# Another class of quadratic APN binomials over $\mathbb{F}_{2^n}$: the case $n$ divisible by 4

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**Abstract**

We exhibit an infinite class of almost perfect nonlinear quadratic binomials from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^n}$ with $n = 4k$ and $k$ odd. We prove that these functions are CCZ-inequivalent to known APN power functions when $k \neq 1$. In particular it means that for $n = 12, 20, 28$, they are CCZ-inequivalent to any power function.

**Keywords.** Affine equivalence, Almost bent, Almost perfect nonlinear, CCZ-equivalence, Differential uniformity, Nonlinearity, S-box, Vectorial Boolean function.

## 1 Introduction

A function $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is called APN if, for every $a \neq 0$ and every $b$ in $\mathbb{F}_2^n$, the equation $F(x)+F(x+a) = b$ admits at most two solutions (it is also called differentially 2-uniform). Vectorial Boolean functions used as S-boxes in block ciphers must have low differential uniformity to prevent from the differential cryptanalysis (see [5, 33]). In this sense almost perfect nonlinear (APN) functions are optimal. The notion of APN function is closely connected to the notion of almost bent (AB) function. A function $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is called AB if the minimum Hamming distance between all Boolean functions $v \cdot F$, $v \in \mathbb{F}_2^n \setminus \{0\}$ (where "$\cdot$" denotes the usual inner product in $\mathbb{F}_2^n$, note that any other choice of an inner product would lead to the same notion) and all affine Boolean functions on $\mathbb{F}_2^n$ is maximal (this distance is called the nonlinearity of $F$ and this maximum equals $2^{n-1} - 2^{\frac{n-1}{2}}$). AB functions oppose an optimum resistance to the linear cryptanalysis (see [31, 15]). Besides, every AB function is APN [15] and any quadratic APN function is AB [14].

Until recently the only known constructions of APN and AB functions were EA-equivalent to power functions over finite fields. Recall that functions $F$ and $F'$ are called extended affine equivalent (EA-equivalent) if $F' = A_1 \circ F \circ A_2 + A$, where the mappings

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$A, A_1, A_2$ are affine, and where $A_1, A_2$ are permutations. Table 1 gives all known values of exponents $d$ (up to multiplication by a power of 2 modulo $2^n - 1$, and up to taking the inverse when a function is a permutation) such that the power function $x^d$ over $\mathbb{F}_{2^n}$ is APN. For $n$ odd the Gold, Kasami, Welch and Niho APN functions from Table 1 are also AB (for the proofs of AB property see [11, 12, 25, 27, 29, 33]).

Table 1
Known APN power functions $x^d$ on $\mathbb{F}_{2^n}$.

| Functions | Exponents $d$ | Conditions | Proven in |
|---|---|---|---|
| Gold | $2^i + 1$ | $\gcd(i, n) = 1$ | [25, 33] |
| Kasami | $2^{2i} - 2^i + 1$ | $\gcd(i, n) = 1$ | [28, 29] |
| Welch | $2^t + 3$ | $n = 2t + 1$ | [20] |
| Niho | $2^t + 2^{\frac{t}{2}} - 1$, $t$ even<br>$2^t + 2^{\frac{3t+1}{2}} - 1$, $t$ odd | $n = 2t + 1$ | [19] |
| Inverse | $2^{2t} - 1$ | $n = 2t + 1$ | [4, 33] |
| Dobbertin | $2^{4t} + 2^{3t} + 2^{2t} + 2^t - 1$ | $n = 5t$ | [21] |

When using S-boxes EA-equivalent to power functions the advantage is the low implementation complexity in hardware environments. On the other hand the properties of power functions could be exploited in an attack (see [1]). A first well known property of a power permutation $F$ is that all its component functions $\mathrm{tr}(cF)$, $c \in \mathbb{F}_{2^n}^*$, are affine equivalent. A second consequence is that the rich algebraic structure of the field $\mathbb{F}_{2^n}$ can be extensively used, probably in a simpler manner for a power function than for a polynomial with many terms. The impact of the choice of power functions on algebraic attacks is another open question [16]. Probably, some of the potential weaknesses of S-boxes based on power functions can be avoided by using S-boxes EA-inequivalent or even CCZ-inequivalent (see below) to power mappings.

Applying the stability properties studied in [14] and more recently called CCZ-equivalence (cf. definition at Section 2), classes of APN functions EA-inequivalent to power functions are constructed in [9, 8]. They are presented in Table 2. When $n$ is odd these functions are also AB. However they are, by construction, CCZ-equivalent to Gold mappings.

Table 2
Known APN functions EA-inequivalent to power functions on $\mathbb{F}_{2^n}$.

| Functions | Conditions | Alg. degree |
|---|---|---|
| $x^{2^i+1} + (x^{2^i} + x\,\mathrm{tr}(1) + 1)\,\mathrm{tr}(x^{2^i+1} + x\,\mathrm{tr}(1))$ | $n \geq 4$<br>$\gcd(i, n) = 1$ | 3 |
| $[x + \mathrm{tr}_{n/3}(x^{2(2^i+1)} + x^{4(2^i+1)}) + \mathrm{tr}(x)\,\mathrm{tr}_{n/3}(x^{2^i+1} + x^{2^{2i}(2^i+1)})]^{2^i+1}$ | $n$ divisible by 6<br>$\gcd(i, n) = 1$ | 4 |
| $x^{2^i+1} + \mathrm{tr}_{n/m}(x^{2^i+1}) + x^{2^i}\,\mathrm{tr}_{n/m}(x) + x\ \mathrm{tr}_{n/m}(x)^{2^i}$<br>$+[\mathrm{tr}_{n/m}(x)^{2^i+1} + \mathrm{tr}_{n/m}(x^{2^i+1}) + \mathrm{tr}_{n/m}(x)]^{\frac{1}{2^i+1}}(x^{2^i} + \mathrm{tr}_{n/m}(x)^{2^i} + 1)$<br>$+[\mathrm{tr}_{n/m}(x)^{2^i+1} + \mathrm{tr}_{n/m}(x^{2^i+1}) + \mathrm{tr}_{n/m}(x)]^{\frac{2^i}{2^i+1}}(x + \mathrm{tr}_{n/m}(x))$ | $m \neq n$<br>$n$ odd<br>$n$ divisible by $m$<br>$\gcd(n, i) = 1$ | $m + 2$ |

The first examples of APN functions CCZ-inequivalent to power mappings are introduced in [24]. These are two quadratic binomials:

- $x^3 + wx^{36}$ over $\mathbb{F}_{2^{10}}$, where $w$ has the order 3 or 93,
- $x^3 + wx^{528}$ over $\mathbb{F}_{2^{12}}$, where $w$ has the order 273 or 585.

The second of these two functions has been proven being part of an infinite sequence of quadratic APN binomials given in Table 3 which represents by the only known classes of APN functions CCZ-inequivalent to power functions. Note that the first function from [24] is not explained yet by any infinite family.

Table 3
Known APN functions CCZ-inequivalent to power functions on $\mathbb{F}_{2^n}$.

| | Functions | Conditions | Proven in |
|---|---|---|---|
| The case $n$ divisible by 3 | $x^{2^s+1} + wx^{2^{ik}+2^{mk+s}}$ | $n = 3k$, $\gcd(k, 3) = \gcd(s, 3k) = 1$<br>$k \geq 4$, $i = sk \mod 3$, $m = 3 - i$<br>$w$ has the order $2^{2k} + 2^k + 1$ | [6, 7] |
| The case $n$ divisible by 4 | $x^{2^s+1} + wx^{2^{ik}+2^{mk+s}}$ | $n = 4k$, $\gcd(k, 2) = \gcd(s, 2k) = 1$<br>$k \geq 3$, $i = sk \mod 4$, $m = 4 - i$<br>$w$ has the order $2^{3k} + 2^{2k} + 2^k + 1$ | Theorem 1 of the present paper |

The functions from Table 3 which correspond to the case $n$ divisible by 3 are proven to be APN for $n$ even and in case $n$ odd they are AB permutations [6, 7]. The present paper introduces a new infinite family of quadratic APN binomials which corresponds to the case $n$ divisible by 4 in Table 3. It is proven (in [6] for $n$ divisible by 3 and in the present paper for $n$ divisible by 4) that all these functions are EA-inequivalent to power functions and CCZ-inequivalent to the Gold and Kasami mappings. This implies that for $n$ even they are CCZ-inequivalent to all known APN functions and for $n = 12, 15, 20, 24, 28$, they are CCZ-inequivalent to any power mappings. We conjecture CCZ-inequivalence of these functions to any power functions for all $n \geq 12$.

Though quadratic APN functions are used in some Feistel ciphers (see for instance [32]) functions of low algebraic degree are not the best choices for S-boxes. However, the APN functions from Table 3 can be viewed as the first necessary steps to construct maximum nonlinear S-boxes of a larger algebraic degree CCZ-inequivalent to power functions. Note that, applying CCZ-equivalence to quadratic APN functions it is possible to construct nonquadratic APN mappings CCZ-inequivalent to power functions. The existence of APN functions CCZ-inequivalent to power functions and to quadratic functions is still an open problem.

# 2 Preliminaries

Let $\mathbb{F}_2^n$ be the $n$-dimensional vector space over the field $\mathbb{F}_2$. Any function $F$ from $\mathbb{F}_2^n$ to itself can be uniquely represented as a polynomial on $n$ variables with coefficients in $\mathbb{F}_2^n$, whose degree with respect to each coordinate is at most 1:

$$F(x_1, ..., x_n) = \sum_{u \in \mathbb{F}_2^n} c(u)\Big(\prod_{i=1}^{n} x_i^{u_i}\Big), \qquad c(u) \in \mathbb{F}_2^n.$$

This representation is called the *algebraic normal form* of $F$ and its degree $d^\circ(F)$ the *algebraic degree* of the function $F$.
Besides, the field $\mathbb{F}_{2^n}$ can be identified with $\mathbb{F}_2^n$ as a vector space. Then, viewed as a function from this field to itself, $F$ has a unique representation as a univariate polynomial over $\mathbb{F}_{2^n}$ of degree smaller than $2^n$:

$$F(x) = \sum_{i=0}^{2^n-1} c_i x^i, \quad c_i \in \mathbb{F}_{2^n}.$$

For any $k$, $0 \leq k \leq 2^n - 1$, the number $w_2(k)$ of the nonzero coefficients $k_s \in \{0, 1\}$ in the binary expansion $\sum_{s=0}^{n-1} 2^s k_s$ of $k$ is called the 2-*weight* of $k$. The algebraic degree of $F$ is equal to the maximum 2-weight of the exponents $i$ of the polynomial $F(x)$ such that $c_i \neq 0$, that is $d^\circ(F) = \max_{0 \leq i \leq n-1, c_i \neq 0} w_2(i)$ (see [14]).

A function $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is *linear* if and only if $F(x)$ is a linearized polynomial over $\mathbb{F}_{2^n}$, that is,

$$\sum_{i=0}^{n-1} c_i x^{2^i}, \quad c_i \in \mathbb{F}_{2^n}.$$

The sum of a linear function and a constant is called an *affine function.*

Let $F$ be a function from $\mathbb{F}_{2^n}$ to itself and $A_1$, $A_2 : \mathbb{F}_{2^n} \longrightarrow \mathbb{F}_{2^n}$ be affine permutations. The functions $F$ and $A_1 \circ F \circ A_2$ are then called *affine equivalent.* Affine equivalent functions have the same algebraic degree (i.e. the algebraic degree is *affine invariant*).

As recalled in introduction, we say that the functions $F$ and $F'$ are *extended affine equivalent* if $F' = A_1 \circ F \circ A_2 + A$ for some affine permutations $A_1$, $A_2$ and an affine function $A$. If $F$ is not affine, then $F$ and $F'$ have again the same algebraic degree.

Two mappings $F$ and $G$ from $\mathbb{F}_{2^n}$ to itself are called *Carlet-Charpin-Zinoviev equivalent* (CCZ-equivalent) if the graphs of $F$ and $G$, that is, the subsets $\{(x, F(x)) \mid x \in \mathbb{F}_{2^n}\}$ and $\{(x, G(x)) \mid x \in \mathbb{F}_{2^n}\}$ of $\mathbb{F}_{2^n} \times \mathbb{F}_{2^n}$, are affine equivalent. Hence, $F$ and $G$ are CCZ-equivalent if and only if there exists an affine automorphism $\mathcal{L} = (L_1, L_2)$ of $\mathbb{F}_{2^n} \times \mathbb{F}_{2^n}$ such that

$$y = F(x) \Leftrightarrow L_2(x, y) = G(L_1(x, y)).$$

Note that since $\mathcal{L}$ is a permutation then the function $L_1(x, F(x))$ has to be a permutation too (see [6]). As shown in [14], EA-equivalence is a particular case of CCZ-equivalence and any permutation is CCZ-equivalent to its inverse.

For a function $F : \mathbb{F}_{2^n} \longrightarrow \mathbb{F}_{2^n}$ and any elements $a, b \in \mathbb{F}_{2^n}$ we denote

$$\delta_F(a, b) = |\{x \in \mathbb{F}_2^n : F(x + a) + F(x) = b\}|$$

and

$$\Delta_F = \{\delta_F(a,b) : a, b \in \mathbb{F}_{2^n}, a \neq 0\}.$$

$F$ is called a *differentially $\delta$-uniform* function if $\max_{a \in \mathbb{F}_{2^n}^*, b \in \mathbb{F}_{2^n}} \delta_F(a,b) \leq \delta$. Note that $\delta \geq 2$ for any function over $\mathbb{F}_{2^n}$. Differentially 2-uniform mappings are called *almost perfect nonlinear*.

For any function $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ we denote

$$\lambda_F(a,b) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{\mathrm{tr}(bF(x)+ax)}, \qquad a, b \in \mathbb{F}_{2^n},$$

where $tr(x) = x + x^2 + x^4 + ... + x^{2^{n-1}}$ is the trace function from $\mathbb{F}_{2^n}$ into $\mathbb{F}_2$. The set $\Lambda_F = \{\lambda_F(a,b) : a, b \in \mathbb{F}_{2^n}, b \neq 0\}$ is called the *Walsh spectrum* of $F$ and the value

$$\mathcal{NL}(F) = 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_{2^n}, b \in \mathbb{F}_{2^n}^*} |\lambda_F(a,b)|$$

equals the *nonlinearity* of the function $F$. The nonlinearity of any function $F$ satisfies the inequality

$$\mathcal{NL}(F) \leq 2^{n-1} - 2^{\frac{n-1}{2}}$$

([15, 35]) and in case of equality $F$ is called *almost bent* or *maximum nonlinear.*

It is shown in [14] that, if $F$ and $G$ are CCZ-equivalent, then $F$ is APN (resp. AB) if and only if $G$ is APN (resp. AB). More general, CCZ-equivalent functions have the same nonlinearity and differential uniformity.

Obviously, AB functions exist only for $n$ odd. It is proven in [15] that every AB function is APN and its Walsh spectrum equals $\{0, \pm 2^{\frac{n+1}{2}}\}$. If $n$ is odd, every APN mapping which is quadratic (that is, whose algebraic degree equals 2) is AB [14], but this is not true for nonquadratic cases: the Dobbertin and the inverse APN functions are not AB (see [14],[12]). When $n$ is even, the inverse function $x^{2^n-2}$ is a differentially 4-uniform permutation [33] and has the best known nonlinearity [30], that is $2^{n-1} - 2^{\frac{n}{2}}$ (see [12, 18]). This function has been chosen as the basic S-box, with $n = 8$, in the Advanced Encryption Standard (AES), see [17]. A comprehensive survey on APN and AB functions can be found in [13].

# 3 A new family of APN functions

**Theorem 1** *Let $s$ and $k$ be positive integers such that $s \leq 4k-1$, $\gcd(k,2) = \gcd(s,2k) = 1$, and $i = sk \mod 4$, $m = 4 - i$, $n = 4k$. If $w \in \mathbb{F}_{2^n}$ has the order $2^{3k} + 2^{2k} + 2^k + 1$ then the function $F(x) = x^{2^s+1} + wx^{2^{ik}+2^{mk+s}}$ is APN on $\mathbb{F}_{2^n}$.*

*Proof.* Without loss of generality we can assume that $w = \alpha^{2^k-1}$ where $\alpha$ is a primitive element of $\mathbb{F}_{2^n}^*$. We have to show that for every $u, v \in \mathbb{F}_{2^n}$, $u \neq 0$, the equation

$$F(x) + F(x+u) = v \qquad (1)$$

has at most 2 solutions. We have

$$\begin{aligned}
F(x)+F(x+u) &= \alpha^{2^k-1}\left(x^{2^{ik}+2^{mk+s}}+(x+u)^{2^{ik}+2^{mk+s}}\right)+x^{2^s+1}+(x+u)^{2^s+1} \\
&= \alpha^{2^k-1}u^{2^{ik}+2^{mk+s}}\left(\left(\frac{x}{u}\right)^{2^{ik}}+\left(\frac{x}{u}\right)^{2^{mk+s}}\right) \\
&\quad +u^{2^s+1}\left(\left(\frac{x}{u}\right)^{2^s}+\left(\frac{x}{u}\right)\right)+\alpha^{2^k-1}u^{2^{ik}+2^{mk+s}}+u^{2^s+1}
\end{aligned}$$

As this is a linear equation in $x$ it is sufficient to study the kernel. To simplify notation we denote

$$a=\alpha^{2^k-1}u^{2^{ik}+2^{mk+s}-2^s-1}.$$

After replacing $x$ by $ux$ and dividing by $u^{2^s+1}$, we see the equation (1) admits 0 or 2 solutions for every $u\in\mathbb{F}_{2^n}^*$ if and only if, denoting

$$\Delta_a(x)=a\left(x^{2^{ik}}+x^{2^{mk+s}}\right)+x^{2^s}+x,$$

the equation $\Delta_a(x)=0$ has the only solutions 0 and 1.

From now on we consider the cases $i=1$ and $i=3$ separately.

**Case 1** $(i=3, m=1)$**:** If we denote $y=x^{2^k}$, $z=y^{2^k}$, $t=z^{2^k}$ and $b=a^{2^k}$, $c=b^{2^k}$, $d=c^{2^k}$ the equation $\Delta_a(x)=0$ can be rewritten as

$$a(t+y^{2^s})+x^{2^s}+x=0.$$

Since $2^{ik}+2^{mk+s}-2^s-1=2^{3k}+2^{k+s}-2^s-1=(2^k-1)(2^{2k}+2^k+2^s+1)$ then $a$ is always a $(2^k-1)$-th power and thus $abcd=1$. Considering also the conjugated equations we derive the following system of equations

$$\begin{aligned}
f_1 = &\quad a(t+y^{2^s})+x^{2^s}+x = &&0 \\
f_2 = &\quad b(x+z^{2^s})+y^{2^s}+y = &&0 \\
f_3 = &\quad c(y+t^{2^s})+z^{2^s}+z = &&0 \\
f_4 = &\quad z+x^{2^s}+abc(t^{2^s}+t) = &&0.
\end{aligned}$$

The aim is now to eliminate $y$, $z$ and $t$ from these equations and to get an equation in $x$ only. First we compute

$$\begin{aligned}
R_1 &= bcf_1+abcf_2+abf_3+f_4 \\
&= ab(bc+1)z^{2^s}+(ab+1)z+(bc+1)x^{2^s}+bc(ab+1)x
\end{aligned}$$

and

$$\begin{aligned}
R_2 &= cf_1^{2^s}+a^{2^s}c(f_2^{2^s}+f_2)+a^{2^s}f_3 \\
&= a^{2^s}b^{2^s}cz^{2^{2s}}+a^{2^s}(bc+1)z^{2^s}+a^{2^s}z+cx^{2^{2s}}+c(ab+1)^{2^s}x^{2^s}+a^{2^s}bcx
\end{aligned}$$

to eliminate $t$ and $y$. To eliminate $z^{2^{2s}}$ we compute

$$\begin{aligned} R_3 &= cR_1^{2^s} + (bc+1)^{2^s} R_2 \\ &= (c(ab+1)^{2^s} + a^{2^s}(bc+1)^{2^s+1})z^{2^s} + a^{2^s}(bc+1)^{2^s} z + c(ab+1)^{2^s} x^{2^s} + a^{2^s} bc(bc+1)^{2^s} x. \end{aligned}$$

Using equations $R_1$ and $R_3$ we can eliminate $z^{2^s}$ by computing

$$\begin{aligned} R_4 &= ab(bc+1)R_3 + (c(ab+1)^{2^s} + a^{2^s}(bc+1)^{2^s+1})R_1 \\ &= P(a)(z + (bc+1)x^{2^s} + bcx), \end{aligned}$$

where

$$P(a) = c(ab+1)^{2^s+1} + a^{2^s}(bc+1)^{2^s+1}.$$

Below we shall show that $P(a) \neq 0$, thus we can denote

$$R_5 = \frac{R_4}{P(a)} = z + (bc+1)x^{2^s} + bcx.$$

Computing

$$\begin{aligned} R_6 &= R_1 + ab(bc+1)R_5^{2^s} \\ &= (ab+1)z + ab(bc+1)^{2^s+1}x^{2^{2s}} + (ab^{2^s+1}c^{2^s}+1)(bc+1)x^{2^s} + bc(ab+1)x \end{aligned}$$

we finally get our desired equation

$$\begin{aligned} R_7 &= (ab+1)R_5 + R_6 \\ &= ab(bc+1)^{2^s+1}\left(x^{2^{2s}} + x^{2^s}\right). \end{aligned}$$

Obviously if $x$ is a solution of $\Delta_a(x) = 0$ then $R_7(x) = 0$. For $P(a) \neq 0$ and $bc+1 \neq 0$ this is equivalent to $x = 0, 1$. Thus to prove the theorem we have to show that $P(a)$ and $bc+1$ do not vanish for elements $a$ fulfilling the equation

$$a = \alpha^{2^k-1}u^{2^{3k}+2^{k+s}-2^s-1}. \tag{2}$$

Assume $bc = 1$, that is, $a^{2^{2k}+2^k} = 1$ and then $a^{2^k+1} = 1$. We have

$$a^{2^k+1} = \left(\alpha u^{2^k+2^s}\right)^{2^{2k}-1}$$

because

$$(2^{3k} + 2^{k+s} - 2^s - 1)(2^k+1) = (2^{2k}-1)(2^k+2^s) \mod (2^{4k}-1).$$

Since $a^{2^k+1} = 1$ then $\alpha u^{2^k+2^s}$ should be $(2^{2k}+1)$-th power of an element of the field. We have

$$2^k + 2^s = 2^s(2^{k-s}+1) = 2^s(2^{2p}+1)$$

with some $p$ odd. Indeed, $ks \mod 4 = 3$, then

$$k \mod 4 \neq s \mod 4$$

for odd $k, s$, and $k - s = 2p$ for some $p$ odd.

Numbers $2^{2p} + 1$ and $2^{2k} + 1$ are divisible by 5 because $p, k$ are odd. We get that $u^{2^k+2^s}$ is 5-th power of an element of the field and $\alpha u^{2^k+2^s}$ is not. Therefore $\alpha u^{2^k+2^s}$ is not $(2^{2k}+1)$-th power of an element of the field. A contradiction.

Let $c(ab+1)^{2^s+1} + a^{2^s}(bc+1)^{2^s+1} = 0$. Since $bc + 1 \neq 0$ then $ab + 1 \neq 0$ and we get

$$\frac{c}{a^{2^s}} = \left(\frac{bc+1}{ab+1}\right)^{2^s+1}.$$

Note that since $n$ is even and $s$ is odd then $2^n - 1$ and $2^s + 1$ are divisible by 3. Therefore $c/a^{2^s}$ is third power of an element of the field. We have

$$c/a^{2^s} = a^{2^{2k}-2^s} = a^{2^s(2^{2k-s}-1)}$$

and

$$(2^{3k}+2^{k+s}-2^s-1)(2^{2k-s}-1) = -(2^{2k}-1)-2^{k-s}(2^{2s}-1)-2^s(2^{2(k-s)}-1) \mod (2^{4k}-1).$$

The numbers $2^{2k} - 1$, $2^{2s} - 1$ and $2^{2(k-s)} - 1$ are divisible by 3. On the other hand $2^k - 1$ and $2^{2k-s} - 1$ are not divisible by 3 since $k$ and $2k - s$ are odd. We get

$$a^{2^s(2^{2k-s}-1)} = \alpha^{2^s(2^{2k-s}-1)(2^k-1)} u^{2^s\left(-(2^{2k}-1)-2^{k-s}(2^{2s}-1)-2^s(2^{2(k-s)}-1)\right)}.$$

Obviously $c/a^{2^s}$ is not third power of an element of the field and therefore it is not $(2^s+1)$-th power.

**Case 2** $(i = 1, m = 3)$**:** Since $2^{ik} + 2^{mk+s} - 2^s - 1 = 2^k + 2^{3k+s} - 2^s - 1 = (2^k - 1)(1 + 2^{2k+s} + 2^{k+s} + 2^s)$ then $a$ is always a $(2^k - 1)$-th power and thus again $abcd = 1$.

In this case the equation $\Delta_a(x) = 0$ can be transformed into the following system of equations

$$\begin{aligned}
f_1 = &\quad a(y + t^{2^s}) + x^{2^s} + x = &\quad 0 \\
f_2 = &\quad b(z + x^{2^s}) + y^{2^s} + y = &\quad 0 \\
f_3 = &\quad c(t + y^{2^s}) + z^{2^s} + z = &\quad 0 \\
f_4 = &\quad x + z^{2^s} + abc(t^{2^s} + t) = &\quad 0.
\end{aligned}$$

We get

$$\begin{aligned}
R_1 &= bcf_1 + abcf_2 + abf_3 + f_4 \\
&= (ab+1)z^{2^s} + ab(bc+1)z + bc(ab+1)x^{2^s} + (bc+1)x, \\
R_2 &= c^{2^s}f_1 + ac^{2^s}(f_2^{2^s} + f_2) + af_3^{2^s} \\
&= az^{2^{2s}} + a(bc+1)^{2^s}z^{2^s} + abc^{2^s}z + ab^{2^s}c^{2^s}x^{2^{2s}} + c^{2^s}(ab+1)x^{2^s} + c^{2^s}x, \\
R_3 &= aR_1^{2^s} + (ab+1)^{2^s}R_2 \\
&= a(bc+1)^{2^s}z^{2^s} + abc^{2^s}(ab+1)^{2^s}z + (a(bc+1)^{2^s} + c^{2^s}(ab+1)^{2^s+1})x^{2^s} + c^{2^s}(ab+1)^{2^s}x, \\
R_4 &= (ab+1)R_3 + a(bc+1)^{2^s})R_1 \\
&= P(a)(abz + (ab+1)x^{2^s} + x),
\end{aligned}$$

where

$$P(a) = c^{2^s}(ab+1)^{2^s+1} + a(bc+1)^{2^s+1}.$$

Assuming that $P(a) \neq 0$ we continue

$$\begin{aligned}
R_5 &= \frac{R_4}{P(a)} = abz + (ab+1)x^{2^s} + x, \\
R_6 &= a^{2^s}b^{2^s}R_1 + (ab+1)R_5^{2^s} \\
&= a^{2^s+1}b^{2^s+1}(bc+1)z + (ab+1)^{2^s+1}x^{2^{2s}} + (a^{2^s}b^{2^s+1}c+1)(ab+1)x^{2^s} + a^{2^s}b^{2^s}(bc+1)x, \\
R_7 &= a^{2^s}b^{2^s}(bc+1)R_5 + R_6 \\
&= (ab+1)^{2^s+1}\left(x^{2^{2s}} + x^{2^s}\right).
\end{aligned}$$

We see now that the equation $\Delta_a(x) = 0$ has the only solutions 0 and 1 if $P(a) \neq 0$ and $ab+1 \neq 0$.

Assume that $ab = 1$, that is, $a^{2^k+1} = 1$. We have

$$(2^k + 2^{3k+s} - 2^s - 1)(2^k+1) = (2^{2k}-1)(2^{k+s}+1) \mod (2^{4k}-1)$$

and

$$a^{2^k+1} = \left(\alpha^{2^k-1}u^{2^k+2^{3k+s}-2^s-1}\right)^{2^k+1} = \left(\alpha u^{2^{k+s}+1}\right)^{2^{2k}-1}.$$

Because $a^{2^k+1} = 1$, the element $\alpha u^{2^{k+s}+1}$ should be $(2^{2k}+1)$-th power of an element of the field. Since $ks \mod 4 = 1$ then $k \mod 4 = s \mod 4$ and $2^{k+s}+1 = 2^{2p}+1$ for some $p$ odd. Thus $2^{k+s}+1$ and $2^{2k}+1$ are divisible by 5. Therefore $\alpha u^{2^{k+s}+1}$ is not fifth power of an element of the field and then it is not $(2^{2k}+1)$-th power. A contradiction.

Let $c^{2^s}(ab+1)^{2^s+1} + a(bc+1)^{2^s+1} = 0$. Since $ab+1 \neq 0$ then

$$c^{2^s}/a = \left(\frac{bc+1}{ab+1}\right)^{2^s+1}.$$

We show that the element $c^{2^s}/a = a^{2^{2k+s}-1}$ is not third power of an element of the field. A contradiction.

Indeed, for $n$ even and $s$ odd the numbers $2^s+1$ and $2^n-1$ are divisible by 3. On the other hand

$$a^{2^{2k+s}-1} = \left(\alpha^{2^k-1}u^{2^k+2^{3k+s}-2^s-1}\right)^{2^{2k+s}-1} = \alpha^{(2^k-1)(2^{2k+s}-1)}u^{(2^k+2^{3k+s}-2^s-1)(2^{2k+s}-1)}$$

and

$$(2^k+2^{3k+s}-2^s-1)(2^{2k+s}-1) = 2^s(1-2^{2k})+(1-2^{2(k+s)})+2^k(2^{2s}-1) \mod (2^{4k}-1).$$

Since $2^{2k}-1$, $2^{2(k+s)}+1$ and $2^{2s}-1$ are divisible by 3 then $u^{(2^k+2^{3k+s}-2^s-1)(2^{2k+s}-1)}$ is third power of an element of the field. The number $(2^k-1)(2^{2k+s}-1)$ is not divisible by 3 because $k$ and $2k+s$ are odd. Therefore, $a^{2^{2k+s}-1}$ is not third power of an element of the field. □

# 4 On CCZ-inequivalence of the introduced APN functions to power functions

To prove CCZ-inequivalence of APN functions of Theorem 1 to the Gold and Kasami functions we use results from [6].

**Theorem 2** ([6]) *Let $n$ be a positive integer and let $s, j, q$ be three nonzero elements of $\mathbb{Z}/n\mathbb{Z}$ such that $q \neq \pm s$, $j \neq \pm s, \pm q, 2s, s \pm q$. Then the function $F(x) = x^{2^s+1} + ax^{2^j(2^q+1)}$ with $a \in \mathbb{F}_{2^n}^*$ is EA-inequivalent to power functions on $\mathbb{F}_{2^n}$.*

**Theorem 3** ([6]) *Let $n$ be a positive integer and $r, s, q$ be three nonzero elements of $\mathbb{Z}/n\mathbb{Z}$ and $j$ an element of $\mathbb{Z}/n\mathbb{Z}$ such that $s \neq \pm q$, $j \neq s-r$, $j \neq -r$, $j+q \neq s-r$, $j+q \neq -r$. If for $a \in \mathbb{F}_{2^n}^*$ the function $F(x) = x^{2^s+1} + ax^{2^j(2^q+1)}$ is APN on $\mathbb{F}_{2^n}$ and it is CCZ-equivalent to the function $G(x) = x^{2^r+1}$ then $F$ and $G$ are EA-equivalent.*

**Theorem 4** ([6]) *Let $n$ be a positive integer and $r, s, q, j$ be nonzero elements of $\mathbb{Z}/n\mathbb{Z}$ such that $\gcd(r,n) = 1$, $n > 4$, $s \neq \pm q$, $s \neq \pm 3q$, $q \neq \pm 3s$, $s \neq \pm j$, $q \neq \pm j$, $3q + j \neq 0$, $j + q \neq \pm s$, $j \neq s + q$, $2q \neq \pm j$, $2q \neq s - j$, $2s \neq j$, $2s \neq j + q$. Then for $a \in \mathbb{F}_{2^n}^*$ the functions $F(x) = x^{2^s+1} + ax^{2^j(2^q+1)}$ and $K(x) = x^{4^r-2^r+1}$ are CCZ-inequivalent on $\mathbb{F}_{2^n}$.*

**Proposition 1** *The function $F$ of Theorem 1 is EA-inequivalent to power functions when $k \geq 3$.*

*Proof.* The function $F$ satisfies the conditions of Theorem 2. If $i = 1$ then $j = k$ and $q = 2k+s$. The conditions $q \neq \pm s$, $j \neq \pm s, \pm q, \pm 2s, s \pm q$ are satisfied when $k \geq 3$ because $k, s$ are odd, $n = 4k$, $\gcd(s, 4k) = 1$. The same is with the case $i = 3$. □

**Proposition 2** *The function $F$ of Theorem 1 is CCZ-inequivalent to the Gold mappings when $k \geq 3$.*

*Proof.* The proof is based on Proposition 1 and Theorem 3. Let $i = 1$, then $j = k$ and $q = 2k + s$ satisfy the conditions $q \neq \pm s, j \neq s - r, j \neq -r, j + q \neq s - r, j + q \neq -r$ for any $r$ satisfying $1 \leq r < n/2$ and $\gcd(r, n) = 1$. Indeed, $q = \pm s$ is in contradiction with $\gcd(s, 4k) = 1$, $n = 4k$. If $k = s - r$ then it contradicts to the fact that $k$ is odd and $s - r$ is even. If $k = -r$ then it would contradict to $\gcd(r, 4k) = 1$. If $3k + s = s - r$ then $3k = -r$ and $\gcd(r, k) \neq 1$, a contradiction. If $3k + s = -r$ then $s + r = k$ while $s, r, k$ are odd. By Theorem 3 and Proposition 1 the function $F$ is CCZ-inequivalent to $x^{2^r+1}$. For the case $i = 3$ the proof is similar. $\square$

**Proposition 3** *The function $F$ of Theorem 1 is CCZ-inequivalent to the Kasami mappings when $k \geq 3$.*

*Proof.* Obviously, when $k \geq 3$ the function $F$ satisfies the conditions of Theorem 4 because $k, s$ are odd, $n = 4k$, $\gcd(s, 4k) = 1$. $\square$

If $n$ is even then for any quadratic APN mapping $F$ the number $2^{n/2}$ divides all the values in the Walsh spectrum of $F$ (see [34]). Besides, it is proven in [11] that $2^{\frac{2n}{5}+1}$ cannot be a divisor of all the values in the Walsh spectrum of the Dobbertin function. Since the Walsh spectrum of a function is invariant (up to the sign of the values in it) under CCZ-equivalence then we can make the following conclusion from Propositions 1-3.

**Corollary 1** *The function $F$ of Theorem 1 is CCZ-inequivalent to all known power APN functions when $k \geq 3$.*

For $n = 12, 20, 28$ Corollary 1 implies that the introduced APN binomials are CCZ-inequivalent to all power functions. When $n \geq 20$ and $n$ is not divisible by 3 then the function $F$ is CCZ-inequivalent to all known APN functions.

**Problem 1** *Construct APN polynomials CCZ-inequivalent to power functions and to quadratic functions.*

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