Towards Provably Secure Group Key Agreement Building on Group Theory

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Abstract. Known proposals for key establishment schemes basing on combinatorial group theory are often formulated in a rather informal manner. Typically, issues like the choice of a session identifier and parallel protocol executions are not addressed, and no security proof in an established model is provided. Successful attacks against proposed parameter sets for braid groups further decreased the attractivity of combinatorial group theory as a candidate platform for cryptography.
We present a 2-round group key agreement protocol that can be proven secure in the random oracle model if a certain group-theoretical problem is hard. The security proof builds on a framework of Bresson et al., and explicitly addresses some issues concerning malicious insiders. While being designed as a tool for basing group key agreement on non-abelian groups, our framework also yields a provably secure 2-round group key agreement basing on a Computational Diffie-Hellman assumption.

Keywords: group key establishment, provable security, conjugacy problem, automorphisms of groups

1 Introduction

While in recent years cryptographic proposals building on combinatorial group theory, in particular braid groups, proliferated, repeated cryptanalytic success also diminished the initial optimism on the subject significantly. Dehornoy's paper [11] gives a good survey on the state of the subject, and evidently significant research is still needed to reach a definite conclusion on the cryptographic potential of braid groups. As far as key establishment is concerned, especially an idea of Anshel et al. [2, 1] received a lot of attention (e. g., [12, 15, 21]). Several further ideas for deriving a key establishment scheme from combinatorial group theory have been put forward, including the work in [16, 17, 20, 22, 19]. Unfortunately, to the best of our knowledge for none of these proposals a modern security analysis in an established cryptographic framework like [5, 3, 18, 4, 8, 9] has been carried out, and schemes are often analyzed and specified on a rather

informal level. In particular, topics like session identifers, parallel protocol executions or malicious insiders are usually not addressed at all. Thus, even if the non-trivial issue of finding "good" parameters can be settled, the situation is cryptographically rather unsatisfying.

In this contribution we want to show that once suitable group-theoretical tools are available, a 2-round group[1] key agreement scheme meeting modern cryptographic requirements can be derived. So far, the only known proposals for basing a group key establishment on non-abelian groups we are aware of are due to Lee et al. [17] and Grigoriev and Ponomarenko [14]. Unfortunately the former builds on ideas from a two-party protocol presented in [16], so Cheon and Jun's polynomial time solution to the Braid Diffie-Hellman conjugacy problem [10] impairs the attractivity of Lee et al.'s scheme. Grigoriev and Ponomarenko use a different approach to build a group key establishment. They build on ideas from [2, 1] and apply a 2-party protocol linear often in the number of participants.

While our contribution does not solve the problem of fixing concrete "non-abelian parameters", basing on the assumption that certain group-theoretical preconditions can be met, we offer a security proof in the random oracle model and using a framework along the lines of [8]. More specifically, we use the model variant in [6] which allows us to conveniently address some security issues concerning malicious insiders. In addition to this, instances of our protocol can be derived from a Computational Diffie-Hellman (CDH) assumption in a cyclic group.

Our approach can be seen in the spirit of [13], which takes a similar effort to identify requirements on finite non-abelian groups that allow to implement a provably IND-CCA secure public key encryption scheme: Certainly our contribution does not solve the problem of using combinatorial group theory as platform for a (group) key establishment, but it demonstrates that once certain group-theoretical tools can be provided, an efficient and provably secure key agreement in a modern sense can be derived. On the informal level, for the special case of the automorphisms in our scheme to be inner ones, the group-theoretical assumptions underlying [2, 1] look similar to the ones we need, so it appears plausible that finding instances of our proposal is of comparable difficulty as finding instances for the construction of Anshel et al.

2 Preliminaries

Giving a general introduction to the existing models for group key establishment is beyond the scope of this paper, and we refer to the standard reference [7] for this. Moreover, the background in group theory needed for describing our protocol is extremely modest. Hence, we restrict to recalling some details of the cryptographic proof model used for the security proof below. A more detailed discussion of this model can be found in [4, 8, 6].

[1] Unfortunately, in this paper the term *group* has to express two different meanings; here it refers to a set of principals.

2.1 Security model

Participants. A finite set $\mathcal{P}$ of probabilistic polynomial time (ppt) Turing machines U_i models the users that constitute the (potential) protocol participants. Each user $U_i \in \mathcal{P}$ may execute a polynomial number of protocol instances in parallel. We denote the instance $s \in \mathbb{N}$ of principal $U_i \in \mathcal{P}$ by Π_i^s. Each instance Π_i^s may be taken for a process executed by U_i and has assigned seven variables state_i^s, sid_i^s, pid_i^s, sk_i^s, term_i^s, used_i^s and acc_i^s:

used_i^s is initialized with false and set to true as soon as the instance begins a protocol run triggered by a call to the $\mathsf{Execute}$-oracle or a call to the Send-oracle (see below);
state_i^s stores the state information during the protocol execution;
term_i^s is initialized with false and set to true when the execution has terminated;
sid_i^s holds the (non-secret) session identifier that serves as identifier for the session key sk_i^s and is initialized with a distinguished NULL value;
pid_i^s stores the set of user identities that Π_i^s aims at establishing a key with—it also includes U_i itself;
acc_i^s is initialized with false and set to true if the protocol execution terminated successfully. I. e. the principal accepted the session key for use with users pid_i^s in session sid_i^s;
sk_i^s contains the session key after the execution is accepted by instance Π_i^s. Before acceptance, it stores a distinguished NULL value.

Initialization. In a one-time initialization phase, before the first execution of the key establishment protocol, for each user $U_i \in \mathcal{P}$ a secret key/public key pair (SK_i, PK_i) is generated. The secret key SK_i is only revealed to U_i, the corresponding public key PK_i is given to all users.[2]

Communication network. We assume arbitrary point-to-point connections to be available between the users. However, the connections are insecure and fully asynchronous, modeled by an active adversary with full control over the network (cf. the adversarial model below).

Adversarial model. The adversary $\mathcal{A}$ interacts with the user instances via a set of oracles $\mathsf{Execute}$, Send, Reveal, $\mathsf{Corrupt}$ and Test. We call the adversary *passive* if no access to the Send- and $\mathsf{Corrupt}$-oracle is granted.

$\mathsf{Execute}(\{U_1, U_2, \ldots, U_r\})$ This query executes a protocol run between unused instances Π_i^s of the specified users and returns a transcript of all messages sent during the protocol execution.
$\mathsf{Send}(U_i, s, M)$ This query sends the message M to instance Π_i^s and returns the reply generated by this instance. A special message $M = \{U_1, \ldots, U_r\}$ sent to an unused instance will set $\mathsf{pid}_i^s := M$, $\mathsf{used}_i^s := \mathsf{true}$ and provoke Π_i^s to begin with the protocol execution.

[2] We assume these keys to be generated and distributed honestly by a trusted party.

$\mathsf{Reveal}(U_i, s)$ returns the session key sk_i^s and the session identifier sid_i^s.

$\mathsf{Corrupt}(U_i)$ returns the long-term secret key SK_i that U_i holds. We will refer to a user U_i as *honest* if no query of the form $\mathsf{Corrupt}(U_i)$ was made.

$\mathsf{Test}(U_i, s)$ The adversary is allowed to use this query only once. Provided that $\mathsf{sk}_i^s \neq \text{NULL}$, a random bit b is drawn and depending on b with probability $1/2$ the session key sk_i^s and with probability $1/2$ a uniformly chosen random session key is returned. The adversary is allowed to query other oracles after its Test-query, but no query that would repeal the freshness of Π_i^s is allowed.

Correctness. To exclude "useless" protocols, we take a group key establishment protocol P for *correct* if in the presence of a passive adversary a single execution of P among arbitrary participants $U_1, \ldots, U_r$ involves r instances $\Pi_1^{s_1}, \ldots, \Pi_r^{s_r}$ and ensures that with overwhelming probability all instances accept a matching session key with a common partner identifier and a common and unique session identifier. More formally, with overwhelming probability the following conditions have to hold:

- $\mathsf{used}_1^{s_1} = \cdots = \mathsf{used}_r^{s_r} = \mathsf{true}$;
- $\mathsf{acc}_1^{s_1} = \cdots = \mathsf{acc}_r^{s_r} = \mathsf{true}$;
- $\mathsf{sk}_1^{s_1} = \cdots = \mathsf{sk}_r^{s_r}$;
- $\mathsf{sid}_1^{s_1} = \cdots = \mathsf{sid}_r^{s_r}$ globally unique;
- $\mathsf{pid}_1^{s_1} = \mathsf{pid}_2^{s_2} = \cdots = \mathsf{pid}_r^{s_r} = \{U_1, \ldots, U_r\}$.

Freshness. For the security definition, we have to specify which instances are fresh, i. e., hold a session key that should be unknown to the adversary. As a first step we define the notion of partnering.

Definition 1 (Partnering). *Two instances $\Pi_i^{s_i}$, $\Pi_j^{s_j}$ are* partnered *if $\mathsf{sid}_i^{s_i} = \mathsf{sid}_j^{s_j}$, $\mathsf{acc}_i^{s_i} = \mathsf{acc}_j^{s_j} = \mathsf{true}$ and both $U_j \in \mathsf{pid}_i^{s_i}$ and $U_i \in \mathsf{pid}_j^{s_j}$.*

Now the freshness of an instance is defined as follows.

Definition 2. *We call a user instance $\Pi_i^{s_i}$ that has accepted, i. e., $\mathsf{acc}_i^{s_i} = \mathsf{true}$,* fresh *if none of the following two conditions holds:*

- *For a $U_j \in \mathsf{pid}_i^{s_i}$ a $\mathsf{Corrupt}(U_j)$ query was executed before a query of the form $\mathsf{Send}(U_\ell, s_\ell, M)$ with $U_\ell \in \mathsf{pid}_i^s$ has taken place.*
- *A $\mathsf{Reveal}(U_j, s_j)$ was executed where $\Pi_i^{s_i}$ and $\Pi_j^{s_j}$ are partnered.*

We say that an adversary $\mathcal{A}$ was successful if $\mathcal{A}$, after interacting with the oracles including one $\mathsf{Test}(\Pi_i^{s_i})$ query for a fresh oracle $\Pi_i^{s_i}$, outputs a bit d and it holds that $d = b$ for the bit b used by the Test-oracle. We denote this probability by Succ and define $\mathcal{A}$'s advantage to be

$$\mathsf{Adv}_{\mathcal{A}} := |2 \cdot \mathsf{Succ} - 1|.$$

Definition 3 ((Basic) security). *We call the group key establishment protocol* P secure *if for all ppt adversaries $\mathcal{A}$ the function $\mathsf{Adv}_{\mathcal{A}} = \mathsf{Adv}_{\mathcal{A}}(k)$ is negligible in the security parameter k.*

The following extended security properties aim at avoiding further attacks imposed by malicious participants:

Definition 4 (Strong entity authentication). Strong entity authentication *to an oracle* $\Pi_i^{s_i}$ *is provided if both* $\mathsf{acc}_i^{s_i} = \mathsf{true}$ *and for all honest* $U_j \in \mathsf{pid}_i^{s_i}$ *with overwhelming probability there exists an oracle* $\Pi_j^{s_j}$ *with* $\mathsf{sid}_j^{s_j} = \mathsf{sid}_i^{s_i}$ *and* $U_i \in \mathsf{pid}_j^{s_j}$.

Definition 5 (Integrity). *We say a correct group key establishment protocol fulfills* integrity *if with overwhelming probability all oracles of honest principals that have accepted with the same session identifier* $\mathsf{sid}_j^{s_j}$

- *hold identical session keys* $\mathsf{sk}_j^{s_j}$*, and*
- *hold a* $\mathsf{pid}_j^{s_j}$*-value encompassing the identities of all honest parties having accepted with session identifier* $\mathsf{sid}_j^{s_j}$.

2.2 Assumptions on the underlying group

For the security proof of our protocol, the underlying group G (resp. family of groups $G = G(k)$, indexed by the security parameter) has to satisfy certain requirements. In particular, we assume products and inverses of group elements to be computable by ppt algorithms. For the sake of simplicity, we also assume that G allows a ppt computable canonical representation of elements, so that we can identify group elements with their canonical representation. To generate the group elements needed in a protocol execution, we rely on the existence of three algorithms, that capture the problem of creating "good instances":

- DomPar denotes a (stateless) ppt *domain parameter generation* algorithm that upon input of the security parameter 1^k outputs a finite sequence S of elements in G. The subgroup $\langle S\rangle$ of G spanned by S will be publically known. For the special case of applying our framework to a CDH-assumption, S specifies a public generator of a cyclic group.
- SamAut denotes a (stateless) ppt *automorphism group sampling* algorithm that upon input of the security parameter 1^k and a sequence S output by DomPar returns a description of an automorphism $\phi \in \mathrm{Aut}(G)$ of G, so that both ϕ and ϕ^{-1} can be evaluated efficiently. E. g., for a cyclic group, ϕ could be given as an exponent, or for an inner automorphism the conjugating group element could be specified.
- SamSub denotes a (stateless) ppt *subgroup sampling* algorithm that upon input of the security parameter 1^k and a sequence S output by DomPar returns a word $x(S)$ in the generators S (and their inverses) representing an element $x \in \langle S\rangle$. Intuitively, SamSub chooses a random $x \in \langle S\rangle$, so that it is hard to recognize x if we know elements of x's orbit under $\mathrm{Aut}(G)$. Our protocol needs an explicit representation of x in terms of the generators S.

With this notation, we can define a computational problem of parallel automorphism application, where $o \leftarrow \mathsf{A}(i)$ denotes that algorithm A outputs o upon receiving input i:

Definition 6 (Parallel automorphism application). *Let $r \in \mathbb{N}_{>0}$ be a natural number. By the* problem of r-fold parallel automorphism application (r-PAA) *w. r. t. the quadruple* $(G, \mathsf{DomPar}, \mathsf{SamAut}, \mathsf{SamSub})$ *we mean the task of finding an algorithm $\mathcal{A}$ which on input of S, $\phi_i(S) := (\phi_i(s))_{s \in S}$ for $i = 1, \ldots, r$ and $\phi_1(x), \ldots, \phi_r(x)$ outputs the group element x represented by the word $x(S)$, where*

- $S \leftarrow \mathsf{DomGen}(1^k)$,
- $x(S) \leftarrow \mathsf{SamSub}(1^k, S)$,
- $\phi_i \leftarrow \mathsf{SamAut}(1^k, S)$ $(i = 1, \ldots, r)$.

To capture the assumption needed in the security proof below, we also define the advantage of an adversary in solving the above problem:

Definition 7 (r-PAA advantage). *For an algorithm $\mathcal{A}$ trying to solve r-PAA, we denote its advantage as a function in the security parameter k and its runtime t by* $\mathsf{Adv}_{\mathcal{A}}^{r-\mathrm{PAA}} = \mathsf{Adv}_{\mathcal{A}}^{r-\mathrm{PAA}}(k, t) =$

$$\Pr\left(x \leftarrow \mathcal{A}(S, (\phi_i(S), \phi_i(x))_{1 \le i \le r}) \;\middle|\; \begin{array}{c} S \leftarrow \mathsf{DomGen}(1^k), x(S) \leftarrow \mathsf{SamSub}(1^k, S), \\ \phi_i \leftarrow \mathsf{SamAut}(1^k, S)\ (i = 1, \ldots, r) \end{array}\right).$$

Our security proof builds on the assumption that for any ppt adversary $\mathcal{A}$ the advantage $\mathsf{Adv}_{\mathcal{A}}^{r-\mathrm{PAA}}$ is negligible. For the case of ϕ being an inner automorphism, r-PAA expresses a kind of parallel conjugacy problem. At an intuitive level, finding conjugacy-problem based instances for the group key agreement described in the next section seems to be of comparable difficulty as finding instances for the two-party key establishment put forward by Anshel et al [2, 1]. Note however, that instead of looking for concrete instances building on a non-abelian group, we may apply our framework to an "ordinary" Computational Diffie-Hellman (CDH) setting, too:

Example 1 (Basing on CDH). Let G be a cyclic group and choose for $S := (g)$ an element $g \in G$ of prime order q. Now let SamSub choose uniformly at random an exponent $x \in \{1, \ldots, q\}$. Similarly, we specify SamAut to choose uniformly at random an exponent $\phi \in \{1, \ldots, q-1\}$. Then r-PAA is polynomial time equivalent to the CDH-problem in $\langle g \rangle$:

"CDH solution $\Rightarrow$ r-PAA solution": A CDH-oracle allows to find g^x from a single pair $(g^{\phi}, g^{x\phi})$ as follows. First compute $g^{\phi^{q-2} \bmod q} = g^{\phi^{-1} \bmod q}$ by using the CDH-oracle as "square- and multiple unit for the exponent". Next we can obtain g^x by applying the CDH-oracle to $g^{\phi^{-1} \bmod q}$ and $g^{x\phi}$.

"CDH solution $\Leftarrow$ r-PAA solution": Given g^{ϕ_1}, g^v $(\phi_1, v \in \{1, \ldots, q-1\})$, we can use an oracle solving r-PAA to compute $g^{v\phi_1^{-1}}$: We can interpret v as having the form $v = x \cdot \phi_1 \bmod q$, and by raising g^{ϕ_1} and g^v to uniformly at random chosen powers $\phi_1^{-1}\phi_i \in \{1, \ldots, q-1\}$ $(i = 2, \ldots, r)$, we obtain the input needed by an oracle solving r-PAA to compute $g^x = g^{v\phi_1^{-1}}$. Hence, given a pair (g^u, g^v) we can compute g^{uv} as follows:

1. Apply the above method to (g^u, g^v), yielding $g^{vu^{-1}}$.
2. Apply the above method to $(g^v, g^{vu^{-1}})$, yielding $g^{u^{-1}}$.
3. Apply the above method to $(g^{u^{-1}}, g^v)$, yielding g^{uv}.

3 A 2-Round Protocol for Group Key Agreement

To discuss our group key agreement protocol we adopt the common assumption that, from some protocol-external context, the set of protocol participants $\mathcal{U} \subseteq \mathcal{P}$ is known to all $U_i \in \mathcal{U}$. To simplify notation, w. l. o. g. we assume $\mathcal{U} = \{U_1, \ldots, U_r\}$. Moreover, we assume that an asymmetric signature scheme is available that is existentially unforgeable under adaptive chosen message attacks. The respective signing and verification keys are to be fixed and distributed throughout the initialization phase mentioned in Section 2.1, and we denote a signature of a protocol participant U_i on a message M by $\mathsf{Sig}_i(M)$.

3.1 Description of the protocol

Having fixed the security parameter k, first we have to run $\mathsf{DomGen}(1^k)$ to generate the public subgroup generators S. Hereafter, for an instance $\Pi_i^{s_i}$ of a protocol participant U_i a single protocol run can be described as shown in Figure 1. At this, *Broadcast:* M means that message M is sent to all other participants $U_j \in \mathcal{U}$ over *point-to-point* connections, i. e., the adversary is allowed to delay, suppress or modify some or all of the transmitted messages. Moreover, H denotes a cryptographic hash function which will be modeled as a random oracle.

Round 1: Initialization Set $\mathsf{pid}_i^{s_i} := \mathcal{U}$, $\mathsf{used}_i^{s_i} := \mathsf{true}$.
Choose $\phi_i^{s_i} \leftarrow \mathsf{SamAut}(1^k)$, $x_i^{s_i}(S) \leftarrow \mathsf{SamSub}(1^k, S)$, and compute the message $m_1^{s_i}(U_i) := ((\phi_i^{s_i}(t))_{t \in S}, H(x_i^{s_i}))$.
Broadcast: $m_1^{s_i}(U_i)$.

Round 2: Key Exchange Set $\mathsf{sid}_i^{s_i} := H\left(m_1^{s_1}(U_1), \ldots, m_1^{s_r}(U_r), \mathsf{pid}_i^{s_i}\right)$.
Compute and send $m_2^{s_i}(U_i, U_j) := \left(\phi_j^{s_j}(x_i^{s_i}), \mathsf{Sig}_i(\mathsf{sid}_i^{s_i})\right)$ to each participant $U_j \in \mathsf{pid}_i^{s_i}$, $j \neq i$. (To compute $\phi_j^{s_j}(x_i^{s_i})$ use the representation of $x_i^{s_i} = x_i^{s_i}(S)$ in terms of the generators S.)

Key Generation Compute from $\phi_i^{s_i}(x_j^{s_j})$ the original $x_j^{s_j}$ for all $j \neq i$ by applying the inverse of $\phi_i^{s_i}$.
Compute the common session key $K := H\left(x_1^{s_1}, \ldots, x_r^{s_r}, \mathsf{pid}_i^{s_i}\right)$.

Verification Check for all $U_j \in \mathsf{pid}_i^{s_i}$ if $\mathsf{Sig}_j(\mathsf{sid}_j^{s_j})$ is a valid signature for $\mathsf{sid}_i^{s_i}$ and if for $x_j^{s_j}$ the received hash value $H(x_j^{s_j})$ in $m_1^{s_j}(U_j)$ was correct.
If true, set $\mathsf{acc}_i^{s_i} := \mathsf{term}_i^{s_i} := \mathsf{true}$, and $\mathsf{sk}_i^{s_i} := K$.
Else set $\mathsf{acc}_i^{s_i} := \mathsf{false}$, $\mathsf{term}_i^{s_i} := \mathsf{true}$.

Fig. 1. A group key agreement protocol basing on r-PAA

Remark 1. Having in mind instances of this protocol where the ϕ_i are inner automorphisms, it is worth noting that the protocol is symmetric in the sense that all participants perform the same steps: Differing from Anshel et al.'s 2-party construction, the key computation for the initiator is the same as for the other protocol participants.

3.2 Security analysis

Correctness of the protocol in Figure 1 is immediate. To prove its security, we first observe that the constructed session identifier is with overwhelming probability globally unique:

Lemma 1. *If for all ppt adversaries $\mathcal{A}$ the advantage $\mathsf{Adv}_{\mathcal{A}}$ in solving r-PAA is negligible, then the session identifier $\mathsf{sid}_i^{s_i}$ constructed in the above protocol is with overwhelming probability globally unique.*

Proof. The assumption of the lemma implies in particular that the probability of SamSub outputing twice the same value in a ppt number of executions is negligible. Thus the collision-freeness of H yields the desired uniqueness of the session identifier. □

Next, before looking at (basic) security, we note that the above protocol also offers strong entity authentication and integrity:

Proposition 1. *The protocol provides strong entity authentication according to Definition 4 and integrity according to Definition 5.*

Proof. Strong entity authentication. Consider an arbitrary instance $\Pi_i^{s_i}$ of an uncorrupted participant U_i that has accepted with session identifier $\mathsf{sid}_i^{s_i}$. Let $U_j \in \mathsf{pid}_i^{s_i}$ be some other uncorrupted participant. Instance $\Pi_i^{s_i}$ must have received a message of U_j with a signature on U_j's session identifier $\mathsf{sid}_j^{s_j}$. By unforgeability of the signature scheme, uniqueness of the session identifier $\mathsf{sid}_j^{s_j} = \mathsf{sid}_i^{s_i}$, and the collision resistance of the hash function we obtain $\mathsf{pid}_j^{s_j} = \mathsf{pid}_i^{s_i}$ with overwhelming probability.

Integrity. Consider any two instances $\Pi_i^{s_i}$ and $\Pi_j^{s_j}$ that both have accepted with $\mathsf{sid} = \mathsf{sid}_i^{s_i} = \mathsf{sid}_j^{s_j}$ and where the participants U_i and U_j are honest. By unforgeability of the signature scheme, uniqueness of the session identifier sid, and the collision resistance of the hash function, with overwhelming probability we get $\mathsf{pid}_i^{s_i} = \mathsf{pid}_j^{s_j}$ and the equivalence of the messages $m_1^{s_\ell}(U_\ell)$ they received in Round 1. Those messages include hash values $H(x_\ell^{s_\ell})$ from all protocol participants and before accepting, all participants check if the computed values $x_\ell^{s_\ell}$ in Round 2 are consistent with the $H(x_\ell^{s_\ell})$. Unless a collision of H occurs they compute the same key. □

For the ease of presentation, in the proof of the basic security property we imagine the protocol without the hash value $H(x_i^{s_i})$ in Round 1. This simplification can be justified with a standard random oracle argument as in the proof of Lemma 3.

Proposition 2. *Denote the maximal number of protocol participants by $n = |\mathcal{P}|$, and let $\mathcal{A}$ be an adversary that is allowed at most $q_{\mathrm{s}}, q_{\mathrm{ex}}, q_H$ queries to the Send, $\mathsf{Execute}$ and hash oracle, respectively. Moreover, let $\mathsf{Adv}^{(n-1)-\mathrm{PAA}}$ resp. $\mathsf{Adv}^{\mathrm{Sig}}$ be the maximum advantage of a ppt algorithm solving $(n-1)$-PAA resp. achieving an existential forgery in running time t. Then*

$$\mathsf{Adv}_{\mathcal{A}} = |\mathsf{Succ} - 1/2| \leq n \cdot (q_{\mathrm{s}} + q_{\mathrm{ex}})^n \cdot q_H \cdot \mathsf{Adv}^{(n-1)-\mathrm{PAA}} + n \cdot \mathsf{Adv}^{\mathrm{Sig}}.$$

Proof. Let $\mathsf{Succ} := (\mathsf{Adv}_{\mathcal{A}} + 1)/2$ be the success probability of adversary $\mathcal{A}$ to win the experiment. Imagine $\mathcal{A}$ now to be connected to a simulator *Sim* that simulates the oracles. We consider a sequence of games and bound the difference of the adversary's success probability between subsequent games.

In Game 0 the simulator *Sim* simulates the oracles and principals' instances faithfully. Thus, there is no difference for the adversary and denoting $\mathcal{A}$'s success probability in Game i by $\mathsf{Succ}_{\text{Game } i}$, we have $\mathsf{Succ}_{\text{Game } 0} = \mathsf{Succ}$.

In Game 1 the simulator will keep a list with an entry $(i, \mathsf{sid}_i^{s_i})$ for every session identifier $\mathsf{sid}_i^{s_i}$ the simulator signs with the secret key of user U_i and returns it in a Round 2 message to $\mathcal{A}$ following an Execute-query or on a Send-query. We define the event Forge to occur, if $\mathcal{A}$ comes up with a query $\mathsf{Send}(*, *, M)$ where M includes a signature $\mathsf{Sig}(\mathsf{sid}_i^{s_i})$, signed by an uncorrupted principal U_i and $(i, \mathsf{sid}_i^{s_i})$, does not appear in the simulator's list. In this case we abort the experiment and count it as success for the adversary. Thus we have:

$$|\mathsf{Succ}_{\text{Game } 1} - \mathsf{Succ}_{\text{Game } 0}| \leq \Pr(\mathsf{Forge}).$$

Lemma 2. *If the signature scheme is existentially unforgeable, the probability of* Forge *is negligible. Formally:*

$$\Pr(\mathsf{Forge}) \leq n \cdot \mathsf{Adv}^{\mathrm{Sig}}$$

Proof. The simulator can use an adversary that can reach Forge with a non-negligible probability as black box to forge a signature from the underlying signature scheme.

The simulator is given a public key PK and a signing oracle. In the initialization it will uniformly choose one user U_i and assign the key PK as PK_i to U_i. If in the following simulation *Sim* has to generate a signed message for U_i it will use the signing oracle to sign the message. If $\mathcal{A}$ will send a message $(*, \sigma)$, where σ is a signature of a session identifier $\mathsf{sid}_i^{s_i}$ that is not in the simulator's list, the simulator will return $(\mathsf{sid}_i^{s_i}, \sigma)$ as existential forgery. Otherwise the simulator returns $\perp$. As i was chosen uniformly the simulator will succeed with a probability of $1/n \cdot \Pr(\mathsf{Forge})$, thus $\Pr(\mathsf{Forge}) \leq n \cdot \mathsf{Adv}^{\mathrm{Sig}}$. □

In Game 2 the simulation of the Test oracle is modified. On a query $\mathsf{Test}(U_i, s_i)$, the simulator checks if $\Pi_i^{s_i}$ is fresh. If so, then *Sim* will not query the random oracle, but return a random value in any case. As now no information about the Test-oracle's secret bit b is given to $\mathcal{A}$ in Game 2, the success probability is $\mathsf{Succ}_{\text{Game } 2} = 1/2$.

Now we have to determine the difference in the adversary's success probability between Game 1 and Game 2. For $\mathcal{A}$, a random value and the random oracle's answer are indistinguishable as long as $\mathcal{A}$ does not know the actual query to the random oracle. The success probabilities can only differ, if $\mathcal{A}$ queries $H(x_1^{s_1}, \ldots, x_r^{s_r}, \mathsf{pid}_i^{s_i})$ to the random oracle. Denoting this event by Random, we have

$$|\mathsf{Succ}_{\text{Game } 2} - \mathsf{Succ}_{\text{Game } 1}| \leq \Pr(\mathsf{Random}).$$

Lemma 3. *The probability* $\Pr(\mathsf{Random})$ *of the event* Random *to occur is negligible if* n *is constant and* $\mathsf{Adv}^{(n-1)-\mathrm{PAA}}$ *is negligible.*

Proof. The simulator is given an instance $(S, (\phi_i(S), \phi_i(x))_{1\leq i\leq n-1})$ of the $(n-1)$-PAA problem. In the initialization phase, the simulator will give S as parameter to $\mathcal{A}$ and uniformly choose n random numbers $\alpha_i \in \{1, q_\mathrm{s}+q_\mathrm{ex}\}$ $(i = 1, \ldots, n)$ to point to the instances $\Pi_i^{\alpha_i}$. The simulator will choose uniformly at random $\beta \in \{1, \ldots, n\}$ to select one distinguished instance $\Pi_\beta^{\alpha_\beta}$ among them.

When the simulator has to process Round 1 for one instance $\Pi_i^{\alpha_i}, i = 1, \ldots, n$, $i \neq \beta$, *Sim* will use the given $\phi_i(S)$ instead of computing a new ϕ_i with SamAut. For instance $\Pi_\beta^{\alpha_\beta}$ the simulator will use the given $x(S)$. If instance $\Pi_\beta^{\alpha_\beta}$ does not only get messages containing $\phi_i(S), (i = 1, \ldots, n, i \neq \beta)$ the simulator aborts and outputs $\perp$. Also, if the simulator ever has to apply a ϕ_i^{-1} it aborts and outputs $\perp$ (this will only happen from a Reveal-query).

Because $\Pi_\beta^{\alpha_\beta}$ is uniformly selected out of a set of $n \cdot (q_\mathrm{s} + q_\mathrm{ex})$ potential instances, it will be used in the Test-query with a probability of $(n \cdot (q_\mathrm{s}+q_\mathrm{ex}))^{-1}$. To be able to apply the Test-query the adversary has to let $\Pi_\beta^{\alpha_\beta}$ accept. All $U_i \in \mathsf{pid}_\beta^{\alpha_\beta}$ have to be uncorrupted. Then by uniqueness of the session identifier (Lemma 1) the messages $\Pi_\beta^{\alpha_\beta}$ must have got in Round 1 were generated by the same instances as the messages $\Pi_\beta^{\alpha_\beta}$ received in Round 2. These have to be the distinguished oracles $\Pi_i^{\alpha_i}$ for $U_i \in \mathsf{pid}_\beta^{\alpha_\beta}$. For the principals $U_j \notin \mathsf{pid}_\beta^{\alpha_\beta}$, $\Pi_j^{\alpha_j}$ must be an oracle that was not revealed. There must be at least one potential instance that is not used for each $U_j \notin \mathsf{pid}_\beta^{\alpha_\beta}$. Consequently, with a probability of $1/n \cdot (q_\mathrm{s} + q_\mathrm{ex})^n$ the principals are distributed as needed.

If $\mathcal{A}$ halts, the simulator chooses uniformly one of the at most q_H queries to the random oracle, extracts x_β (assuming it is of the form $H(x_1^{s_1}, \ldots, x_r^{s_r}, \mathsf{pid}_i^{s_i})$) and answers this to the (n-1)-PAA challenge. The probability to pick the correct query is $1/q_H \cdot \Pr(\mathsf{Random})$. With a probability of at least

$$\Pr((n-1)\text{−}\mathsf{PAAsolved}) \geq \frac{1}{n \cdot (q_\mathrm{s} + q_\mathrm{ex})^n \cdot q_H} \cdot \Pr(\mathsf{Random})$$

the simulator solves the challenge. □

Putting it all together we obtain

$$\mathsf{Adv}_\mathcal{A} = |\mathsf{Succ} - 1/2| \leq n \cdot (q_\mathrm{s} + q_\mathrm{ex})^n \cdot q_H \cdot \mathsf{Adv}^{(n-1)-\mathrm{PAA}} + n \cdot \mathsf{Adv}^{\mathrm{Sig}}.$$

□

Remark 2. If instead of a constant number n of potential protocol participants, we want to allow a size $\mathcal{P}$ of polynomial size, the bound in Proposition 2 is in general no longer negligible. However, if we base on a CDH-assumption as in Example 1, we can allow for a set $\mathcal{P}$ of polynomial size: With the argument given in Example 1 we see that in this case solving 1-PAA is equivalent to solving r-PAA for an arbitrary r of polynomial size. In the security proof, this reduction allows us to replace the exponent n by the constant 2.

4 Conclusion

In this contribution we have described a 2-round group key agreement and showed it to be secure under the assumption that certain group-theoretical tools are available. In addition to the "standard" security requirement, the proposed protocol also offers strong entity authentication and integrity. While our framework is primarily geared towards building a provably secure group key agreement on non-abelian groups, it also allows to derive a 2-round group key agreement from a CDH assumption.

Acknowledgment

We are indebted to Dennis Hofheinz for valuable discussions and comments.

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