On construction of non-weakly-normal functions

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Abstract

Given two non-weakly k-normal Boolean functions on n-variables a method is proposed to construct a non-weakly $(k+1)$-normal Boolean function on $(n+2)$-variables.

1 Introduction

Let $n = 2m$ be an even positive integer. Any function from $\mathbb{F}_2^n$ into $\mathbb{F}_2$ is called a Boolean functions on n variables. A bent function on n variables is a function whose Hamming distance is maximum from the set of all affine functions. The Walsh Hadamard transform of f at at $\lambda \in \mathbb{F}_2^n$ is given by $W_f(\lambda) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)+\langle \lambda, x \rangle}$ where $\langle x, \lambda \rangle$ is an inner product of x and λ on $\mathbb{F}_2^n$. A Boolean function $f : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2$ is bent if and only if $|W_f(\lambda)| = 2^m$ for all $\lambda \in \mathbb{F}_2^n$. Bent functions were first constructed by Rothaus [8] and Dillon [4, 5].

Definition 1 *[3] A Boolean function $f : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2$ is called k-normal (weakly k-normal) if f is constant (affine) on a k-dimensional flat of $\mathbb{F}_2$.*

Dobbertin [6] introduced the notion of normality and used normal bent functions to construct balanced functions with high nonlinearity. A bent function is called (weakly)-normal if it is (weakly) $\frac{n}{2}$-normal. However for a long time no non-normal or non-weakly-normal bent function was known. Non-normal and non-weakly-normal bent functions for $n = 10$ and $n = 14$, respectively were first constructed by Canteaut, Daum, Dobbertin and Leander [1]. Further they proved that

Theorem 1 *(Lemma 10, [1]) Let $f : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2$ be a Boolean function. The following properties are equivalent:*

1. *f is (weakly) normal.*
2. *The function*
$$g : \mathbb{F}_2^n \times \mathbb{F}_2 \times \mathbb{F}_2 \longrightarrow \mathbb{F}_2$$
defined by
$$g(x, y, z) = f(x) + yz$$
is (weakly) normal.

By using this result it is possible to construct non-(weakly)-normal bent functions on higher dimensional spaces starting from a non-(weakly)-normal function on $\mathbb{F}_n$ for some n. Carlet, Dobbertin and Leander [2] proved that the direct sum of a non-(weakly)-normal bent function with any (weakly)-normal one is non-(weakly)-normal. Considering the difficulty of deciding non-(weak)-normality of bent functions any such secondary construction which guarantees non-(weak)-normality is extremely important.

2 Main Result

In this section we present our main result, a generalization of theorem 1.

Theorem 2 *Let $f_1, f_2 : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2$ be two Boolean functions. The following statements are equivalent:*

1. *f_1 or f_2 is weakly k-normal.*
2. *The function*
$$g : \mathbb{F}_2^n \times \mathbb{F}_2 \times \mathbb{F}_2 \longrightarrow \mathbb{F}_2$$
defined by
$$g(x, y, z) = f_1(x) + yz + (y + z)(f_1(x) + f_2(x))$$
is weakly $(k + 1)$-normal.

Proof : Suppose g is weakly $(k+1)$-normal. Therefore there exists an $k+1$ dimensional flat E, $\gamma \in \mathbb{F}_2^n$ and $\alpha, \beta \in \mathbb{F}_2$ such that

$$h(x, y, z) = g(x, y, z) + \alpha y + \beta z + \langle \gamma, x \rangle$$

takes the same value, c, on E. We claim that either $f_1(x)$ of $f_2(x)$ is weakly normal.

For $a, b \in \mathbb{F}_2$ we define

$$E_{ab} = \{x \in \mathbb{F}_2^n | (x, a, b) \in E\}.$$

Suppose $x \in E_{ab}$, then

$$c = h(x, a, b) = f_1(x) + ab + (a + b)(f_1(x) + f_2(x)) + \alpha a + \beta b + \langle \gamma, x \rangle$$

i.e.,

$$f_1(x) + (a + b)(f_1(x) + f_2(x)) = c + ab + \alpha a + \beta b + \langle \gamma, x \rangle.$$

Note that

$$f_1(x) + (a + b)(f_1(x) + f_2(x)) = \begin{cases} f_1(x) & \text{i}f a + b = 0 \\ f_2(x) & \text{i}f a + b = 1 \end{cases}$$

Therefore if $x \in E_{ab}$ then either $f_1(x)$ or $f_2(x)$ is affine on E_{ab}.

If one of the flats E_{ab} has dimension $\geq k$ then we are done. If this is not true, all the flats E_{ab} have dimension $k - 1$. Furthermore since the union of all E_{ab} is a flat, all E_{ab} are cosets of the same subspace U, we write $E_{ab} = x_{ab} + U$. Moreover, $x_{\alpha,\overline{\beta}} \neq x_{\overline{\beta},\alpha}$. Otherwise for any element $(x, \overline{\alpha}, \beta) \in E$ the element $(x, \alpha, \overline{\beta}) \in E$. Then, if we consider two elements $(x, \overline{\alpha}, \beta)$ and (x', α, β) in E, we obtain that,

$$(x, \overline{\alpha}, \beta) + (x, \alpha, \overline{\beta}) + (x', \alpha, \beta) = (x', \overline{\alpha}, \overline{\beta})$$

belongs to E implying that $h(x', \alpha, \beta) = h(x', \overline{\alpha}, \overline{\beta})$. But,

$$\begin{aligned} h(x', \overline{\alpha}, \overline{\beta}) &= f_1(x') + \overline{\alpha}\overline{\beta} + (\overline{\alpha} + \overline{\beta})(f_1(x') + f_2(x')) + \alpha\overline{\alpha} + \beta\overline{\beta} + \langle \gamma, x \rangle \\ &= f_1(x') + \alpha\beta + (\alpha + \beta)(f_1(x') + f_2(x')) + \alpha + \beta + \langle \gamma, x \rangle \\ &= h(x', \alpha, \beta) + 1 \end{aligned}$$

which leads to a contradiction. Therefore since $x_{\alpha,\overline{\beta}} \neq x_{\overline{\alpha},\beta}$, the set $E_{\alpha,\overline{\beta}} \cup E_{\overline{\alpha},\beta}$ is a flat of dimension k. Moreover we deduce the following:

For all $x \in E_{\alpha,\overline{\beta}}$

$c = h(x, \alpha, \overline{\beta}) = f_1(x) + \alpha\overline{\beta} + (\alpha + \overline{\beta})(f_1(x) + f_2(x)) + \alpha\alpha + \beta\overline{\beta} + \langle \gamma, x \rangle$

i.e., $f_1(x) + (\alpha + \beta + 1)(f_1(x) + f_2(x)) = c + \alpha\beta + \langle \gamma, x \rangle$.

Similarly for all $x \in E_{\overline{\alpha},\beta}$

$c = h(x, \overline{\alpha}, \beta) = f_1(x) + \overline{\alpha}\beta + (\overline{\alpha} + \beta)(f_1(x) + f_2(x)) + \alpha\overline{\alpha} + \beta\beta + \langle \gamma, x \rangle$

i.e., $f_1(x) + (\alpha + \beta + 1)(f_1(x) + f_2(x)) = c + \alpha\beta + \langle \gamma, x \rangle$.

Therefore when $x \in E_{\alpha,\overline{\beta}} \cup E_{\overline{\alpha},\beta}$

$f_1(x) + (\alpha + \beta + 1)(f_1(x) + f_2(x)) = c + \alpha\beta + \langle \gamma, x \rangle$.

Thus either $f_1(x)$ or $f_2(x)$ is weakly normal.

Conversely suppose $f_1(x)$ is weakly normal which implies that there exists an k dimensional space E on which $f_1(x)$ is affine. Suppose $f_1(x) = \langle \gamma, x \rangle + c$ on E. Consider the $k+1$ dimensional subspace

$E' = E \times \{0\} \times \{0\} \cup E \times \{1\} \times \{1\}$.

It can be checked that

$g(x, 0, 0) = f_1(x) = \langle \gamma, x \rangle + c$

and

$g(x, 1, 1) = f_1(x) + 1 = \langle \gamma, x \rangle + c + 1$

Therefore we can write $g(x, y, z) = \langle \gamma, x \rangle + y + c$ for all $(x, y, z) \in E'$. Thus g is weakly $(k + 1)$-normal. ■

Thus if we start with two non-weakly-normal bent functions f_1 and f_2 on n variables then the function $g(x, y, z)$ is a non-weakly-normal function on $n + 2$ variables. The construction given in 1 cannot increase the algebraic degree of the bent function on $n + 2$ variables whereas our construction increases degree by 1 if algebraic degree of f_1 and $f_1 + f_2$ are same. Thus unlike the construction in [1] starting from two non-weakly-normal bent functions on n variables and algebraic degree $\frac{n}{2}$ with $deg(f_1) = deg(f_1 + f_2)$ if is possible to construct a non-weakly-normal bent function of algebraic degree $\frac{n+2}{2}$, which is optimal. It is to be noted that the bent function generated by this method is affinely equivalent to the function constructed in Proposition 8 [2].

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