr [A_{hF}^{G4} \Rightarrow 1] \quad (15)$$

$$\leq \Pr [A_{hF}^{G4} \text{ sets bad}]. \quad (16)$$

Above, (15) is by Claim 3.10. Since  $G3, G4$  differ only in statements that follow the setting of **bad**, (16) follows from the Fundamental Lemma of Game Playing [9].

We define au-adversary  $A'_F$  against  $F$ , as follows: It runs  $A_{hF}^{G5}$ , then picks at random  $i, j$  subject to  $1 \leq i < j \leq q$ , and finally outputs the messages  $M_i, M_j$ . In other words, it runs  $A_{hF}$ , replying to the oracle queries of the latter with random values, and then outputs a random pair of messages that  $A_{hF}$  queries to its oracle. (In order for  $M_i, M_j$  to always be defined, we assume  $A_{hF}$  always makes *exactly*  $q$  oracle queries rather than at most  $q$  where by “always” we mean no matter how its oracle queries are replied to. This is wlog.) We claim that

$$\Pr [A_{hF}^{G4} \text{ sets bad}] \leq \binom{q}{2} \cdot \mathbf{Adv}_F^{\text{au}}(A'_F). \quad (17)$$

Combining (16) and (17) yields (13). We now justify (17). Intuitively, it is true because  $i, j$  are chosen at random after the execution of  $A_{hF}$  is complete, so  $A_{hF}$  has no information about them. A rigorous proof however needs a bit more work. Consider the experiment defining the au-advantage of  $A'_F$ . (Namely, we run  $A_{hF}^{G5}$ , pick  $i, j$  at random subject to  $1 \leq i < j \leq q$ , and then pick  $K_{\text{in}} \stackrel{\$}{\leftarrow} \{0, 1\}^k$ .) In this experiment, consider the following events defined for  $1 \leq \alpha < \beta \leq q$ :

$$C_{\alpha, \beta} : F(K_{\text{in}}, M_\alpha) = F(K_{\text{in}}, M_\beta)$$

$$C : \bigvee_{1 \leq \alpha < \beta \leq q} C_{\alpha, \beta}.$$

Notice that the events “ $C_{\alpha, \beta} \wedge (i, j) = (\alpha, \beta)$ ” ( $1 \leq \alpha < \beta \leq q$ ) are disjoint. (Even though the events  $C_{\alpha, \beta}$  for  $1 \leq \alpha < \beta \leq q$  are not.) Thus:

$$\begin{aligned} \mathbf{Adv}_F^{\text{au}}(A'_F) &= \Pr [\bigvee_{1 \leq \alpha < \beta \leq q} (C_{\alpha, \beta} \wedge (i, j) = (\alpha, \beta))] \\ &= \sum_{1 \leq \alpha < \beta \leq q} \Pr [C_{\alpha, \beta} \wedge (i, j) = (\alpha, \beta)]. \end{aligned}$$



Since  $i, j$  are chosen at random after the execution of  $A_{hF}^{G5}$  is complete, the events “ $(i, j) = (\alpha, \beta)$ ” and  $C_{\alpha, \beta}$  are independent. Thus the above equals

$$\begin{aligned} \sum_{1 \leq \alpha < \beta \leq q} \Pr[C_{\alpha, \beta}] \cdot \Pr[(i, j) = (\alpha, \beta)] &= \sum_{1 \leq \alpha < \beta \leq q} \Pr[C_{\alpha, \beta}] \cdot \frac{1}{\binom{q}{2}} \\ &= \frac{1}{\binom{q}{2}} \cdot \sum_{1 \leq \alpha < \beta \leq q} \Pr[C_{\alpha, \beta}] \\ &\geq \frac{1}{\binom{q}{2}} \cdot \Pr[C]. \end{aligned}$$

The proof of (17) is concluded by noting that  $\Pr[C]$  equals  $\Pr[G4 \text{ sets bad}]$ .

Finally, note that

$$\mathbf{Adv}_F^{\text{au}}(A'_F) = \mathbf{E}_{M_1, M_2} [\text{Coll}_F(M_1, M_2)]$$

where the expectation is over  $(M_1, M_2) \xleftarrow{\$} A'_F$ . Thus there must exist  $M_1, M_2 \in B^+$  such that  $\text{Coll}_F(M_1, M_2) \geq \mathbf{Adv}_F^{\text{au}}(A'_F)$ . (And these messages are distinct because  $A_{hF}$  never repeats an oracle query.) Let  $A_F$  be the au-adversary that has  $M_1, M_2$  hardwired as part of its code and, when run, simply outputs these messages and halts. Then (14) follows. Furthermore the time complexity of  $A_F$  is  $O(mT_h)$ . (Remember that by our convention the time-complexity is that of the overlying experiment, so includes the time to compute  $T_F$  on the messages that  $A_F$  outputs.)

## 4 MAC-security of NMAC under weaker assumptions

Since any PRF is a secure MAC [5, 7], Theorem 3.3 implies that NMAC is a secure MAC if the compression function is a PRF. Here we show however that it is possible to establish that NMAC is secure MAC under a weaker assumption on the compression function. We introduce the notion of a privacy-preserving MAC, and show that this, coupled with a weak assumption on the hash function, implies NMAC is a secure MAC. This is of interest given the still numerous usages of HMAC as a MAC (rather than as a PRF).

### 4.1 Privacy preserving MACs

MAC FORGERY. Recall that the mac-advantage of mac-adversary  $A$  against a family of functions  $f: Keys \times Dom \rightarrow Rng$  is

$$\mathbf{Adv}_f^{\text{mac}}(A) = \Pr \left[ A^{f(K, \cdot), \text{VF}_f(K, \cdot, \cdot)} \text{ forges} : K \xleftarrow{\$} Keys \right].$$

The verification oracle  $\text{VF}_f(K, \cdot, \cdot)$  associated to  $f$  takes input  $M, T$ , returning 1 if  $f(K, M) = T$  and 0 otherwise. Queries to the first oracle are called mac queries, and ones to the second are called verification queries.  $A$  is said to forge if it makes a verification query  $M, T$  the response to which is 1 but  $M$  was not previously a mac query. Note we allow multiple verification queries [7].

PRIVACY-PRESERVING MACS. We define privacy for MACs by adapting the notion of left-or-right indistinguishability of encryption [4] to functions that are deterministic. An oracle query of an ind-adversary  $A$  against family  $f: \{0, 1\}^k \times \{0, 1\}^l \rightarrow \{0, 1\}^L$  is a pair of  $l$ -bit strings. The reply is provided by one or the other of the following games:

<b>Game Left</b> $K \xleftarrow{\$} \{0, 1\}^c$ <b>On query</b> $(x_0, x_1)$ : Reply $f(K, x_0)$	<b>Game Right</b> $K \xleftarrow{\$} \{0, 1\}^c$ <b>On query</b> $(x_0, x_1)$ : Reply $f(K, x_1)$
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Each game has an initialization step in which it picks a key; it then uses this key in computing replies to all the queries made by  $A$ . The ind-advantage of  $A$  is

$$\mathbf{Adv}_f^{\text{ind}}(A) = \Pr[A^{\text{Right}} \Rightarrow 1] - \Pr[A^{\text{Left}} \Rightarrow 1].$$

However, unlike for encryption, the oracles here are deterministic. So  $A$  can easily win (meaning, obtain a high advantage), for example by making a pair of queries of the form  $(x, z), (y, z)$ , where  $x, y, z$  are distinct, and then returning 1 iff the replies returned are the same. (We expect that  $h(K, x) \neq h(K, y)$  with high probability over  $K$  for functions  $h$  of interest, for example compression functions.) We fix this by simply outlawing such behavior. To be precise, let us say that  $A$  is *legitimate* if for any sequence  $(x_0^1, x_1^1), \dots, (x_0^q, x_1^q)$  of oracle queries that it makes,  $x_0^1, \dots, x_0^q$  are all distinct  $l$ -bit strings, and also  $x_1^1, \dots, x_1^q$  are all distinct  $l$ -bit strings. (As a test, notice that the adversary who queried  $(x, z), (y, z)$  was not legitimate.) It is to be understood henceforth that an ind-adversary means a legitimate one. When we say that  $f$  is privacy-preserving, we mean that the ind-advantage of any (legitimate) practical ind-adversary is low.

Privacy-preservation is not, by itself, a demanding property. For example, it is achieved by a constant family such as the one defined by  $f(K, x) = 0^L$  for all  $K, x$ . We will however want the property for families that are also secure MACs, in which case it becomes non-trivial.

PP-MAC  $<$  PRF. We claim that a privacy-preserving MAC (PP-MAC) is strictly weaker than a PRF, in the sense that any PRF is (a secure MAC [5, 7] and) privacy-preserving, but not vice-versa. This means that when (below) we assume that a compression function  $h$  is a PP-MAC, we are indeed assuming less of it than that it is a PRF. Let us now provide some details about the claims made above. First, the following is the formal statement corresponding to the claim that any PRF is privacy-preserving:

**Proposition 4.1** *Let  $f: \{0, 1\}^k \times \{0, 1\}^l \rightarrow \{0, 1\}^L$  be a family of functions, and  $A_{\text{ind}}$  an ind-adversary against it that makes at most  $q$  oracle queries and has time-complexity at most  $t$ . Then there is a prf-adversary  $A_{\text{prf}}$  against  $f$  such that  $\mathbf{Adv}_f^{\text{ind}}(A_{\text{ind}}) \leq 2 \cdot \mathbf{Adv}_f^{\text{prf}}(A_{\text{prf}})$ . Furthermore,  $A_{\text{prf}}$  makes at most  $q$  oracle queries and has time-complexity at most  $t$ . ■*

The proof is a simple exercise and is omitted. Next we explain why a PP-MAC need not be a PRF. The reason (or one reason) is that if the output of a family of functions has some structure, for example always ending in a 0 bit, it would disqualify the family as a PRF but need not preclude its being a PP-MAC. To make this more precise, let  $f: \{0, 1\}^k \times \{0, 1\}^l \rightarrow \{0, 1\}^L$  be a PP-MAC. Define  $g: \{0, 1\}^k \times \{0, 1\}^l \rightarrow \{0, 1\}^{L+1}$  by  $g(K, x) = f(K, x) \| 0$  for all  $K \in \{0, 1\}^k$  and  $x \in \{0, 1\}^l$ . Then  $g$  is also a PP-MAC, but is clearly not a PRF.

## 4.2 Results

The following implies that if  $h$  is a PP-MAC and  $F$  is cAU then their composition  $hF$  is a secure MAC.

**Lemma 4.2** *Let  $B = \{0, 1\}^b$ . Let  $h: \{0, 1\}^c \times B \rightarrow \{0, 1\}^c$  and  $F: \{0, 1\}^k \times D \rightarrow B$  be families of functions, and let  $hF: \{0, 1\}^{c+k} \times D \rightarrow \{0, 1\}^c$  be defined by*

$$hF(K_{\text{out}} \| K_{\text{in}}, M) = h(K_{\text{out}}, F(K_{\text{in}}, M))$$

for all  $K_{\text{out}} \in \{0, 1\}^c$ ,  $K_{\text{in}} \in \{0, 1\}^k$  and  $M \in D$ . Let  $A_{hF}$  be a mac-adversary against  $hF$  that makes at most  $q_{\text{mac}}$  mac queries and at most  $q_{\text{vf}}$  verification queries, with the messages in each of these queries being of length at most  $n$ . Suppose  $A_{hF}$  has time-complexity at most  $t$ . Let  $q = q_{\text{mac}} + q_{\text{vf}}$  and assume  $2 \leq q < 2^b$ . Then there exists a mac-adversary  $A_1$  against  $h$ , an ind-adversary  $A_2$  against  $h$ , and an au-adversary  $A_F$  against  $F$  such that

$$\mathbf{Adv}_{hF}^{\text{mac}}(A_{hF}) \leq \mathbf{Adv}_h^{\text{mac}}(A_1) + \mathbf{Adv}_h^{\text{ind}}(A_2) + \binom{q}{2} \cdot \mathbf{Adv}_F^{\text{au}}(A_F). \quad (18)$$

Furthermore,  $A_1$  makes at most  $q_{\text{mac}}$  mac queries and at most  $q_{\text{vf}}$  verification queries and has time-complexity at most  $t$ ;  $A_2$  makes at most  $q$  oracle queries and has time-complexity at most  $t$ ; and  $A_F$  outputs messages of length at most  $n$ , makes 2 oracle queries, and has time-complexity  $O(T_F(n))$ , where  $T_F(n)$  is the time to compute  $F$  on an  $n$ -bit input. ■

The proof is in Section 4.3. As a corollary we have the following, which says that if  $h$  is a PP-MAC and  $h^*$  is cAU then GNMAC is a secure MAC.

**Theorem 4.3** Assume  $b \geq c$  and let  $B = \{0, 1\}^b$ . Let  $h: \{0, 1\}^c \times B \rightarrow \{0, 1\}^c$  be a family of functions and let  $\text{fpad} \in \{0, 1\}^{b-c}$  be a fixed padding string. Let  $\text{GNMAC}: \{0, 1\}^{2c} \times B^+ \rightarrow \{0, 1\}^c$  be defined by

$$\text{GNMAC}(K_{\text{out}} \| K_{\text{in}}, M) = h(K_{\text{out}}, h^*(K_{\text{in}}, M) \| \text{fpad})$$

for all  $K_{\text{out}}, K_{\text{in}} \in \{0, 1\}^c$  and  $M \in B^+$ . Let  $A_{\text{GNMAC}}$  be a mac-adversary against GNMAC that makes at most  $q_{\text{mac}}$  mac queries and at most  $q_{\text{vf}}$  verification queries, with the messages in each of these queries being of at most  $m$  blocks. Suppose  $A_{\text{GNMAC}}$  has time-complexity at most  $t$ . Let  $q = q_{\text{mac}} + q_{\text{vf}}$  and assume  $2 \leq q < 2^b$ . Then there exists a mac-adversary  $A_1$  against  $h$ , an ind-adversary  $A_2$  against  $h$ , and an au-adversary  $A^*$  against  $h^*$  such that

$$\mathbf{Adv}_{\text{GNMAC}}^{\text{mac}}(A_{\text{GNMAC}}) \leq \mathbf{Adv}_h^{\text{mac}}(A_1) + \mathbf{Adv}_h^{\text{ind}}(A_2) + \binom{q}{2} \cdot \mathbf{Adv}_{h^*}^{\text{au}}(A^*). \quad (19)$$

Furthermore,  $A_1$  makes at most  $q_{\text{mac}}$  mac queries and at most  $q_{\text{vf}}$  verification queries and has time-complexity at most  $t$ ;  $A_2$  makes at most  $q$  oracle queries and has time-complexity at most  $t$ ; and  $A^*$  outputs messages of at most  $m$  blocks, makes 2 oracle queries, and has time-complexity  $O(mT_h)$ , where  $T_h$  is the time for one computation of  $h$ . ■

We remark that Lemma 4.2 can be extended to show that  $hF$  is not only a MAC but itself privacy-preserving. (This assumes  $h$  is privacy-preserving and  $F$  is cAU. We do not prove this here.) This implies that GNMAC is privacy-preserving as long as  $h$  is privacy-preserving and  $h^*$  is cAU. This is potentially useful because it may be possible to show that a PP-MAC is sufficient to ensure security in some applications where HMAC is currently assumed to be a PRF. (But we do not know of such an application at this time.)

### 4.3 Proof of Lemma 4.2

A mac-adversary against  $h$  gets a mac oracle  $h(K_{\text{out}}, \cdot)$  and corresponding verification oracle  $\text{VF}_h(K_{\text{out}}, \cdot, \cdot)$ . By itself picking key  $K_{\text{in}}$  and invoking these oracles, it can easily simulate the mac oracle  $h(K_{\text{out}}, F(K_{\text{in}}, \cdot))$  and verification oracle  $\text{VF}_h(K_{\text{out}}, F(K_{\text{in}}, \cdot), \cdot)$  required by a mac-adversary against  $hF$ . This leads to the following natural construction of  $A_1$ :

$$\mathbf{Adversary} \ A_1^{h(K_{\text{out}}, \cdot), \text{VF}_h(K_{\text{out}}, \cdot, \cdot)}$$

<p><b>Game <math>G1</math></b></p> <p style="text-align: center;"><math>K_{\text{out}} \xleftarrow{\\$} \{0, 1\}^c ; K_{\text{in}} \xleftarrow{\\$} \{0, 1\}^k ; i \leftarrow 0</math></p> <p><b>On mac query <math>M</math> or verification query <math>M, T</math>:</b></p> <p style="padding-left: 20px;"><math>i \leftarrow i + 1 ; M_i \leftarrow M ; y_i \leftarrow F(K_{\text{in}}, M)</math></p> <p style="padding-left: 20px;">If <math>\exists j &lt; i : y_i = y_j</math> and <math>M_i \neq M_j</math> then <math>\text{bad} \leftarrow \text{true}</math></p> <p style="padding-left: 20px;">If <math>\exists j &lt; i : M_i = M_j</math> then <math>T_i \leftarrow T_j</math></p> <p style="padding-left: 20px;">Else <math>T_i \leftarrow h(K_{\text{out}}, y_i)</math></p> <p style="padding-left: 20px;">If mac query <math>M</math> then reply <math>T_i</math></p> <p style="padding-left: 20px;">If verification query <math>M, T</math> then</p> <p style="padding-left: 40px;">If <math>T_i = T</math> then reply 1 else reply 0</p> <p><b>Game <math>G2</math></b></p> <p style="text-align: center;"><math>K_{\text{out}} \xleftarrow{\\$} \{0, 1\}^c ; K_{\text{in}} \xleftarrow{\\$} \{0, 1\}^k ; i \leftarrow 0</math></p> <p><b>On mac query <math>M</math> or verification query <math>M, T</math>:</b></p> <p style="padding-left: 20px;"><math>i \leftarrow i + 1 ; M_i \leftarrow M ; y_i \leftarrow F(K_{\text{in}}, M)</math></p> <p style="padding-left: 20px;">If <math>\exists j &lt; i : y_i = y_j</math> and <math>M_i \neq M_j</math> then <math>\text{bad} \leftarrow \text{true}</math></p> <p style="padding-left: 20px;">If <math>\exists j &lt; i : M_i = M_j</math> then <math>T_i \leftarrow T_j</math></p> <p style="padding-left: 20px;">Else <math>T_i \leftarrow h(K_{\text{out}}, \langle i \rangle_b)</math></p> <p style="padding-left: 20px;">If mac query <math>M</math> then reply <math>T_i</math></p> <p style="padding-left: 20px;">If verification query <math>M, T</math> then</p> <p style="padding-left: 40px;">If <math>T_i = T</math> then reply 1 else reply 0</p>
--

Figure 6: Games for the proof of Lemma 4.2.

$K_{\text{in}} \xleftarrow{\$} \{0, 1\}^k ; i \leftarrow 0$

Run  $A_{hF}$ , replying to its oracle queries as follows:

**On mac query  $M$  or verification query  $M, T$ :**

$i \leftarrow i + 1 ; M_i \leftarrow M ; y_i \leftarrow F(K_{\text{in}}, M)$

If mac query  $M$  then reply  $h(K_{\text{out}}, y_i)$  to  $A_{hF}$

If verification query  $M, T$  then reply  $\text{VF}_h(K_{\text{out}}, y_i, T)$  to  $A_{hF}$

Consider the experiment defining the mac-advantage of  $A_1$ . Namely, choose  $K_{\text{out}} \xleftarrow{\$} \{0, 1\}^c$  and run  $A_1$  with oracles  $h(K_{\text{out}}, \cdot)$  and  $\text{VF}_h(K_{\text{out}}, \cdot, \cdot)$ . Let  $\text{Coll}$  (for ‘‘collision’’) be the event that there exist  $j, l$  such that  $y_j = y_l$  but  $M_j \neq M_l$ . Then

$$\begin{aligned}
\text{Adv}_h^{\text{mac}}(A_1) &= \Pr[A_1 \text{ forges}] \\
&\geq \Pr[A_{hF} \text{ forges} \wedge \overline{\text{Coll}}] \\
&\geq \Pr[A_{hF} \text{ forges}] - \Pr[\text{Coll}] \\
&= \text{Adv}_{hF}^{\text{mac}}(A_{hF}) - \Pr[\text{Coll}].
\end{aligned} \tag{20}$$

The rest of the proof is devoted to upper bounding  $\Pr[\text{Coll}]$ . Consider the games  $G1, G2$  of Figure 6, where we denote by  $\langle i \rangle_b$  the representation of integer  $i$  as a  $b$ -bit string. (The assumption  $q < 2^b$  made in the lemma statement means that we can always represent  $i$  this way in  $G2$ .) These games differ only in the manner in which tag  $T_i$  is computed. In  $G1$ , it is equal to  $hF(K_{\text{out}} \| K_{\text{in}}, M_i)$ .

<p><b>Adversary <math>A_2^{g(\cdot, \cdot)}</math></b>  <math>K_{\text{in}} \xleftarrow{\\$} \{0, 1\}^k ; i \leftarrow 0</math>  Run <math>A_{hF}</math>, replying to its oracle queries as follows:  <b>On mac query <math>M</math> or verification query <math>M, T</math>:</b>  <math>i \leftarrow i + 1 ; M_i \leftarrow M ; y_i \leftarrow F(K_{\text{in}}, M)</math>  If <math>\exists j &lt; i : y_i = y_j</math> and <math>M_i \neq M_j</math> then return 1  If <math>\exists j &lt; i : M_i = M_j</math> then <math>T_i \leftarrow T_j</math>  Else <math>T_i \leftarrow g(\langle i \rangle_b, y_i)</math>  If mac query <math>M</math> then reply <math>T_i</math> to <math>A_{hF}</math>  If verification query <math>M, T</math> then  If <math>T_i = T</math> then reply 1 to <math>A_{hF}</math> else reply 0 to <math>A_{hF}</math>  Return 0</p> <p><b>Adversary <math>A'_F</math></b>  <math>K_{\text{out}} \xleftarrow{\\$} \{0, 1\}^c ; i \leftarrow 0</math>  Run <math>A_{hF}</math>, replying to its oracle queries as follows:  <b>On mac query <math>M</math> or verification query <math>M, T</math>:</b>  <math>i \leftarrow i + 1 ; M_i \leftarrow M</math>  If <math>\exists j &lt; i : M_i = M_j</math> then <math>T_i \leftarrow T_j</math>  Else <math>T_i \leftarrow h(K_{\text{out}}, \langle i \rangle_b)</math>  If mac query <math>M</math> then reply <math>T_i</math> to <math>A_{hF}</math>  If verification query <math>M, T</math> then  If <math>T_i = T</math> then reply 1 to <math>A_{hF}</math> else reply 0 to <math>A_{hF}</math>  Pick <math>i, j</math> at random subject to <math>1 \leq i &lt; j \leq q</math>  Return <math>M_i, M_j</math></p>
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Figure 7: Ind-adversary  $A_2$  against  $h$ , taking an oracle  $g$  that on input a pair of  $b$ -bit strings returns a  $c$ -bit string, and au-adversary  $A'_F$  against  $F$ , that outputs a pair of strings in  $D$ .

In  $G_2$ , however, it is the result of applying  $h(K_{\text{out}}, \cdot)$  to the current value of the counter  $i$ , and as such does not depend on  $K_{\text{in}}$ . Now note that

$$\begin{aligned} \Pr[\text{Coll}] &= \Pr[A_{hF}^{G_1^1} \text{ sets bad}] \\ &= \left( \Pr[A_{hF}^{G_1^1} \text{ sets bad}] - \Pr[A_{hF}^{G_2^2} \text{ sets bad}] \right) + \Pr[A_{hF}^{G_2^2} \text{ sets bad}]. \end{aligned} \quad (21)$$

Now consider the adversaries  $A_2, A'_F$  described in Figure 7. We claim that

$$\Pr[A_{hF}^{G_1^1} \text{ sets bad}] - \Pr[A_{hF}^{G_2^2} \text{ sets bad}] \leq \mathbf{Adv}_h^{\text{ind}}(A_2) \quad (22)$$

$$\Pr[A_{hF}^{G_2^2} \text{ sets bad}] \leq \binom{q}{2} \cdot \mathbf{Adv}_F^{\text{au}}(A'_F). \quad (23)$$

Adversary  $A'_F$ , however, has time-complexity  $t$  (and outputs messages of at most  $n$  bits). A standard “coin-fixing” argument will be used to derive from  $A'_F$  an au-adversary  $A_F$  that has time-complexity  $O(T_F(n))$  (and also outputs messages of  $n$  bits) such that

$$\mathbf{Adv}_F^{\text{au}}(A'_F) \leq \mathbf{Adv}_F^{\text{au}}(A_F). \quad (24)$$

Combining (24), (23), (22), (21) and (20) yields (18) and completes the proof of Lemma 4.2. It remains to prove (22), (23) and (24). We begin with the first of these.

Recall that an ind-adversary against  $h$  is given an oracle that takes as input a pair of  $b$ -bit strings  $x_0, x_1$ . We are denoting this oracle by  $g$ . Now it is easy to see that

$$\begin{aligned}\Pr \left[ A_2^{\text{Right}} \Rightarrow 1 \right] &= \Pr \left[ A_{hF}^{G1} \text{ sets bad} \right] \\ \Pr \left[ A_2^{\text{Left}} \Rightarrow 1 \right] &= \Pr \left[ A_{hF}^{G2} \text{ sets bad} \right],\end{aligned}$$

which implies (22). However, there is one important thing we still need to verify, namely that  $A_2$  is legitimate. So consider the sequence  $(x_1, y_1), (x_2, y_2), \dots$  of oracle queries it makes. The left halves  $x_1, x_2, \dots$  are values of the counter  $i$  in different loop iterations and are thus strictly increasing (although not necessarily successive) and in particular different. On the other hand the right half values  $y_1, y_2, \dots$  are distinct because as soon as  $y_i = y_j$  for some  $j < i$ , adversary  $A_2$  halts (and returns 1), never making an oracle query whose right half is  $y_i$ .

Next we turn to  $A'_F$ . In order for the messages  $M_i, M_j$  it returns to always be defined, we assume wlog that  $A_{hF}$  always makes exactly, rather than at most,  $q_{\text{mac}}$  mac queries and exactly, rather than at most,  $q_{\text{vf}}$  verification queries. Intuitively, (23) is true because  $i, j$  are chosen at random after the execution of  $A_{hF}$  is complete, so  $A_{hF}$  has no information about them. This can be made rigorous just as in the proof of Lemma 3.2, and the details follow. Consider the experiment defining the au-advantage of  $A'_F$ . In this experiment, consider the following events defined for  $1 \leq \alpha < \beta \leq q$ :

$$\begin{aligned}C_{\alpha, \beta} &: F(K_{\text{in}}, M_\alpha) = F(K_{\text{in}}, M_\beta) \wedge M_\alpha \neq M_\beta \\ C &: \bigvee_{1 \leq \alpha < \beta \leq q} C_{\alpha, \beta}.\end{aligned}$$

Notice that the events “ $C_{\alpha, \beta} \wedge (i, j) = (\alpha, \beta)$ ” ( $1 \leq \alpha < \beta \leq q$ ) are disjoint. (Even though the events  $C_{\alpha, \beta}$  for  $1 \leq \alpha < \beta \leq q$  are not.) Thus:

$$\begin{aligned}\text{Adv}_F^{\text{au}}(A'_F) &= \Pr \left[ \bigvee_{1 \leq \alpha < \beta \leq q} (C_{\alpha, \beta} \wedge (i, j) = (\alpha, \beta)) \right] \\ &= \sum_{1 \leq \alpha < \beta \leq q} \Pr [C_{\alpha, \beta} \wedge (i, j) = (\alpha, \beta)].\end{aligned}$$

Since  $i, j$  are chosen at random after the execution of  $A_{hF}$  is complete, the events “ $(i, j) = (\alpha, \beta)$ ” and  $C_{\alpha, \beta}$  are independent. Thus the above equals

$$\begin{aligned}\sum_{1 \leq \alpha < \beta \leq q} \Pr [C_{\alpha, \beta}] \cdot \Pr [(i, j) = (\alpha, \beta)] &= \sum_{1 \leq \alpha < \beta \leq q} \Pr [C_{\alpha, \beta}] \cdot \frac{1}{\binom{q}{2}} \\ &= \frac{1}{\binom{q}{2}} \cdot \sum_{1 \leq \alpha < \beta \leq q} \Pr [C_{\alpha, \beta}] \\ &\geq \frac{1}{\binom{q}{2}} \cdot \Pr [C].\end{aligned}$$

The proof of (23) is concluded by noting that  $\Pr [C]$  equals  $\Pr [G2 \text{ sets bad}]$ .

Finally, au-adversary  $A_F$  of time-complexity  $O(T_F(n))$  satisfying (24) can be obtained from  $A'_F$  just as in the proof of Lemma 3.2.

## 5 Security of HMAC

In this section we show how our security results about NMAC lift to corresponding ones about HMAC. We begin by recalling the observation of [2] as to how this works for HMAC with two independent keys, and then discuss how to extend this to the single-keyed version of HMAC.

THE CONSTRUCTS. Let  $h: \{0, 1\}^c \times \{0, 1\}^b \rightarrow \{0, 1\}^c$  as usual denote the compression function. Let  $\text{pad}$  be the padding function as described in Section 3, so that  $s^* = s \parallel \text{pad}(|s|) \in B^+$  for any string  $s$ . Recall that the cryptographic hash function  $H$  associated to  $h$  is defined by  $H(M) = h^*(\text{IV}, M^*)$ , where  $\text{IV}$  is a  $c$ -bit initial vector that is fixed as part of the description of  $H$  and  $M$  is a string of any length up to some maximum length that is related to  $\text{pad}$ . (This maximum length is  $2^{64}$  for current hash functions.) Then  $\text{HMAC}(K_{\text{out}} \parallel K_{\text{in}}, M) = H(K_{\text{out}} \parallel H(K_{\text{in}} \parallel M))$ , where  $K_{\text{out}}, K_{\text{in}} \in \{0, 1\}^b$ . If we write this out in terms of  $h^*$  alone we get

$$\text{HMAC}(K_{\text{out}} \parallel K_{\text{in}}, M) = h^*(\text{IV}, K_{\text{out}} \parallel h^*(\text{IV}, K_{\text{in}} \parallel M \parallel \text{pad}(b + |M|)) \parallel \text{pad}(b + c)).$$

As with NMAC, the details of the padding conventions are not important to the security of HMAC as a PRF, and we will consider the more general construct GHMAC:  $\{0, 1\}^{2b} \times B^+ \rightarrow \{0, 1\}^c$  defined by

$$\text{GHMAC}(K_{\text{out}} \parallel K_{\text{in}}, M) = h^*(\text{IV}, K_{\text{out}} \parallel h^*(\text{IV}, K_{\text{in}} \parallel M) \parallel \text{fpad}) \quad (25)$$

for all  $K_{\text{out}}, K_{\text{in}} \in \{0, 1\}^b$  and all  $M \in B^+$ . Here  $\text{IV} \in \{0, 1\}^c$  and  $\text{fpad} \in \{0, 1\}^{b-c}$  are fixed strings. HMAC is a special case, via  $\text{HMAC}(K_{\text{out}} \parallel K_{\text{in}}, M) = \text{GHMAC}(M \parallel \text{pad}(b + |M|))$  with  $\text{fpad} = \text{pad}(b + c)$ , and thus security properties of GHMAC (as a PRF or MAC) are inherited by HMAC, allowing us to focus on the former.

THE DUAL FAMILY. To state the results, it is useful to define  $\bar{h}: \{0, 1\}^b \times \{0, 1\}^c \rightarrow \{0, 1\}^c$ , the dual of family  $h$ , by  $\bar{h}(x, y) = h(y, x)$ . The assumption that  $h$  is a PRF when keyed by its data input is formally captured by the assumption that  $\bar{h}$  is a PRF.

### 5.1 Security of GHMAC

Let  $K'_{\text{out}} = h(\text{IV}, K_{\text{out}})$  and  $K'_{\text{in}} = h(\text{IV}, K_{\text{in}})$ . The observation of [2] is that

$$\begin{aligned} \text{GHMAC}(K_{\text{out}} \parallel K_{\text{in}}, M) &= h(K'_{\text{out}}, h^*(K'_{\text{in}}, M) \parallel \text{fpad}) \\ &= \text{GNMAC}(K'_{\text{out}} \parallel K'_{\text{in}}, M). \end{aligned} \quad (26)$$

This effectively reduces the security of GHMAC to GNMAC. Namely, if  $\bar{h}$  is a PRF and  $K_{\text{out}}, K_{\text{in}}$  are chosen at random, then  $K'_{\text{out}}, K'_{\text{in}}$  will be computationally close to random. Now (26) implies that if GNMAC is a PRF then so is GHMAC. The formal statement follows.

**Lemma 5.1** *Assume  $b \geq c$  and let  $B = \{0, 1\}^b$ . Let  $h: \{0, 1\}^c \times B \rightarrow \{0, 1\}^c$  be a family of functions. Let  $\text{fpad} \in \{0, 1\}^{b-c}$  be a fixed padding string and  $\text{IV} \in \{0, 1\}^c$  a fixed initial vector. Let  $\text{GHMAC}: \{0, 1\}^{2b} \times B^+ \rightarrow \{0, 1\}^c$  be defined by (25) above. Let  $A$  be a prf-adversary against GHMAC that has time-complexity at most  $t$ . Then there exists a prf-adversary  $A_{\bar{h}}$  against  $\bar{h}$  such that*

$$\text{Adv}_{\text{GHMAC}}^{\text{prf}}(A) \leq 2 \cdot \text{Adv}_{\bar{h}}^{\text{prf}}(A_{\bar{h}}) + \text{Adv}_{\text{GNMAC}}^{\text{prf}}(A).$$

Furthermore,  $A_{\bar{h}}$  makes only 1 oracle query and has time-complexity at most  $t$ .  $\blacksquare$

The proof is simple and is omitted. Combining this with Theorem 3.3 yields the result that GHMAC is a PRF assuming  $h, \bar{h}$  are both PRFs. Note that the PRF assumption on  $\bar{h}$  is mild because  $A_{\bar{h}}$  makes only one oracle query.

## 5.2 Single-keyed HMAC

HMAC, GHMAC as described and analyzed above use two keys that are assumed to be chosen independently at random. However, HMAC is in fact usually implemented with these keys derived from a single  $b$ -bit key. Here we provide the first security proofs for the actually-implemented single-key version of HMAC.

Specifically, let  $\text{opad}, \text{ipad} \in \{0, 1\}^b$  be distinct, fixed and known constants. (Their particular values can be found in [2] and are not important here.) Then the single-key version of HMAC is defined by  $\text{HMAC-1}(K, M) = \text{HMAC}(K \oplus \text{opad} \| K \oplus \text{ipad}, M)$ . As before, we look at this as a special case of a more general construct, namely GHMAC-1:  $\{0, 1\}^b \times B^+ \rightarrow \{0, 1\}^c$ , defined by

$$\text{GHMAC-1}(K, M) = \text{GHMAC}(K \oplus \text{opad} \| K \oplus \text{ipad}, M) \quad (27)$$

for all  $K \in \{0, 1\}^b$  and all  $M \in B^+$ . We now focus on GHMAC-1. We will show that GHMAC-1 inherits the security of GNMAC as long as  $\bar{h}$  is a PRF against an appropriate class of related key attacks. In such an attack, the adversary can obtain input-output examples of  $h$  under keys related to the target key. Let us recall the formal definitions following [6].

A related-key attack on a family of functions  $\bar{h}: \{0, 1\}^b \times \{0, 1\}^c \rightarrow \{0, 1\}^c$  is parameterized by a set  $\Phi \subseteq \text{Maps}(\{0, 1\}^b, \{0, 1\}^b)$  of *key-derivation* functions. We define the function  $\text{RK}: \Phi \times \{0, 1\}^b \rightarrow \{0, 1\}^b$  by  $\text{RK}(\phi, K) = \phi(K)$  for all  $\phi \in \Phi$  and  $K \in \{0, 1\}^b$ . A rka-adversary  $A$  may make an oracle query of the form  $\phi, x$  where  $\phi \in \Phi$  and  $x \in \{0, 1\}^c$ . Its rka-advantage is defined by

$$\text{Adv}_{h, \Phi}^{\text{rka}}(A) = \Pr \left[ A^{\bar{h}(\text{RK}(\cdot, K), \cdot)} \Rightarrow 1 \right] - \Pr \left[ A^{G(\text{RK}(\cdot, K), \cdot)} \Rightarrow 1 \right].$$

In the first case,  $K$  is chosen at random from  $\{0, 1\}^b$  and the reply to query  $\phi, x$  of  $A$  is  $\bar{h}(\phi(K), x)$ . In the second case,  $G \stackrel{\$}{\leftarrow} \text{Maps}(\{0, 1\}^b \times \{0, 1\}^c, \{0, 1\}^c)$  and  $K \stackrel{\$}{\leftarrow} \{0, 1\}^b$ , and the reply to query  $\phi, x$  of  $A$  is  $G(\phi(K), x)$ . For any string  $s \in \{0, 1\}^b$  let  $\Delta_s: \{0, 1\}^b \rightarrow \{0, 1\}^b$  be defined by  $\Delta_s(K) = K \oplus s$ .

**Lemma 5.2** *Assume  $b \geq c$  and let  $B = \{0, 1\}^b$ . Let  $h: \{0, 1\}^c \times B \rightarrow \{0, 1\}^c$  be a family of functions. Let  $\text{fpad} \in \{0, 1\}^{b-c}$  be a fixed padding string,  $\text{IV} \in \{0, 1\}^c$  a fixed initial vector, and  $\text{opad}, \text{ipad} \in \{0, 1\}^b$  fixed, distinct strings. Let GHMAC-1:  $\{0, 1\}^b \times B^+ \rightarrow \{0, 1\}^c$  be defined by (27) above. Let  $\Phi = \{\Delta_{\text{opad}}, \Delta_{\text{ipad}}\}$ . Let  $A$  be a prf-adversary against GHMAC-1 that has time-complexity at most  $t$ . Then there exists a rka-adversary  $A_{\bar{h}}$  against  $\bar{h}$  such that*

$$\text{Adv}_{\text{GHMAC-1}}^{\text{prf}}(A) \leq \text{Adv}_{h, \Phi}^{\text{rka}}(A_{\bar{h}}) + \text{Adv}_{\text{GNMAC}}^{\text{prf}}(A).$$

Furthermore,  $A_{\bar{h}}$  makes 2 oracle queries and has time-complexity at most  $t$ .  $\blacksquare$

The proof is simple and is omitted. Combining this with Theorem 3.3 yields the result that GHMAC-1 is a PRF assuming  $h$  is a PRF and  $\bar{h}$  is a PRF under  $\Phi$ -restricted related-key attacks, where  $\Phi$  is as in Lemma 5.2. We remark that  $\Phi$  is a small set of simple functions, which is important because it is shown in [6] that if  $\Phi$  is too rich then no family can be a PRF under  $\Phi$ -restricted related-key attacks. Furthermore, the assumption on  $\bar{h}$  is rendered milder by the fact that  $A_{\bar{h}}$  makes only two oracle queries.

## 5.3 Lifting the results of Section 4

The procedure above to lift the NMAC results of Section 3 to HMAC applies also to lift the results of Section 4 to HMAC. Specifically, if  $h$  is a PP-MAC,  $h^*$  is AU and  $\bar{h}$  is a PRF then GHMAC is a (privacy-preserving) MAC. Also if  $h$  is a PP-MAC,  $h^*$  is AU and  $\bar{h}$  is a PRF under  $\Phi$ -restricted related-key attacks, with  $\Phi$  as in Lemma 5.2, then GHMAC-1 is a (privacy-preserving) MAC. Note that the assumption on  $\bar{h}$  continues to be that it is a PRF or PRF against  $\Phi$ -restricted related-key



attacks. (Namely, this has not been reduced to its being a PP-MAC.) This assumption is however mild in this context since (as indicated by Lemmas 5.2 and 5.1) it need only hold with respect to adversaries that make very few queries.

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## A Attacking weak collision-resistance

Recall that  $H$  represents the cryptographic hash function (eg. MD5, SHA-1) while  $H^*$  is the extended hash function, which is the hash function with the initial vector made explicit as an (additional) first input. Let us use the term *general collision-finding attack* to refer to an attack that finds collisions in  $H^*(IV, \cdot)$  for an arbitrary but given  $IV$ . As we discussed in Section 1, it was noted in [2, 19] that any general collision-finding attack can be used to compromise the weak collision-resistance (WCR) of  $H$ . (And since the known collision-finding attacks on MD5 and SHA-1 [31, 30] do extend to general ones, the WCR of these functions is no more than their CR.) Here we recall the argument that shows this. It is a simple extension attack, and works as follows.

To compromise WCR of  $H$ , an attacker given an oracle for  $H^*(K, \cdot)$  under a hidden key  $K$  must output distinct  $M_1, M_2$  such that  $H^*(K, M_1) = H^*(K, M_2)$ . Our attacker picks some string  $x$  and calls its oracle to obtain  $IV = H^*(K, x)$ . Then it runs the given general collision-finding attack on input  $IV$  to obtain a collision  $X_1, X_2$  for  $H^*(IV, \cdot)$ . (That is,  $X_1, X_2$  are distinct strings such that  $H^*(IV, X_1) = H^*(IV, X_2)$ .) Now let  $M_1 = x \parallel \text{pad}(|x|) \parallel X_1 \parallel \text{pad}(|X_1|)$  and  $M_2 = x \parallel \text{pad}(|x|) \parallel X_2 \parallel \text{pad}(|X_2|)$ . (Here  $\text{pad}(n)$  is a padding string that when appended to a string of length  $n$  results in a string whose length is a positive multiple of  $b$  bits where  $b$  is the block-length of the underlying compression function. The function  $\text{pad}$  is part of the description of the cryptographic hash function.) Then it follows that  $H^*(K, M_1) = H^*(K, M_2)$ .

## B The reduction-from-pf-PRF proof

We sketch how one can obtain the result that  $h$  a PRF implies  $h^*$  is cAU by using the result of [3] that says that  $h$  a PRF implies  $h^*$  is a pf-PRF (a PRF as long as no query of the adversary is a prefix of another query). We then compare this with the direct proof given in Section 3.3 .

THE RESULT OF [3]. A pf-adversary is said to be prefix-free if no query it makes is a prefix of another. The result of [3] is that if  $D$  is a prefix-free pf-adversary against  $h^*$  that makes at most  $q$  queries, each of at most  $m$  blocks, then there is a pf-adversary  $A$  against  $h$  such that

$$\mathbf{Adv}_{h^*}^{\text{prf}}(D) \leq qm \cdot \mathbf{Adv}_h^{\text{prf}}(A) \quad (28)$$

and  $A$  has about the same time-complexity as  $D$ . (We remark that there is a typo in the statement of Theorem 3.1 of the proceedings version of [3] in this regard: the factor  $q$  is missing from the bound. This is however corrected in the on-line version of the paper.)

THE REDUCTION-FROM-PF-PRF. The result obtained via the reduction-from-pf-PRF proof will be slightly worse than the one of Lemma 3.1. Namely we claim that, under the same conditions as in that lemma, (2) is replaced by

$$\mathbf{Adv}_{h^*}^{\text{au}}(A^*) \leq 2 \cdot [\max(n_1, n_2) + 1] \cdot \mathbf{Adv}_h^{\text{prf}}(A) + \frac{1}{2^c}, \quad (29)$$

and the time-complexity of  $A$  increases from  $(n_1 + n_2)$  computations of  $h$  to  $2 \max(n_1, n_2)$  computations of  $h$ . For some intuition about the proof, imagine  $h^*$  were a PRF. (It is not.) Then  $\text{Coll}_{h^*}(M_1, M_2)$  would be about the same as the probability that  $f(M_1) = f(M_2)$  for a random function  $f$ , because otherwise the pf-adversary who queried its oracle with  $M_1, M_2$  and accepted iff the replies were the same would be successful. With  $h^*$  in fact only a pf-PRF, the difficulty is the case that  $M_1 \subseteq M_2$ , which renders the adversary just described not prefix-free. There is a simple (and well-known) observation—we will call it the extension trick—to get around this. Namely, assuming wlog  $M_1 \neq M_2$  and  $\|M_1\|_b \leq \|M_2\|_b$ , let  $x \in B$  be a block different from  $M_2[\|M_1\|_b + 1]$ , and let  $M'_1 = M_1 \|x$  and  $M'_2 = M_2 \|x$ . Then  $\text{Coll}_{h^*}(M_1, M_2) \leq \text{Coll}_{h^*}(M'_1, M'_2)$  but  $M_1$  is not a prefix of  $M_2$ . This leads to the prefix-free pf-adversary against  $h^*$  below:

**Adversary  $D^f$**

$(M_1, M_2) \xleftarrow{\$} A^*$

If  $M_1 \subseteq M_2$  then  $x \xleftarrow{\$} B \setminus \{M_2[\|M_1\|_b + 1]\}$ ;  $M'_1 \leftarrow M_1 \|x$ ;  $M'_2 \leftarrow M_2 \|x$

Else  $M'_1 \leftarrow M_1$ ;  $M'_2 \leftarrow M_2$

If  $f(M'_1) = f(M'_2)$  then return 1 else return 0

Here  $f: B^+ \rightarrow \{0, 1\}^c$  and we assume wlog that  $M_1, M_2 \in B^+$  are distinct messages with  $\|M_1\|_b \leq \|M_2\|_b$ . Now the result of [3] gives us a pf-adversary  $A$  against  $h$  such that (28) holds. Thus:

$$\begin{aligned} \mathbf{Adv}_{h^*}^{\text{au}}(A^*) - 2^{-c} &\leq \mathbf{Adv}_{h^*}^{\text{prf}}(D) \\ &\leq 2 \cdot [\max(n_1, n_2) + 1] \cdot \mathbf{Adv}_h^{\text{prf}}(A) + 2^{-c}. \end{aligned}$$

Re-arranging terms yields (29).

The time-complexity of  $A$  as per [3] is (essentially) the time-complexity  $t$  of  $A^*$ , rather than being a small quantity independent of  $t$  as in Lemma 3.1. It is not clear whether or not the coin-fixing argument of our proof of Section 3.3 can be applied to  $A$  to reduce this time-complexity. (One would have to enter into the details of the proof of [3] to check.) However, instead, we can first modify  $A^*$  to an adversary that has embedded in its code a pair  $M_1, M_2 \in B^+$  of distinct messages that

maximize  $\text{Coll}_{h^*}(M_1, M_2)$ . It just outputs these messages and halts. We then apply the argument above to this modified  $A^*$ , and now the  $A$  we will have time-complexity that of  $2 \max(n_1, n_2)$  computations of  $h$  plus minor overhead.

COMPARISONS. For the case that  $M_1 \subseteq M_2$ , our direct proof (meaning the one of Section 3.3 ) uses a different (and novel) idea as opposed to the extension trick, which leads to a factor of only  $n_1 + 1$  in the bound (Claim 3.6) in this case, as opposed to the  $2[\max(n_1, n_2) + 1]$  factor obtained via the reduction-from-pf-PRF proof. The difference can be significant in the case that  $M_2$  is long and  $M_1$  is short. In the case  $M_1 \not\subseteq M_2$  our direct proof relies heavily on ideas of [3], but avoids the intermediate reduction to the multi-oracle model they use and exploits the non-adaptive nature of the setting to improve the factor in the bound from  $2 \max(n_1, n_2)$  to  $n_1 + n_2$ . We clarify that in this case we do not think the improvement is significant in practice, but we think it is aesthetically and theoretically nice to prove the “right” bounds, meaning those that are intuitively what should be there.

The reduction-from-pf-PRF proof is certainly simpler than our direct proof if one is willing to take the result of [3] as given. However, especially for a construct that is as widely standardized as HMAC, we think it is useful to have from-scratch proofs that are as easily verifiable as possible. If the measure of complexity is that of a from-scratch (i.e. self-contained) proof, we contend that our direct one (although not trivial) is simpler than that of [3]. (In particular because we do not use the multi-oracle model.) We remark that if a reader’s primary interest is the simplest possible self-contained proof regardless of the quality of the bound, the way to get it is to use our direct proof for the case  $M_1 \not\subseteq M_2$  and then the extension trick (as opposed to our direct proof) for the case  $M_1 \subseteq M_2$ .