

# The Design Principle of Hash Function with Merkle-Damgård Construction

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**Abstract.** The paper discusses the security of hash function with Merkle-Damgård construction and provides the complexity bound of finding a collision and preimage of hash function based on the condition probability of compression function  $y = F(x, k)$ . we make a conclusion that in Merkle-Damgård construction, the requirement of free start collision resistant and free start collision resistant on compression function is not necessary and it is enough if the compression function with properties of fix start collision resistant and fix start preimage resistant. However, the condition probability  $P_{Y|X=x}(y)$  and  $P_{Y|K=k}(y)$  of compression function  $y = F(x, k)$  have much influence on the security of the hash function. The best design of compression function should have properties of that  $P_{Y|X=x}(y)$  and  $P_{Y|K=k}(y)$  are both uniformly distributed for all  $x$  and  $k$ . At the end of the paper, we discussed the block cipher based hash function, point out among the the 20 schemes, selected by PGV[2] and BPS[12], the best scheme is block cipher itself, if the block cipher with perfect security and perfect key distribution.

## 1 Introduction

Most of hash functions iterated a compression function by Merkle-Damgård construction with constant IV[3]. A well known approach for building hash function is the compression function out of a block cipher which have been discussed sine Rabin[10] given the first model of that kind of structure. As BRS point out the block cipher approach has been less widely used for variety of reasons, and the emergence of the AES[13] has somewhat modified this landscape, especially recently the MD5 and SHA1 are attacked[8][9][14][16].

The topics of building hash function based on block cipher have been systematically analyzed in paper [2] [4][7][12][15]. The PGV paper considered turning a block cipher  $E : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}^n$  into a hash function  $H : (\{0, 1\}^n)^* \rightarrow \{0, 1\}^n$  using a compression function  $F : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}^n$  derived from  $E$ . For  $v$  is a fixed  $n$ -bit constant, PGV considers all 64 compression functions

$F$  of the form  $F(h_{i-1}, m_i) = E_a(b) \oplus c$  where  $a, b, c \in \{h_{i-1}, m_i, h_{i-1} \oplus m_i, v\}$ , then define the iterated hash of  $F$  as  $H(M_1, \dots, M_t, IV) = h_t, h_0 = IV, h_i = F(h_{i-1}, M_i), i \in [1..t], |M_i| = n$ . Of the 64 such schemes, the authors of [2] regarded 12 as secure. In fact, PGV[2] implied the condition of free start not fix start. The authors of [12] taken a more proof-centric look at the schemes from PGV, proved additional 8 schemes were collision resistant, divided the 20 schemes into two group where the *group* – 1 were the 12 schemes picked by PGV and the *group* – 2 were the new founded 8 schemes. For the new founded schemes, the hash function  $H$  immune to collision attack within the Merkle-Damgård paradigm, the compression functions were not immune to collision attack, the proves of collision resistant of *group* – 2 used the assumptions of  $E$  with black box model and  $H$  with fix start model. They also provided both upper and lower bounds for each scheme.

This paper takes a proof-centric look at the schemes based on the probability theory, providing the exact probability of finding a collision or preimage based on the assumption of known condition probability of block cipher  $E$ . Let  $F : \{0, 1\}^t \times \{0, 1\}^k \rightarrow \{0, 1\}^n, x \in \{0, 1\}^t, y \in \{0, 1\}^n, k \in \{0, 1\}^k$  and Hash function  $H$  is Merkle-Damgård construction hash function with compression function  $F$ .

Firstly, the probability of finding a collision or a preimage is defined by complexity of finding collision in one time computation of compression function  $F$ , based on which a more precise definitions of free start and fix start of collision resistant and preimage resistant are given about compression function and Hash function. In our point of view, if we have no way to find the collision or preimage except exhaustive search then the function is called collision resistant or preimage resistant. Secondly, the upper bound probabilities of finding collision and preimage about the compression function  $F$  and the Hash function with M-D paradigm are given which is based on the condition probability of  $P_{Y|X=x}(y)$  and  $P_{Y|K=k}(y)$ . At last we analyze the 64 schemes with M-D construction, in fact the best compression function to build M-D hash function is block cipher  $E$  itself, the best block cipher is the block cipher designed with perfect security and perfect key distribution.

The paper is organized as follows. The mathematical preliminaries and notation employed are described in section 2. The preimage resistance and collision resistance of compression function are given in section 3, that of a hash function are presented in section 4. Section 5 describe the preimage resistance and collision resistance of PGV schemes. Section 6 is our conclusion.

## 2 Definition

### 2.1 Basic Definition

A discrete random variable  $X$  is a mapping from the sample space  $\Omega$  to an alphabed  $\mathcal{X}$ .  $X$  assigns a value  $x \in \mathcal{X}$  to each elementary event in the  $\Omega$  and the probability distribution of  $X$  is the function[5]

$$P_X : \mathcal{X} \rightarrow \mathfrak{R} : x \mapsto P_X(x) = P[X = x] = \sum_{\omega \in \Omega : X(\omega) = x} P[\omega].$$

If the conditioning event involves another random variable  $Y$  defined on the same sample space, the conditional probability distribution of  $X$  given that  $Y$  takes on a value  $y$  is:

$$P_{X|Y=y}(x) = \frac{P_{XY}(x, y)}{P_Y(y)}$$

whenever  $P_Y(y)$  is positive. Two random variables  $X$  and  $Y$  are called independent if for all  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$ :

$$P_{XY}(x, y) = P_X(x) \cdot P_Y(y).$$

Let  $F : \{0, 1\}^\ell \times \{0, 1\}^\kappa \rightarrow \{0, 1\}^n$ ,  $x \in \{0, 1\}^\ell$ ,  $y \in \{0, 1\}^n$ ,  $k \in \{0, 1\}^\kappa$  and  $y = F_k(x)$ . If  $F : \{0, 1\}^\ell \times \{0, 1\}^\kappa \rightarrow \{0, 1\}^n$  is a compression function of hash function  $H_X$  and  $H_K$ :

$$H_X : \{0, 1\}^n \times \{0, 1\}^{\kappa^*} \rightarrow \{0, 1\}^n, x \in \{0, 1\}^n, m \in \{0, 1\}^{\kappa^*}, y \in \{0, 1\}^n$$

$$z = H_X(m, x) \triangleq H_X(m_t \| \dots \| m_1, x) = F_{m_t}(F_{m_{t-1}}(\dots(F_{m_1}(x))\dots))$$

$$H_K : \{0, 1\}^{\ell^*} \times \{0, 1\}^n \rightarrow \{0, 1\}^n, k \in \{0, 1\}^n, m \in \{0, 1\}^{\ell^*}, y \in \{0, 1\}^n$$

$$z = H_K(m, k) \triangleq H_K(m_t \| \dots \| m_1, k) = F_{\dots F_{F_k(m_1)}(m_2)\dots}(m_t)$$

$$H_K(m_2 \| m_1, k) = F_{F_k(m_1)}(m_2).$$

In this paper, if no special statement are given, the function  $H_K, H_X$  and  $F$  are defined as above and  $x, k$  are uniformly distributed in definition domain.  $q_k, q_x$  and  $q_y$  denote  $q_k \triangleq \max_y P_{Y|K=k}(y)2^\ell$ ,  $q_x \triangleq \max_y P_{Y|X=x}(y)2^\kappa$  and  $q_y \triangleq P_Y(y)2^\ell 2^\kappa$ .

In hash function attack, the probability of finding a preimage or collision is different from tradition point of view of probability. If the compression function  $F$  is block cipher  $E$  with form of  $E_k(x) = y$ , then the probabilities of  $P_{X|Y=y, K=k}(x)$ ,  $P_{K|Y=y, X=x}(k)$  are both equal 0 or 1 (assume the cipher with perfect key distribution). However, for given  $y, k$  the value  $x$  satisfying  $y = E_k(x)$  can be found directly by computing  $x = E_k^{-1}(y)$ , but for given  $y, x$  the value  $k$  satisfying  $y = E_k(x)$  can be found only by exhaustive search of  $k$ , that implies we should compute  $E$  for each guessing  $k$ . So we consider giving new definition about the probability of finding collision or preimage based on the times computing  $F$  being made.

**Definition 1.** Let  $x \in \mathcal{X}$ ,  $y \in \mathcal{Y}$ ,  $k \in \mathcal{K}$ ,  $\Lambda \subset \mathcal{X}$ ,  $\Gamma \subset \mathcal{K}$ ,  $G : \mathcal{X} \times \mathcal{K} \rightarrow \mathcal{Y}$ , Let  $\Omega_{\{XYK\}^G} = \{(x, y, k) | x \in \{0, 1\}^\ell, y \in \{0, 1\}^n, k \in \{0, 1\}^\kappa, y = G(x, y)\}$ . The  $\sigma$ -algebra  $\mathcal{F}$  is the subsets of  $\Omega$ ,  $\omega \in \Omega$  and  $A \in \mathcal{F}$ . Let  $A_{\mathcal{E}^G}$  to denote a set in  $\mathcal{F}$  with restriction  $\mathcal{E}^G$ . Let  $A_{\mathcal{E}_1^G, \dots, \mathcal{E}_t^G}$  to denote a serious of set  $\{A_{\mathcal{E}_1^G}, \dots, A_{\mathcal{E}_t^G}\}$ ,

$$\text{denoted } A_{\mathcal{E}_1^G, \dots, \mathcal{E}_t^G} \triangleq \bigsqcup_{i=1}^t A_{\mathcal{E}_i^G}$$

The example of restriction  $\mathcal{E}$  on  $A$  are as followings:

- $A_{\{XKY_{y_0}\}}^G = \{(x, y, k) | (x, y, k) \in \Omega^G, G(x, k) = y_0\}$ ;
- $A_{\{X_\Lambda YK\}}^G = \{(x, y, k) | (x, y, k) \in \Omega^G, x \in \Lambda\}$

The example of restriction  $\mathcal{E}$  on  $\mathcal{A}$  are as followings:

- $\mathcal{A}_{\{X_\Lambda KY\}}^G = \{A_{\{X_\Lambda KY\}}^G\}$ ;
- $\mathcal{A}_{\{X_x KY\}}_{x \in \Lambda}^G = \bigsqcup_{x \in \Lambda} A_{\{X_x YK\}}^G$

**Definition 2.** Let  $\mathcal{T}$  is some algorithm,  $A_\mathcal{E}^\mathcal{T}$  to denote using algorithm  $\mathcal{T}$  to finding  $\omega \in A_{\mathcal{E}G}$ .  $A_{\{X|KY\}}^G$  means that the algorithm is given  $k, y$  find  $x$  satisfying  $y = G(x, k)$ .

**Definition 3.** Let  $F : \{0, 1\}^t \times \{0, 1\}^k \rightarrow \{0, 1\}^n$ ,  $x \in \{0, 1\}^t$ ,  $y \in \{0, 1\}^n$ ,  $k \in \{0, 1\}^k$  and  $y = F_k(x)$ .  $G$  is function of iterated by  $F$ , assuming  $z = G(m, k)$ ,  $m \in \{0, 1\}^{t*}$ ,  $\tilde{Q}_{\mathcal{E}G}^\mathcal{T}$  is defined as the minimum times of requirement of computing  $F$  with probability of 1 finding a  $\omega \in A$ ,  $A \in \mathcal{A}_{\mathcal{E}G}$  using algorithm  $\mathcal{T}$ ,  $\hat{Q}_{\mathcal{E}G}^\mathcal{T}$  is defined as the minimum times of requirement of computing  $F$  with probability of 1 finding  $\omega_1, \omega_2 \in A$ ,  $A \in \mathcal{A}_{\mathcal{E}G}$  using algorithm  $\mathcal{T}$ . Let  $Q_{\mathcal{E}G} = \max_{\mathcal{T}} \tilde{Q}_{\mathcal{E}G}^\mathcal{T}$  and called Degree of  $\mathcal{E}$ .

More precisely,  $\tilde{Q}_{\{XYK\}}^F$  means times of computing  $F$  required to find values  $x, y, k$  satisfying  $y = F_k(x)$ , in fact for any  $x, k$ , we can compute  $y$  by  $F_k(x)$ , which gives  $\tilde{Q}_{\{XYK\}}^G$ .  $\hat{Q}_{\{X|KY\}}^F$  means the requirement of times of computing  $F$  to find  $x, x'$  satisfying  $F_k(x) = F_k(x')$  for selected  $y, k$  with success probability of 1.

If  $F$  is block cipher, from given  $y, k$  we can compute  $x = F_k^{-1}(y)$  that means  $Q_{XK|Y=y} = Q_{X|K=k, Y=y} = 1$ . But we can't compute  $k$  directly from give  $y, x$ , the only way to find  $k$  is exhaustive search, we have  $Q_{K|X=x, Y=y} \geq P_{Y|X=x}(y)$ .

**Definition 4 (Finding Preimage).** Finding Preimage in  $G$  is for given  $y$  finding  $\omega \in A_{\{XKY_y\}}^G$ .

**Definition 5 (Finding Collision).** Finding Collision of  $G$  is finding  $\omega, \omega' \in A$  and  $A \in \bigsqcup_y A_{\{XKY_y\}}^G$ .

**Definition 6 (Free Start Preimage Resistant).** Let  $y = G(x, k)$ ,  $G$  is preimage resistant if the best way to find  $\omega \in A_{\{XKY_y\}}$  is exhaustive search.

**Definition 7 (Fix Start Preimage Resistant).** Let  $y = G(x, k)$ ,  $A \subset \mathcal{X}$ ,  $G$  is preimage resistant with fix start  $X$ , the best way of finding  $\omega \in A_{\{X_\Lambda KY_Y\}}^G$  is exhaustive search.

**Definition 8 (Free Start Collision Resistant).** Let  $y = G(x, k)$ ,  $G$  is free start collision resistant if the best way to find  $\omega, \omega' \in A$ ,  $A \in \bigsqcup_y A_{\{XKY_y\}}^G$  is exhaustive search.

**Definition 9 (Fix Start Collision Resistant).** Let  $y = G(x, k)$ ,  $\Lambda \subset \mathcal{X}$ ,  $G$  is collision resistant with fix start  $X$ , the best way of finding collision  $\omega, \omega' \in \Lambda$ ,  $A \in \bigsqcup_y \mathcal{A}_{\{X_\Lambda KY_y\}^G}$  is exhaustive search.

**Definition 10 (Perfect Key Distribution).** A cryptosystem has perfect key distribution if

$$P_{K|Y=y}(k) = P_K(k)$$

for all  $x \in \{0, 1\}^n, y \in \{0, 1\}^n$ .

**Definition 11 (Perfect Secrecy[6]).** A cryptosystem has perfect secrecy if

$$P_{X|Y=y}(x) = P_X(x)$$

for all  $x \in \{0, 1\}^n, y \in \{0, 1\}^n$ .

**Definition 12 (Perfect Key Distribution).** A cryptosystem has perfect key distribution if

$$P_{K|Y=y}(k) = P_K(k)$$

for all  $x \in \{0, 1\}^n, y \in \{0, 1\}^n$ .

In fact,  $P_{XY}(xy) = P_{X|Y=y}P_Y(y) = P_{Y|X=x}(y)P_X(x)$ , since  $P_{X|Y=y}(x) = P_X(x)$ , we get  $P_{Y|X=x}(y) = P_Y(y)$ .

**Definition 13 (Random Oracles[12]).** A fixed-size random oracle is a function  $f : \{0, 1\}^l \rightarrow \{0, 1\}^n$ , chosen uniformly at random from the set of all such functions. For interesting sizes  $a$  and  $b$ , it is infeasible to implement such a function, or to store its truth table. Thus, we assume a public oracle which, given  $x \in \{0, 1\}^l$ , computes  $y = f(x) \in \{0, 1\}^n$ .

### 3 Hash Properties of Compression Function

The conclusions of this section are that the best design of  $F$  should satisfy  $y$  is uniformly distributed in  $\{0, 1\}^n$  for each  $k \in \{0, 1\}^k$  and for each  $x \in \{0, 1\}^l$ , no matter for free start or fix start and for preimage resistance or collision resistance. In this paper we make a assumption of  $\frac{1}{0} = 0$

**Lemma 1.** Let  $\Lambda \subset \mathcal{X}$ ,

$$\begin{aligned} \tilde{Q}_{\{XKY_y\}_{y \in \mathcal{Y}}^F} &= \min_y \{ \tilde{Q}_{\{XK|Y_y\}^F}, \tilde{Q}_{\{X|KY_y\}^F}, \tilde{Q}_{\{K|XY_y\}^F} \}. \\ \tilde{Q}_{\{X_\Lambda KY_y\}_{y \in \mathcal{Y}}^F} &= \min_y \{ \min_{x \in \Gamma} \tilde{Q}_{\{K|Y_y X_x\}^F}, \tilde{Q}_{\{X_\Lambda K|Y_y\}^F}, \tilde{Q}_{\{X_\Lambda|KY_y\}^F} \} \\ \hat{Q}_{\langle XKY \rangle^F} &= \min \{ \hat{Q}_{\{XK|Y\}^F}, \hat{Q}_{\{XK\}^F}, \hat{Q}_{\{X|K\}^F}, \hat{Q}_{\{K|X\}^F} \} \\ \hat{Q}_{\langle X_\Lambda KY \rangle^F} &= \min \{ \hat{Q}_{\{X_\Lambda K|Y\}^F}, \hat{Q}_{\{X_\Lambda K\}^F}, \hat{Q}_{\{X_\Lambda|K\}^F}, \min_{x \in \Gamma} \hat{Q}_{\{K|X_x\}^F} \} \end{aligned}$$

### 3.1 Free Start Preimage Resistance

The conclusion of this subsection is Theorem1, the upper bound of free start preimage resistant of  $F$  is  $\max_{x,k} \{\frac{2^\ell}{q_k}, \frac{2^\kappa}{q_x}\}$ , which implies the best selection of free start collision resistant and free start preimage resistant have same requirement on  $F$ .

**Theorem 1.** *If  $F$  is free start Preimage resistance then*

$$\tilde{Q}_{\{XKY_y\}_{y \in \{0,1\}^n}}^F = \min_{x,k} \left\{ \frac{2^\ell}{q_k}, \frac{2^\kappa}{q_x} \right\}. \quad (1)$$

*Proof.*  $F$  is preimage resistant, the only way to get preimage is exhaustive search. Given  $y, k$  searching  $x$  with  $y = F_k(x)$  the success probability is:

$$p = P_{Y|K=k}(y)$$

We get  $\tilde{Q}_{\{X|Y_y K\}}^F = \frac{2^\ell}{q_k}$ .

Similarly we get  $\tilde{Q}_{\{K|Y_y X\}}^F = \frac{2^\kappa}{q_x}$  and  $\tilde{Q}_{\{XK|Y_y\}}^F = \frac{2^\kappa 2^\ell}{q_y}$ .  $\square$

### 3.2 Free Start Collision Resistance

Conclusion of this subsection is Theorem2, upper bound of free start collision resistant of  $F$  is smaller than  $\max_{x,k,y} \left\{ \sqrt{\frac{2^\kappa}{(q_x-1)}}, \sqrt{\frac{2^\ell}{(q_k-1)}}, \sqrt{\frac{2^{\ell+\kappa}}{(q_y-1)}} \right\}$ , which implies the best design of  $F$  should satisfy  $y$  is uniformly distributed in  $\{0,1\}^n$  for each  $k \in \{0,1\}^\kappa$  and for each  $x \in \{0,1\}^\ell$ .

**Lemma 2.**  *$F$  is collision resistant, then*

$$\begin{aligned} - Q_{\{XX'K\}}^F &= \min_k \sqrt{\frac{2^\ell}{(q_k-1)}} \\ - Q_{\{XKK'\}}^F &= \min_x \sqrt{\frac{2^\kappa}{(q_x-1)}} \\ - Q_{\{XX'KK'\}}^F &= \min_y \sqrt{\frac{2^{\ell+\kappa}}{(q_y-1)}} \end{aligned}$$

*Proof.*  $F$  is collision resistant, the collision can be get only by exhaustive search.

- The fastest way to search for collision is the way based on birthday paradox. For random selected  $k$  searching  $x_1, x_2, \dots, x_t$  finding collision of  $F_k(x_i) = F_k(x_j)$ . The max probability of success is

$$p = 1 - \frac{2^\ell (2^\ell - 2^\ell P_{Y|K=k}(y_1)) \dots (2^\ell - \sum_i^{t-1} (2^\ell P_{Y|K=k}(y_i)))}{\binom{2^\ell}{t} t!}$$

Let denote  $n \triangleq 2^l$  and  $q_k \triangleq 2^l \max_y P_{Y|K=k}(y)$  then

$$\begin{aligned} p &\leq 1 - \frac{(n)(n - q_k) \dots (n - q_k(t - 1))}{(n)(n - 1) \dots (n - t + 1)} \\ &= 1 - \prod_{i=0}^{t-1} \frac{n - iq_k}{n - i} = 1 - \prod_{i=0}^{t-1} \left(1 - \frac{iq_k - i}{n - i}\right) = 1 - \prod_{i=0}^{t-1} \left(1 - \frac{i}{n - i}(q_k - 1)\right) \\ &\approx 1 - \prod_{i=0}^{t-1} \exp\left(\frac{i}{n-i}(q_k - 1)\right) \approx 1 - \prod_{i=0}^{t-1} \exp\left(\frac{i}{n} + \frac{i^2}{n^2}\right)(q_k - 1) \end{aligned}$$

Same as birthday paradox, when  $t \geq \sqrt{n/(q_k - 1)}$ ,  $q_k > 1$  the success probability of collision is bigger than 1/2. We get  $\widehat{Q}_{\{X|K\}^F} = \min_k \sqrt{\frac{2^l}{q_k - 1}}$ .

- similar as item 1, we get  $\widehat{Q}_{\{K|X\}^F} = \sqrt{\frac{2^\kappa}{q_x - 1}}$ ;
- similar as item 1, we get  $\widehat{Q}_{\{XK\}^F} = \sqrt{\frac{2^{\kappa+l}}{q_y - 1}}$ . □

**Theorem 2.** *F is collision resistant then*

$$\widehat{Q}_{\{XKY\}^F} = \min_{x,k,y} \left\{ \sqrt{\frac{2^\kappa}{(q_x - 1)}}, \sqrt{\frac{2^l}{(q_k - 1)}}, \sqrt{\frac{2^{\kappa+l}}{(q_y - 1)}} \right\} \quad (2)$$

### 3.3 Fix Start Preimage Resistance

The conclusions of this subsection are Theorem3 and Theorem4.

**Theorem 3.** *If  $\widetilde{Q}_{\{X|Y_y K_k\}^F} = 1 \wedge \widetilde{Q}_{\{K|Y_y X_x\}^F} = \frac{2^\kappa}{q_x}$  then*

$$\widetilde{Q}_{\{XK_\Gamma Y_y\}_{y \in \{0,1\}^n}} = 1. \quad (3)$$

$$\widetilde{Q}_{\{X_\Lambda KY_y\}_{y \in \{0,1\}^n}} \geq \frac{2^\kappa}{\sum_{x \in \Lambda} q_x} \quad (4)$$

*Proof.*

$$\begin{aligned} \widetilde{Q}_{\{XK_\Gamma Y_y\}^F} &\leq \widetilde{Q}_{\{X|Y_y K_\Gamma\}^F} \leq \widetilde{Q}_{\{X|Y_y K_k\}^F} = 1. \\ \widetilde{Q}_{\{X_\Lambda KY_y\}^F} &\geq \frac{2^\kappa}{\sum_{x \in \Lambda} q_x}, \quad \widetilde{Q}_{\{K|X_\Lambda Y_y\}^F} \geq \min_{x \in \Lambda} \frac{2^\kappa}{q_x} \end{aligned}$$

**Theorem 4.** *If  $\widetilde{Q}_{\{K|Y_y X_x\}^F} = 1 \wedge \widetilde{Q}_{\{X|Y_y K_k\}^F} = \frac{2^l}{q_k}$  then:*

$$\widetilde{Q}_{\{X_\Lambda KY_y\}_{y \in \{0,1\}^n}} = 1. \quad (5)$$

$$\widetilde{Q}_{\{XK_\Gamma Y_y\}_{y \in \{0,1\}^n}} \geq \sum_{k \in \Gamma} \frac{2^l}{q_k} \quad (6)$$

### 3.4 Fix Start Collision Resistance

The conclusion of this subsection are Theorem5 and Theorem6, which tell us the best design of  $F$  also should satisfy  $Y$  is uniformly distributed in  $\{0, 1\}^n$  for each  $k \in \{0, 1\}^\kappa$  and for each  $x \in \{0, 1\}^\iota$ .

**Theorem 5.** *If  $\tilde{Q}_{\{X|Y_y K_k\}^F} = 1 \wedge \tilde{Q}_{\{K|Y_y X_x\}^F} = \frac{2^\kappa}{q_x}$  then:*

$$\hat{Q}_{\{XK_\Gamma Y\}^F} = \begin{cases} 2 & |\Gamma| > 1 \text{ or } q_k > 1 \\ 0 & \text{else} \end{cases} \quad (7)$$

$$\hat{Q}_{\{X_\Lambda KY\}^F} \geq \max_{x \in \Lambda} \left\{ \sqrt{\frac{2^\kappa}{(q_x - 1)}}, \frac{2^\kappa}{\sum_{x \in \Lambda} q_x - 1}, \sqrt{\frac{2^{\kappa|\Lambda|}}{\sum_{x \in \Lambda} q_x - 1}} \right\}. \quad (8)$$

*Proof.*  $\hat{Q}_{\{XK_\Gamma Y\}^F} \leq \hat{Q}_{\{X|K_\Gamma Y_y\}^F} = 2$

- Since  $\tilde{Q}_{\{K|Y_y X_x\}^F} = \frac{2^\kappa}{q_x}$ , for  $\hat{Q}_{\{K|X_x\}^F}$ , the fastest way to get collision is for random select a  $k_1, \dots, k_t$  getting  $y = F_{k_i}(x)$ , checking  $F_{k_i}(x) = F_{k_j}(x)$  equals or not, similar as proof of Lemma2,  $\hat{Q}_{\{K|X_x\}^F} = \sqrt{\frac{2^\kappa}{(q_x - 1)}}$ .
- if  $|\Lambda| > 1$ ,  $\hat{Q}_{K|X_\Lambda}$ , for given  $x, x'$  the fastest way to find  $k, k'$  is random select  $k_1, k_2, \dots$ , compute  $y_i = F_{k_i}(x), y'_i = F_{k_i}(x')$  then check  $y_i$  equals  $y'_i$  or not, since  $\tilde{Q}_{\{K|Y_y X_x\}^F} = \frac{2^\kappa}{q_x}$ , from Lemma2 we get  $\hat{Q}_{K|X_\Lambda} \leq \sqrt{\frac{2^{\kappa|\Lambda|}}{\sum_{x \in \Lambda} q_x - 1}}$ .
- for  $\hat{Q}_{\{X_\Lambda|K\}^F}$ ,  $\tilde{Q}_{\{X|Y_y K_k\}^F} = 1 \Rightarrow \hat{Q}_{\{X|K_k Y_{F_k(x_0)}\}^F} = 1$  the probability of  $x \in \Lambda, x \neq x_0$  is at most  $\sum_{x \in \Lambda} \frac{q_x - 1}{2^\kappa}$ , we have  $\hat{Q}_{\{X_\Lambda|K\}^F} = \frac{2^\kappa}{\sum_{x \in \Lambda} q_x - 1}$ .  $\square$

**Theorem 6.** *If  $\tilde{Q}_{\{K|Y_y X_x\}^F} = 1 \wedge \tilde{Q}_{\{X|Y_y K_k\}^F} = \frac{2^\iota}{q_k}$  then:*

$$\hat{Q}_{\{X_\Lambda KY\}^F} = \begin{cases} 2 & |\Lambda| > 1 \text{ or } q_x > 1 \\ 0 & \text{else} \end{cases} \quad (9)$$

$$\hat{Q}_{\{XK_\Gamma Y\}^F} \geq \max_{k \in \Gamma} \left\{ \sqrt{\frac{2^\iota}{(q_k - 1)}}, \frac{2^\iota}{\sum_{k \in \Gamma} q_k - 1}, \sqrt{\frac{2^{\iota|\Gamma|}}{\sum_{k \in \Gamma} q_k - 1}} \right\}. \quad (10)$$

## 4 Hash Properties of Iterated Structure

In this section, we give the proves of that if the compression function is free start preimage resistant and collision resistant, then the hash function is free start preimage resistant and but not free start collision resistant, if the compression function is fix start collision resistant and preimage resistant then the



hash function is fix start collision resistant and preimage resistant, and also the upper bounds of collision resistance and preimage resistance are given based on the condition probabilities  $P_{Y|X=x}(y)$  and  $P_{Y|K=k}(y)$  of compression function  $F$ . And also if the compression function is not immune to free start preimage resistant, then the compression function should be designed with minimum values of  $P_{Y|X=x}(y)$  and  $P_{Y|K=k}(y)$ , which imply the best design require the  $Y$  is uniformly distributed in  $\{0, 1\}^n$  for each  $k$  and each  $x$ , if  $n = \kappa = m$  then the best design of compression function is permutation for each  $k$  and each  $x$ .

**Lemma 3.** *Let  $F : \{0, 1\}^n \times \{0, 1\}^\kappa \rightarrow \{0, 1\}^n$ ,  $y = F_k(x)$ ,  $y \in \{0, 1\}^n$ ,  $x \in \{0, 1\}^n$ ,  $H_X : \{0, 1\}^n \times \{0, 1\}^{\kappa \cdot t} \rightarrow \{0, 1\}^n$ ,  $z = F_{m_t}(\dots F_{m_1}(x) \dots)$ ,  $z \in \{0, 1\}^n$ ,  $m = m_t \parallel \dots \parallel m_1 \in \{0, 1\}^{\kappa \cdot t}$  and  $m_1, \dots, m_t$  are independent from each other then:*

$$\begin{aligned} - P_{Z|M=m}(z) &\leq q_k^t 2^{-n} \\ - P_{Z|X=x}(z) &\leq \frac{q_x}{2^\kappa}. \end{aligned}$$

*Proof.* It is clear  $t = 1$  the inequality is correct, when  $t = 2$ :

$$\begin{aligned} P_{Z|M=m}(z) &= P_{Z|M=m_2 \parallel m_1}(z) \\ &= \sum_x P_X(x) P_{Z|M=m_2 \parallel m_1, X=x}(z = F_{m_2}(F_{m_1}(x))) \\ &= \sum_x \sum_u P_X(x) P_{Z|M=m_2 \parallel m_1, X=x}(z = F_{m_2}(u), u = F_{m_1}(x)) \\ &= \sum_u P_{Z|M_2=m_2, U=u}(z = F_{m_2}(u)) \sum_x P_X(x) P_{U|M_1=m_1, X=x}(u = F_{m_1}(x)) \\ &= \sum_u P_{Z|M_2=m_2, U=u}(z = F_{m_2}(u)) P_{U|M_1=m_1}(u) \\ &\leq q_k \sum_u \frac{1}{2^n} P_{Z|M_2=m_2, U=u}(z = F_{m_2}(u)) \leq q_k P_{Z|M_2=m_2}(z) \end{aligned}$$

$$\begin{aligned} P_{Z|X=x}(z) &= \sum_{m_1, m_2} P_M(m_1) P_M(m_2) P_{Z|M=m_2 \parallel m_1, X=x}(z = F_{m_2}(F_{m_1}(x))) \\ &= \sum_{m_1, m_2} \sum_u P_M(m_1) P_M(m_2) P_{Z|M=m_2 \parallel m_1, X=x}(z = F_{m_2}(u), u = F_{m_1}(x)) \\ &= \sum_{m_2} \sum_u P_M(m_2) P_{Z|M_2, U=u}(z = F_{m_2}(u)) \sum_{m_1} P_M(m_1) P_{U|M_1, X}(u = F_{m_1}(x)) \\ &= \sum_{m_2} \sum_u P_M(m_2) P_{Z|M_2, U=u}(z = F_{m_2}(u)) P_{U|X=x}(u) \\ &= \sum_u P_{Z|U=u}(z) P_{U|X=x}(u) \leq \frac{q_x}{2^\kappa} \sum_u P_{U|X=x}(u) = q_x / 2^\kappa. \end{aligned}$$

Let assume when  $t \leq l - 1$  the inequality is true, when  $t = l$

$$\begin{aligned} P_{Z|M=m}(z) &= \sum_x P_X(x) P_{Z|M'=m' \| m_1, X=x}(z = H_X(m', F_{m_1}(x))) \\ &= \sum_u P_{Z|M'=m', U=u}(z = H_X(m', u)) P_{U|M_1=m_1}(u) \\ &\leq q_k \sum_u \frac{1}{2^n} P_{Z|M'=m', U=u}(z = H_X(m', u)) \leq q_k^l 2^{-n} \end{aligned}$$

$$\begin{aligned} P_{Z|X=x}(z) &= \sum_{m', m_1} P_M(m') P_M(m_1) P_{Z|M=m' \| m_1, X=x}(z = H_X(m', F_{m_1}(x))) \\ &= \sum_{m', m_1, u} P_{M'}(m') P_M(m_1) P_{Z|M=m' \| m_1, X=x, U=u}(z = H_X(m', u), u = F_{m_1}(x)) \\ &= \sum_{m'} \sum_u P_{M'}(m') P_{Z|M'=m', U=u}(z = H(m', u)) P_{U|X=x}(u) \\ &= \sum_u P_{Z|U=u}(z) P_{U|X=x}(u) \leq \frac{q_x}{2^\kappa} \sum_u P_{U|X=x}(u) = \frac{q_x}{2^\kappa}. \end{aligned}$$

From induction principle we get the conclusions.  $\square$

**Lemma 4.** Let  $F : \{0, 1\}^l \times \{0, 1\}^n \rightarrow \{0, 1\}^n$ ,  $H_K : \{0, 1\}^{l-t} \times \{0, 1\}^n \rightarrow \{0, 1\}^n$ ,  $k \in \{0, 1\}^n$ ,  $m \in \{0, 1\}^{l-t}$ ,  $y \in \{0, 1\}^n$ ,  $z \in \{0, 1\}^n$ ,  $y = F_k(x)$ ,  $z = F_{\dots(F_{F_k(m_1)}(m_2)) \dots}(m_t)$  and  $m_1, \dots, m_t$  are independent from each other then:

- $P_{Z|M=m}(z) \leq \max_x q_x^t 2^{-n}$
- $P_{Z|K=k}(z) \leq \max_k \frac{q_k}{2^t}$ .

*Proof.* It is clear  $t = 1$  the inequations are correct. Let assume when  $t \leq l - 1$  the inequations are correct.

$$\begin{aligned} P_{Z|M=m}(z) &= \sum_k P_K(k) P_{Z|M'=m' \| m_1, K=k}(z = H_K(m', F_k(m_1))) \\ &= \sum_u P_{Z|M'=m', U=u}(z = H_K(m', u)) P_{U|M_1=m_1}(u) \\ &\leq q_x \sum_u \frac{1}{2^n} P_{Z|M'=m', U=u}(z = H_X(m', u)) \leq q_x^l 2^{-n} \end{aligned}$$

$$\begin{aligned} P_{Z|K=k}(z) &= \sum_{m', m_1} P_M(m') P_M(m_1) P_{Z|M=m' \| m_1, K=k}(z = H_K(m', F_k(m_1))) \\ &= \sum_{m', m_1, u} P_{M'}(m') P_M(m_1) P_{Z|M=m' \| m_1, K=k, U=u}(z = H_K(m', u), u = F_k(m_1)) \\ &= \sum_{m'} \sum_u P_{M'}(m') P_{Z|M'=m', U=u}(z = H_K(m', u)) P_{U|K=k}(u) \\ &= \sum_u P_{Z|U=u}(z) P_{U|K=k}(u) \leq q_k 2^{-t} \sum_u P_{U|K=k}(u) = \frac{q_k}{2^t}. \end{aligned}$$

From induction principle we get the conclusions.  $\square$

**Theorem 7.** *If  $F : \{0, 1\}^n \times \{0, 1\}^\kappa \rightarrow \{0, 1\}^n$  is preimage resistant and collision resistant,  $H_X : \{0, 1\}^n \times \{0, 1\}^{\kappa \cdot t} \rightarrow \{0, 1\}^n$ ,  $x \in \{0, 1\}^n$ ,  $m \in \{0, 1\}^{\kappa \cdot t}$ ,  $y \in \{0, 1\}^n$ ,  $z \in \{0, 1\}^n$ ,  $y = F_k(x)$  and  $z = F_{m_t}(\dots F_{m_1}(x)\dots)$  then:*

$$\tilde{Q}_{\{MXZz\}_{z \in \{0,1\}^n}}^{H_X} \geq \max_{x,k} \left\{ \frac{2^\kappa}{q_x}, \frac{2^n}{q_k} \right\} + 1 \quad (11)$$

$$\hat{Q}_{\{MXZ\}^{H_X}} = 2 \quad (12)$$

*Proof.*

- Let assume for given  $y$  we find  $m, x$  satisfying  $H_X(m_t \| \dots \| m_1, x) = y$  then we find  $H_X(m_{t-1} \| \dots \| m_1, x)$ ,  $m_t$  satisfying  $F_{m_t}(H_X(m_{t-1} \| \dots \| m_1, x)) = y$ , from Theorem 1 we get the conclusion.
- We get  $H_X(m_2 \| m_1, x) = H_X(m_2, H_X(m_1, x))$ , then we find collision.  $\square$

**Theorem 8.** *If  $F : \{0, 1\}^t \times \{0, 1\}^n \rightarrow \{0, 1\}^n$ , is preimage resistant and collision resistant,  $H_K : \{0, 1\}^{t \cdot t} \times \{0, 1\}^n \rightarrow \{0, 1\}^n$ ,  $k \in \{0, 1\}^n$ ,  $m \in \{0, 1\}^{t \cdot t}$ ,  $y \in \{0, 1\}^n$ ,  $z \in \{0, 1\}^n$ ,  $y = F_k(x)$  and  $z = F_{\dots(F_{F_k(m_1)}(m_2))\dots}(m_t)$  then:*

$$\tilde{Q}_{\{MKZz\}_{z \in \{0,1\}^n}}^{H_K} \geq \min_{x,k} \left\{ \frac{2^\kappa}{q_x}, \frac{2^n}{q_k} \right\} + 1 \quad (13)$$

$$\hat{Q}_{\{MKZ\}^{H_K}} = 2 \quad (14)$$

**Theorem 9.** *If  $F : \{0, 1\}^n \times \{0, 1\}^\kappa \rightarrow \{0, 1\}^n$ ,  $H_X : \{0, 1\}^n \times \{0, 1\}^{\kappa \cdot *} \rightarrow \{0, 1\}^n$ ,  $x \in \{0, 1\}^n$ ,  $m \in \{0, 1\}^{\kappa \cdot *}$ ,  $y \in \{0, 1\}^n$ ,  $z \in \{0, 1\}^n$ ,  $y = F_k(x)$ ,  $z = F_{m_*}(\dots F_{m_1}(x)\dots)$ ,  $m_1, \dots, m_*$  are independent from each other,*

- if  $\tilde{Q}_{\{K|Y_y, X_x\}^F} = 1 \wedge \tilde{Q}_{\{X|Y_y, K_k\}^F} = \frac{2^t}{q_k}$  then

$$\tilde{Q}_{\{X_\Lambda MZz\}_{z \in \{0,1\}^n}}^{H_X} = \frac{|M|}{\kappa}, \quad \hat{Q}_{\{X_\Lambda MZ\}^{H_X}} = \frac{|M| + |M'|}{\kappa}$$

- if  $\tilde{Q}_{\{X|Y_y, K_k\}^F} = 1 \wedge \tilde{Q}_{\{K|Y_y, X_x\}^F} = \frac{2^\kappa}{q_x}$ ,  $\Lambda' \triangleq \{H_X(m', x), x \in \Lambda\} \cup \Lambda$  then:

$$\tilde{Q}_{\{X_\Lambda MZz\}_{z \in \{0,1\}^n}}^{H_X} \geq \min \left\{ \frac{2^\kappa}{\sum_{x \in \Lambda} q_x}, \sqrt{\frac{2^\kappa}{q_x}}, \frac{2^n}{q_k^{\frac{|M|}{\kappa}}} \right\} \quad (15)$$

$$\hat{Q}_{\{X_\Lambda MZ\}^{H_X}} \geq \max_{x \in \Lambda, k} \left\{ \frac{2^n}{q_k^{\frac{|M|}{\kappa}}}, \sqrt{\frac{2^\kappa}{(q_x - 1)}}, \frac{2^\kappa}{\sum_{x \in \Lambda} q_x - 1}, \sqrt{\frac{2^{\kappa|\Lambda'|}}{\sum_{x \in \Lambda'} q_x - 1}} \right\} \quad (16)$$

*Proof.*

– The conclusions  $\tilde{Q}_{\{X_\Lambda M Z_z\}_{z \in \{0,1\}^n}}^{H_X} = \frac{|M|}{\kappa}$ ,  $\hat{Q}_{\{X_\Lambda M Z\}^{H_X}} = \frac{|M|+|M'|}{\kappa}$  can be get by the direct computation, since  $\tilde{Q}_{\{K|Y_y X_x\}^F} = 1$ .

– there are two ways to find the preimage:

- Case 1 : Finding the preimage of  $z$ , directly searching  $m \in \{0,1\}^{\kappa \cdot *}$  satisfying  $z = H_X(m, x)$  where  $x \in \Lambda$ .  $\tilde{Q}_{\{K|Y_y X_x\}^F} = \frac{2^\kappa}{q_x}$  implies for given  $z, x$  the only way of finding  $m$  satisfying  $z = H_X(m, x)$  is exhaustive search, more precisely,  $\tilde{Q}_{\{M|Z_z X_x\}^F} = \frac{2^\kappa |M|}{q_x \kappa}$ . From Lemma3 and Theorem3 we get  $\tilde{Q}_{\{M Z_z X_\Lambda\}^F} \leq \sum_{x \in \Lambda} \frac{2^\kappa}{q_x}$ .
- Case 2 : for given  $z$ , search  $m' \in \{0,1\}^{\kappa \cdot t'}$ ,  $m'' \in \{0,1\}^{\kappa \cdot t''}$ , satisfying  $z = H_X(m'', u)$  and  $u = H_X(m', x)$  where  $x \in \Lambda$ :
  - \* Select  $m'$  randomly, searching  $m''$ , let  $\Lambda' \triangleq \{H_X(m', x), x \in \Lambda\}$ , the problem become case 1;
  - \* Select  $m''$  randomly, get  $u$  from  $z = H_X(m'', u)$ , then searching  $m'$  satisfying  $u = H_X(m', x)$ , equals finding the preimage of  $u$ ;
  - \* Guessing  $m'$  and  $m''$ , compute  $u$  and  $u'$  from  $u = H_X(m', x)$  and  $z = H_X(m'', u')$ , let  $t = |m''|$ , the probability of  $u = u'$  smaller than  $\max\{q_k^t 2^{-n}, \sqrt{\frac{2^\kappa}{q_x}}\}$ , more precisely, if the compression function is designed with property of that,  $\exists \dot{z} \in \{0,1\}^n$ ,  $\dot{m} \in \{0,1\}^{\kappa t}$  satisfy  $P_{Z|M=\dot{m}}(\dot{z}) = q_k^t$  and  $q_k > 1$ , then the complexity of finding preimage of  $\dot{z}$  is  $\frac{q_k^t}{2^n}$ , where we search  $m$  satisfy  $\dot{z} = H_X(\dot{m}||m, x)$ .

From Case 1 and Case 2, we get the conclusion.

– there are three ways to find the collision, let :

- Case 1: Finding collision of  $H_X$  means searching  $m' \in \{0,1\}^{\kappa \cdot t'}$ ,  $m'' \in \{0,1\}^{\kappa \cdot t''}$  satisfying  $H_X(m', x) = H_X(m'', x)$  with  $x \in \Lambda$ .  $\tilde{Q}_{\{K|Y_y X_x\}^F} = \frac{2^\kappa}{q_x}$  implies for given  $z, x$  the only way of finding  $m$  satisfying  $z = H_X(m, x)$  is exhaustive search, more precisely,  $\tilde{Q}_{\{M|Z_z X_x\}^F} = \frac{2^\kappa \cdot \kappa}{q_x |M|}$ . From Lemma3 and Theorem5 we get the conclusion:  $\hat{Q}_{\{X_\Lambda M Z\}^{H_X}} \leq \max_{x \in \Lambda} \left\{ \sqrt{\frac{2^\kappa}{(q_x-1)}}, \sum_{x \in \Lambda} \frac{2^\kappa}{q_x-1}, \sqrt{\frac{2^{\kappa|\Lambda|}}{\sum_{x \in \Lambda} q_x-1}} \right\}$ .
- Case 2: search  $m \in \{0,1\}^{\kappa \cdot t}$ ,  $m' \in \{0,1\}^{\kappa \cdot t'}$ ,  $m'' \in \{0,1\}^{\kappa \cdot t''}$ , satisfying  $H_X(m, x) = H_X(m'', u)$  and  $u = H_X(m', x)$  where  $x \in \Lambda$ :
  - \* if we randomly select  $m$  searching  $m', m''$ , the problem becomes finding a primage of  $z = H_X(m, x)$ ;
  - \* If we randomly select  $m'$  get  $u$  from  $u = H_X(m', x)$ , then search  $m$  and  $m''$  satisfying  $H(m, x) = H_X(m'', u)$ , let  $\Lambda' \triangleq \{H_X(m', x), x \in \Lambda\} \cup \Lambda$ , the problem become case 1 where  $x \in \Lambda'$ ;
  - \* If we randomly select  $m''$  search  $m, m'$  checking  $H_X(m'', H_X(m', x)) = H_X(m, x)$  being satisfied or not which needs more computation than for given  $m''$  finding  $z$  and  $m'$  satisfying  $z = H_X(m'', H_X(m', x))$ .
- Case 3: search  $m \in \{0,1\}^{\kappa \cdot t}$ ,  $m' \in \{0,1\}^{\kappa \cdot t'}$ ,  $\bar{m} \in \{0,1\}^{\kappa \cdot \bar{t}}$ ,  $\bar{m}' \in \{0,1\}^{\kappa \cdot \bar{t}'}$  satisfying  $H_X(m', H_X(m, x)) = H_X(\bar{m}', H_X(\bar{m}, x))$  where  $x \in \Lambda$ , similar as case 2, case 3 needs more computation than case 2.

From Case 1, Case 2 and Case 3, we get the conclusion.  $\square$

**Theorem 10.** Let  $F : \{0, 1\}^n \times \{0, 1\}^\kappa \rightarrow \{0, 1\}^n$ ,  $H_K : \{0, 1\}^{\iota-t} \times \{0, 1\}^n \rightarrow \{0, 1\}^n$ ,  $k \in \{0, 1\}^n$ ,  $m \in \{0, 1\}^{\iota-t}$ ,  $y \in \{0, 1\}^n$ ,  $z \in \{0, 1\}^n$ ,  $y = F_k(x)$ ,  $z = F_{\dots(F_{F_k(m_1)}(m_2))\dots}(m_t)$  and  $m_1, \dots, m_t$  are independent from each other then:

– if  $\tilde{Q}_{\{X|Y_y K_k\}^F} = 1 \wedge \tilde{Q}_{\{K|Y_y X_x\}} = \frac{2^n}{q_x}$  then

$$\tilde{Q}_{\{K_\Gamma M Z_z\}_{z \in \{0,1\}^n}^{H_K}} = \frac{|M|}{\iota}, \quad \hat{Q}_{\{K_\Gamma M Z\}^{H_K}} = \frac{|M| + |M'|}{\iota}$$

– if  $\tilde{Q}_{\{K|Y_y X_x\}} = 1 \wedge \tilde{Q}_{\{X|Y_y K_k\}} = \frac{2^t}{q_x}$ ,  $\Gamma' \triangleq \{H_X(m', x), x \in \Gamma\} \cup \Gamma$  then:

$$\tilde{Q}_{\{K_\Gamma M Z_z\}_{z \in \{0,1\}^n}^{H_K}} = \min_{k \in \Gamma} \left\{ \frac{2^\kappa}{\sum_{k \in \Gamma} q_k}, \sqrt{\frac{2^\kappa}{q_k}}, \frac{2^n}{q_x^\kappa} \right\} \quad (17)$$

$$\hat{Q}_{\{K_\Gamma M Z\}^{H_K}} \leq \max_{k \in \Gamma, x} \left\{ \frac{2^n}{q_x^\kappa}, \sqrt{\frac{2^\kappa}{(q_k - 1)}}, \frac{2^\kappa}{\sum_{k \in \Gamma} q_k - 1}, \sqrt{\frac{2^{\kappa|\Gamma'|}}{\sum_{k \in \Gamma'} q_k - 1}} \right\} \quad (18)$$

**Theorem 11.** Let  $F : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}^n$ ,  $H_K : \{0, 1\}^{n-t} \times \{0, 1\}^n \rightarrow \{0, 1\}^n$ ,  $H_X : \{0, 1\}^{n-t} \times \{0, 1\}^n \rightarrow \{0, 1\}^n$ ,  $m = m_t \| \dots \| m_1$ ,  $m_1, \dots, m_t$  are independent from each other.

$$\tilde{Q}_{\{K_\Gamma M Z_z\}_{z \in \{0,1\}^n}^{H_K}} \geq \min_{k \in \Gamma} \left\{ \frac{2^n}{q_k}, \frac{2^n}{q_x^n} \right\} \quad (19)$$

$$\hat{Q}_{\{K_\Gamma M Z\}^{H_K}} \geq \min_{k \in \Gamma, x} \left\{ \frac{2^n}{q_x^n}, \sqrt{\frac{2^n}{(q_k - 1)}}, \sqrt{\frac{2^{n|\Gamma'|}}{\sum_{k \in \Gamma'} q_k - 1}} \right\} \quad (20)$$

Theorem11 tell us on condition of the compression function  $F$  is free start primage resistant and free start collision resistant, the best design of  $H_X$  and  $H_K$  have properties of  $q_k = 1$  and  $q_x = 1$ .

## 5 Collision Resistance of PGV Schemes

We assume block cipher  $E : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}^n$  has no weakness, that means for given  $y, x$  no ways to finding  $k$  except exhaustive search, we also assume  $P_{Y|X=x}(y) = P_{Y|K=k}(y) = \frac{1}{2^n}$ , the block cipher  $E$  with perfect security and perfect key distribution. The security of 24 PGV schemes is summarized in tables 1 where we give up to consider the constant value  $v$ , in black box model the value  $v$  does not influence the security of compression function and hash function. The functions are numbered in BRS[12], where the  $F_1 \sim F_{12}$

are the group one schemes which is immune to free start collision resistance and  $F_{13} \sim F_{20}$  are the group two schemes which are not immune to free start collision resistance and immune to fix start collision resistance. In fact those 24 schemes are derive from 12 compression function with different fix start and four of which are not immune to fix start collision resistance.

**Theorem 12.** *If block cipher  $E$  is a random oracle model with perfect security and perfect key distribution,  $P_{Y|K=k}(y), P_{Y|X=x}(y)$  of  $F_i$   $1 \leq i \leq 24$  are equal to that of  $E$ .*

*Proof.* Since  $y = E_k(x)$  is random orale,  $x, k$  and  $E_k(x)$  are independent from each other. We give the prove of the most famous mode  $y' = E_k(x) \oplus x$ .

$$\begin{aligned}
P_{Y|K=k}(y) &= \sum_x P_X(x) P_{Y|X=x, K=k}(y = E_k(x) \oplus x) \\
&= \sum_{x, x'} P_X(x) P_{X'}(x') P_{Y|X=x, K=k, X'=x'}(y = E_k(x) \oplus x') \\
&= \sum_{x, x'} \sum_t P_X(x) P_{Y|X=x, T=t}(y = x \oplus t) P_{X'}(x') P_{T|X'=x', K=k}(t) \\
&= \sum_x \sum_t P_X(x) P_{Y|X=x, T=t}(y = x \oplus t) P_{T|K=k}(t) \\
&= p_k \sum_x \sum_t P_X(x) P_{Y|X=x, T=t}(y = x \oplus t) = p_k
\end{aligned}$$

$$\begin{aligned}
P_{Y|X=x}(y) &= \sum_k P_K(k) P_{Y|X=x, K=k}(y = E_k(x) \oplus x) \\
&= \sum_k \sum_t P_K(k) P_{Y|X=x, K=k}(y = x' \oplus t; t = E_k(x)) \\
&= \sum_k \sum_t P_K(k) P_{Y|X=x, T=t}(y = x \oplus t) P_{T|X=x, K=k}(t) \\
&= \sum_t P_{Y|X=x, T=t}(y = x \oplus t) p_x = p_x
\end{aligned}$$

The prove of mode 13 is:

$$\begin{aligned}
P_{Y|K=k}(y) &= \sum_x P_X(x) P_{Y|X=x, K=k}(y = E_{k \oplus x}(x')) \\
&= \sum_{x, t} P_X(x) P_{Y|X=x, K=k}(t = k \oplus x; y = E_t(x')) \\
&= p_k \sum_t P_{Y|X=x, K=k}(t = k \oplus x) = p_k \\
P_{Y|X=x}(y) &= \sum_k P_K(k) P_{Y|X=x, K=k}(y = E_{k \oplus x}(x)) \\
&= \sum_{k, t} P_K(k) P_{Y|X=x, K=k}(t = k \oplus x; y = E_t(x')) = p_x
\end{aligned}$$

other modes can be proved in similar way.  $\square$

**Theorem 13.**  $F_i, 1 \leq i \leq 12$  are free start preimage resistant and free start collision resistant.

**Theorem 14.**  $F_i, 1 \leq 13 \leq 24$  are not free start preimage resistant and not free start collision resistant.

*Proof.*

- $F_{13}:\forall y, k$  compute  $x = E_k^{-1}(y)$ , let  $k' = x \oplus k$  then  $E_{k' \oplus x}(x) = y$ ;
- $F_{14}:\forall y, k$  compute  $x = E_k^{-1}(y \oplus k)$ , let  $k' = x \oplus k$  then  $E_{k' \oplus x}(x) \oplus k = y$ ;
- $F_{15}:\forall y, k$  compute  $x = E_k^{-1}(y \oplus k)$ , let  $k' = x, x' = k$  then  $E_{k'}(x') = y$ ;
- $F_{16}:\forall y, k$  compute  $x = E_k^{-1}(y)$ , let  $k' = x, x' = k \oplus x$  then  $E_{k' \oplus x'}(k') = y$ ;
- $F_{17}:\forall y, k$  compute  $x = E_k^{-1}(y \oplus k)$ , let  $k' = x, x' = k$  then  $E_{x'}(k') \oplus x' = y$ ;
- $F_{18}:\forall y, k$  compute  $x = E_k^{-1}(y \oplus k)$ , let  $k' = x, x' = k \oplus x$  then  $E_{x' \oplus k'}(k') \oplus x' \oplus k' = y$ ;
- $F_{19}:\forall y, k$  compute  $x = E_k^{-1}(y)$ , let  $k' = x \oplus k, x' = k$  then  $E_{x'}(x' \oplus k) = y$ ;
- $F_{20}:\forall y, k$  compute  $x = E_k^{-1}(y \oplus k)$ , let  $k' = x \oplus k$  then  $E_{k' \oplus x}(x) \oplus k = y$ ;
- $F_{21}:\forall y, k$  compute  $x = E_k^{-1}(y)$  then  $E_k(x) = y$ ;
- $F_{22}:\forall y, k$  compute  $x = E_k^{-1}(y \oplus k)$  then  $E_k(x) \oplus k = y$ ;
- $F_{23}:\forall y, k$  compute  $x = E_k^{-1}(y)$ , let  $x' = x \oplus k$  then  $E_k(x' \oplus k) = y$ ;
- $F_{24}:\forall y, k$  compute  $x = E_k^{-1}(y \oplus k)$ , let  $x' = k \oplus k$  then  $E_k(x' \oplus k) \oplus k = y$ .  $\square$

**Theorem 15.** If block cipher  $E$  is a random oracle model with perfect security and perfect key distribution the hash functions  $H_X$  are fix start collision resistant and fix start preimage resistant where compression functions are  $F_i, i \in \{5 \sim 8, 10, 12, 15 \sim 20\}$  and  $H_K$  are fix start preimage resistant and fix start collision resistant where compression functions are  $F_i, i \in \{1 \sim 4, 9, 11, 13, 14\}$ .

The theorem12 needs block cipher is random oracle, or else ( $x$  and  $k$  are not independent from  $E_k(x)$ ). In fact,  $E$  is not a random oracle and only  $F_{15}, F_{21}$  with properties of  $P_{Y|X=x} = P_{Y|K=k} = \frac{1}{2^n}$ , which implies the best compression functions is  $F_{15}$ . So we have conclusion that if  $E$  is designed to be  $P_{Y|X=x} = P_{Y|K=k} = \frac{1}{2^n}$ , then the block cipher can be used to construct a secure hash function.

## 6 Conclusion

The main conclusion of this paper is that if no way to design the compression  $F(k, x)$  immune to free start preimage resistant, then the best design of compression function is a block cipher with perfect key distribution and perfect security where the hash function has M-D structure. So the design of block cipher and hash function can be one problem and the design of key schedule algorithm of block cipher become important than before.

**Table 1.** Summary of results. Column1 is the number of hash function which is given by BRS[12]. Column2 are the compression functions to build hash function. Column3,4,5 are the probabilities of finding the preimages in one time computation of  $F$ . Column 6 are the hash function and column 7 are the compression function  $F_i$ . We write  $w_i = m_i \oplus h_{i-1}$ .

$i$	$y =$	$Q_{K XY}$	$Q_{X KY}$	$Q_{XK Y}$	$z =$	$h_i =$
15	$E_k(x)$	$p_k^{\frac{1}{2}}$	1	1	$H_X$	$E_{m_i}(h_{i-1}) \oplus v$
21	$E_k(x)$				$H_K$	$E_{h_{i-1}}(m_i) \oplus v$
19	$E_k(x \oplus k)$	$p_k^{\frac{1}{2}}$	1	1	$H_X$	$E_{m_i}(w_i) \oplus v$
23	$E_k(x \oplus k)$				$H_K$	$E_{h_{i-1}}(w_i) \oplus v$
5	$E_k(x) \oplus x$	$p_k^{\frac{1}{2}}$	$p_x^{\frac{1}{2}}$	$p^{\frac{1}{2}}$	$H_X$	$E_{m_i}(h_{i-1}) \oplus h_{i-1}$
1	$E_k(x) \oplus x$				$H_K$	$E_{h_{i-1}}(m_i) \oplus m_i$
17	$E_k(x) \oplus k$	$p_k^{\frac{1}{2}}$	1	1	$H_X$	$E_{m_i}(h_{i-1}) \oplus m_i$
22	$E_k(x) \oplus k$				$H_K$	$E_{h_{i-1}}(m_i) \oplus h_{i-1}$
7	$E_k(x) \oplus x \oplus k$	$p_k^{\frac{1}{2}}$	$p_x^{\frac{1}{2}}$	$p^{\frac{1}{2}}$	$H_X$	$E_{m_i}(h_{i-1}) \oplus w_i$
3	$E_k(x) \oplus x \oplus k$				$H_K$	$E_{h_{i-1}}(m_i) \oplus w_i$
8	$E_k(x \oplus k) \oplus x$	$p_k^{\frac{1}{2}}$	$p_x^{\frac{1}{2}}$	$p^{\frac{1}{2}}$	$H_X$	$E_{m_i}(w_i) \oplus h_{i-1}$
4	$E_k(x \oplus k) \oplus x$				$H_K$	$E_{h_{i-1}}(w_i) \oplus m_i$
20	$E_k(x \oplus k) \oplus k$	$p_k^{\frac{1}{2}}$	1	1	$H_X$	$E_{m_i}(w_i) \oplus m_i$
24	$E_k(x \oplus k) \oplus k$				$H_K$	$E_{h_{i-1}}(w_i) \oplus h_{i-1}$
6	$E_k(k \oplus x) \oplus x \oplus k$	$p_k^{\frac{1}{2}}$	$p_x^{\frac{1}{2}}$	$p^{\frac{1}{2}}$	$H_X$	$E_{m_i}(w_i) \oplus w_i$
2	$E_k(k \oplus x) \oplus x \oplus k$				$H_K$	$E_{h_{i-1}}(w_i) \oplus w_i$
16	$E_{k \oplus x}(x)$	$p_k^{\frac{1}{2}}$	$p_x^{\frac{1}{2}}$	1	$H_X$	$E_{w_i}(h_{i-1}) \oplus v$
13	$E_{k \oplus x}(x)$				$H_K$	$E_{w_i}(m_i) \oplus v$
10	$E_{k \oplus x}(x) \oplus x$	$p_k^{\frac{1}{2}}$	$p_x^{\frac{1}{2}}$	$p^{\frac{1}{2}}$	$H_X$	$E_{w_i}(h_{i-1}) \oplus h_{i-1}$
9	$E_{k \oplus x}(x) \oplus x$				$H_K$	$E_{w_i}(m_i) \oplus m_i$
12	$E_{k \oplus x}(x) \oplus k$	$p_k^{\frac{1}{2}}$	$p_x^{\frac{1}{2}}$	$p^{\frac{1}{2}}$	$H_X$	$E_{w_i}(h_{i-1}) \oplus m_i$
11	$E_{k \oplus x}(x) \oplus k$				$H_K$	$E_{w_i}(m_i) \oplus h_{i-1}$
18	$E_{k \oplus x}(x) \oplus x \oplus k$	$p_k^{\frac{1}{2}}$	$p_x^{\frac{1}{2}}$	$p^{\frac{1}{2}}$	$H_X$	$E_{w_i}(h_{i-1}) \oplus w_i$
14	$E_{k \oplus x}(x) \oplus x \oplus k$				$H_K$	$E_{w_i}(m_i) \oplus w_i$



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