Computing the Bilinear Pairing with Efficient Endomorphisms

Chang-An Zhao, Fangguo Zhang, and Jiwu Huang

[1] Department of Electronics and Communication Engineering,
Sun Yat-Sen University, Guangzhou 510275, P.R.China
[2] Guangdong Key Laboratory of Information Security Technology,
Guangzhou 510275, P.R.China
zhcha@student.sysu.edu.cn
isszhfg@mail.sysu.edu.cn
isshjw@mail.sysu.edu.cn

Abstract. In this paper we present a new efficient algorithm for computing the bilinear pairings on a family of non-supersingular elliptic curves with efficient endomorphisms. We obtain a short iteration loop in Miller's algorithm using efficient endomorphisms with degree 1. We show that the proposed algorithm is faster than the previous methods on these curves.

Keywords: Tate pairing, non-suersingluar curves, pairing-based cryptosystems, endomorphism.

1 Introduction

The bilinear pairing is an explicit mapping from a set of certain points on an elliptic curve to a multiplicative subgroup of a finite field. It has been found many interesting applications in elliptic curve cryptography [9].

Many efficient algorithms for implementing the pairings have been proposed [4]. In particular, the eta pairing [3] and the ate pairing [6] are introduced for their efficient computations recently. Their main ideas are to shorten the main iteration loop using some automorphisms of curves in Miller's algorithm [8].

In this paper, a new algorithm is proposed for computing the bilinear pairing on a family of elliptic curves with efficient endomorphisms. The numbers of the main iteration loop in the new algorithm are shortened to a half of the numbers

of the previous main loop in Miller's algorithm. Efficient endomorphisms are used for obtaining a short iteration loop. We show that the proposed algorithm obtains a significant improvement over the previous methods.

This paper is organized as follows: Section 2 explains the Tate pairing and a family of non-supersingular elliptic curves with efficient endomorphisms. Section 3 gives the main results and proposes a new efficient algorithm, and section 4 analyzes the efficiency of the proposed algorithm and compares it with the previous methods. Section 5 gives the conclusions.

2 Mathematical Preliminaries

2.1 The Tate Pairing

Let $\mathbb{F}_q$ be a finite field with $q = p^m$ elements, where p is a prime. Let E be an elliptic curve defined over $\mathbb{F}_q$, and let $\mathcal{O}$ be the point at infinity. Let r be a prime such that $r|\#E(\mathbb{F}_q)$, and let k be the minimal positive integer such that r divides $q^k - 1$. This k is named the embedding degree. We also assume that r^2 does not divide $q^k - 1$ and k is greater than 1.

Let $P \in E[r]$ and $Q \in E(\mathbb{F}_{q^k})$, and let D be the divisor which is equivalent to $(Q) - (\mathcal{O})$. For every integer i and point P, let $f_{i,P}$ be a function such that

$$(f_{i,P}) = i(P) - (iP) - (i-1)(\mathcal{O}).$$

Then the Tate pairing is a map

$$e : E[r] \times E(F_{q^k})/rE(F_{q^k}) \to F_{q^k}^*/(F_{q^k}^*)^r,$$

$$e(P, Q) = f_{r,P}(D).$$

By Theorem 1 in [1], one can define the reduced Tate pairing as

$$e(P, Q) = f_{r,P}(Q)^{\frac{q^k-1}{r}}.$$

The above definition is convenient since we often require a unique element of $\mathbb{F}_{q^k}$ in cryptographic applications. Note that $f_{r,P}(Q)^{a(q^k-1)/r} = f_{ar,P}(Q)^{(q^k-1)/r}$ for any integer a .

2.2 Miller's Algorithm

We recall Miller's algorithm [8] for computing the Tate pairing in polynomial time simply in this section.

Let $P \in E(\mathbb{F}_q)[r]$ and $Q \in E(\mathbb{F}_{q^k})$. Let $l_{R,T}$ be the equation of the line through points R and T, and let v_S be the equation of the vertical line through point S. Then for $i, j \in \mathbb{Z}$, we have

$$f_{i+j,P}(Q) = f_{i,P}(Q) f_{j,P}(Q) \frac{l_{iP,jP}(Q)}{v_{(i+j)P}(Q)}.$$

Miller's algorithm is described as Algorithm 1.

Algorithm 1 Miller's algorithm

Input: $r = \sum_{i=0}^{n} l_i 2^i$, where $l_i \in \{0, 1\}$. $P \in E(F_q)[r]$ and $Q \in E(F_{q^k})$.
Output: $e(P, Q)$
1. $T \leftarrow P$, $f_1 \leftarrow 1$
2. for $i = n-1, n-2, ..., 1, 0$ do
2.1 $f_1 \leftarrow f_1^2 \cdot \frac{l_{T,T}(Q)}{v_{2T}(Q)}$, $T \leftarrow 2T$
2.2 if $l_i = 1$ then
2.3 $f_1 \leftarrow f_1 \cdot \frac{l_{T,P}(Q)}{v_{T+P}(Q)}$, $T \leftarrow T + P$
3. return $f_1^{(q^k-1)/r}$

2.3 A Family of Elliptic Curves with Efficient Endomorphisms

We recall a family of elliptic curves with efficient endomorphisms in this section. Let p be a large prime, consider these ordinary curves over $\mathbb{F}_p$

$$E : y^2 = x^3 + B, \text{where } p \equiv 1 \bmod 3 \tag{1}$$

$$E_1 : y^2 = x^3 + Ax, \text{where } p \equiv 1 \bmod 4 \tag{2}$$

Both of them have efficiently-computable endomorphisms which have been used in fast point multiplication [5] and computations of the Tate pairing [10]. With a small loss of generality, we will mainly consider the first curve (1) for accelerating the computations of the bilinear pairing. Note that some suitable curves of this type have low embedding degrees such that they can be applied in pairing-based cryptography [10].

Suppose that β is an element of order 3 in $\mathbb{F}_p$. An endomorphism of the above curve (1) is defined as

$$\phi : E \to E$$

$$(x, y) \rightarrow (\beta x, y).$$

Since this endomorphism is an isogeny, its dual isogeny is

$$\phi : E \rightarrow E$$

$$\hat{\phi} :\rightarrow (\beta^2 x, y).$$

It is not difficult to show that $\hat{\phi} \circ \phi = [1]$, $\phi^2 = \hat{\phi}$ and $\#ker\phi = 1$ (see Silverman [13] pages 84-86).

We cite some useful facts from [5] since they are necessary in the new algorithm. Let $P \in E(\mathbb{F}_p)$ be a point of prime order r, where r^2 does not divide the order of $E(\mathbb{F}_p)$. Then ϕ and $\hat{\phi}$ act restrictedly on the subgroup $<P>$ as multiplication maps $[\lambda]$ and $[\hat{\lambda}]$ respectively, where λ and $\hat{\lambda}$ are the two roots of the equation: $x^2 + x + 1 = 0 \pmod{r}$. Note that $\lambda P = \phi(P)$ can be computed using one multiplication in $\mathbb{F}_p$.

3 A New Algorithm for Computing the Bilinear Pairing

In this section, we will give the main results for computing the bilinear pairing. As a consequence, A new efficient algorithm will be proposed.

3.1 Main Results

The main results of this paper are summarized in the following theorem.

Theorem 1. *Let E be a non-supersingular curve over $\mathbb{F}_p$ with an efficient endomorphism ϕ and $\hat{\phi}$ defined as above (1). Let k be its embedding degree. Let $Q \in E(\mathbb{F}_{p^k})$ and $P \in E(\mathbb{F}_p)$ be a point of prime order r, where r^2 does not divide $\#E(\mathbb{F}_p)$. Let $[\lambda]$ be the multiplication map of the subgroup $<P>$ defined as above such that $\lambda P = \phi(P)$. Let a be the minimal positive integer such that $\lambda^2 + \lambda + 1 = ar$. Let $l_{R,T}$ be the line through points R and T. Then*

$$e(P, Q)^a = (f_{\lambda,P}(Q)^{\lambda+1} f_{\lambda,P}(\hat{\phi}(Q)) l_{\phi(P),\hat{\phi}(P)}(Q)))^{\frac{p^k-1}{r}}.$$

Note that such a must exist since $\lambda^2 + \lambda + 1 = 0 \pmod{r}$. A second useful remark is that $e(P, Q)^a$ is non-degenerate if r does not divide a. The proof of Theorem 1 is split into three short lemmas.

Lemma 1. *Using the notation as above, we have*

$$e(P,Q)^a = (f_{\lambda^2+\lambda,P}(Q) l_{-P,P}(Q))^{\frac{p^k-1}{r}}.$$

Proof. By definition of the reduced Tate pairing, we have

$$e(P,Q)^a = f_{r,P}(Q)^{\frac{(a)(p^k-1)}{r}} = f_{ar,P}(Q)^{\frac{p^k-1}{r}}.$$

Since $ar = \lambda^2 + \lambda + 1$, we get

$$e(P,Q)^a = f_{ar,P}(Q)^{\frac{p^k-1}{r}} = f_{\lambda^2+\lambda+1,P}(Q)^{\frac{p^k-1}{r}}.$$

Note that

$$(f_{\lambda^2+\lambda+1,P}) = (f_{\lambda^2+\lambda,P} f_{1,P} l_{-P,P})$$

since $(\lambda^2+\lambda)P = -P$. Furthermore, $f_{1,P} = 1$ up to a scalar multiple in $\mathbb{F}_p^*$, so we obtain

$$e(P,Q)^a = f_{\lambda^2+\lambda+1,P}(Q)^{\frac{p^k-1}{r}} = (f_{\lambda^2+\lambda,P}(Q) l_{-P,P}(Q))^{\frac{p^k-1}{r}}$$

which proves the results. □

Lemma 2. *Using the notation as above, we can choose $f_{\lambda^2+\lambda,P} l_{-P,P}$ such that*

$$(f_{\lambda^2+\lambda,P} l_{-P,P}) = (f_{\lambda,P}^{\lambda+1} f_{\lambda,\lambda P} l_{\phi(P),\hat{\phi}(P)}).$$

Proof. We have $(f_{i,P}) = i(P) - (iP) - (i-1)(\mathcal{O})$ and $(\lambda^2+\lambda)P = -P$. Therefore

$$\begin{aligned}(f_{\lambda^2+\lambda,P} l_{-P,P}) =& (f_{\lambda^2,P} f_{\lambda,P} \frac{l_{\lambda^2 P,\lambda P}}{l_{(\lambda^2+\lambda)P,-(\lambda^2+\lambda)P}} l_{-P,P}) \\ =& (f_{\lambda^2,P} f_{\lambda,P} l_{\lambda^2 P,\lambda P}).\end{aligned}$$

Furthermore, since $\lambda P = \phi(P)$ and $\lambda^2 P = \phi^2(P) = \hat{\phi}(P)$, thus

$$l_{\lambda^2 P,\lambda P} = l_{\lambda P,\lambda^2 P} = l_{\phi(P),\hat{\phi}(P)}.$$

Also, (see Lemma 2 in [3])

$$(f_{\lambda^2,P}) = (f_{\lambda,P}^{\lambda} f_{\lambda,\lambda P}).$$

Hence we have

$$(f_{\lambda^2+\lambda,P} l_{-P,P}) = (f_{\lambda^2,P} f_{\lambda,P} l_{\lambda^2 P,\lambda P}) = (f_{\lambda,P}^{\lambda+1} f_{\lambda,\lambda P} l_{\phi(P),\hat{\phi}(P)})$$

which completes the proof . □

Lemma 3. *For $P \in E(\mathbb{F}_p)[r]$ and $Q \in E(\mathbb{F}_{p^k})$, we have $f_{\lambda,\lambda P}(Q) = f_{\lambda,P}(\hat{\phi}(Q))$, with ϕ and $\hat{\phi}$ defined as above.*

Proof. By definition we have $(f_{\lambda,\lambda P}) = \lambda(\lambda P) - (\lambda^2 P) - (\lambda - 1)(\mathcal{O})$. We also have $\phi(P) = \lambda P$ and $\#ker\phi = deg[1] = 1$ (see [13] Chapter III page 85-86). By properties of the pullback we obtain

$$\begin{aligned}\phi^*(f_{\lambda,\lambda P}) =& \phi^*(\lambda(\lambda P) - (\lambda^2 P) - (\lambda - 1)(\mathcal{O})) \\ =& \lambda(P) - (\lambda P) - (\lambda - 1)(\mathcal{O}) \\ =& (f_{\lambda,P}).\end{aligned}$$

Furthermore, $\phi^*(f_{\lambda,\lambda P}) = (f_{\lambda,\lambda P} \circ \phi)$, hence we can take (up to a scalar multiple in $\mathbb{F}_p^*$)

$$f_{\lambda,\lambda P} \circ \phi = f_{\lambda,P}.$$

Applying $\hat{\phi}$ to the above yields

$$f_{\lambda,\lambda P} \circ \phi \circ \hat{\phi} = f_{\lambda,P} \circ \hat{\phi}.$$

Since $\phi \circ \hat{\phi} = [1]$, we have

$$f_{\lambda,\lambda P} = f_{\lambda,P} \circ \hat{\phi}.$$

This completes the proof. □

Proof of Theorem 1: Since $P \in E(\mathbb{F}_p)[r]$, Lemma 3 gives

$$f_{\lambda,\lambda P}(Q) = f_{\lambda,P}(\hat{\phi}(Q)),$$

and applying the above into Lemma 2, we can easily obtain

$$f_{\lambda^2+\lambda,P}(Q) l_{-P,P}(Q) = f_{\lambda,P}^{\lambda+1}(Q) f_{\lambda,P}(\hat{\phi}(Q)) l_{\phi(P),\hat{\phi}(P)}(Q).$$

Substituting the above equality into Lemma 1, we have

$$\begin{aligned}e(P,Q)^a =& (f_{\lambda^2+\lambda,P}(Q) l_{-P,P}(Q))^{\frac{p^k-1}{r}} \\ =& (f_{\lambda,P}(Q)^{\lambda+1} f_{\lambda,P}(\hat{\phi}(Q)) l_{\phi(P),\hat{\phi}(P)}(Q))^{\frac{p^k-1}{r}}.\end{aligned}$$

This completes the whole proof of Theorem 1. □

Note that $e(P,Q)^a$ gives a bilinear pairing since $e(P,Q)$ is bilinear. Furthermore, it is non-degenerate if r does not divide a. Since a is far smaller than r in practice , r indeed does not divide a. Therefore, we obtain a new non-degenerate, bilinear pairing which is equal to a fixed power of the traditional reduced Tate

pairing. Similarly, such a bilinear pairing also exists on the second curve (2). We does not describe it here for simplicity. It should be pointed out that computing the new pairing requires a shorter loop than the traditional Miller's algorithm. In practice, we can make that a is equal to 1 for some elliptic curves with efficient endomorphisms. In this case, we can keep that the value of the new pairing is equal to a correct Tate pairing value.

3.2 The Proposed Algorithm for Computing the Bilinear Pairing

In this section, we will give a new algorithm for computing the pairing $e(P,Q)^a$ by Theorem 1. For simplicity, we only consider these non-supersingular elliptic curves with efficient endomorphisms which have embedding degrees $k = 2$. However, the new method also applies to higher values of k.

Let Q be in the trace-zero subgroup $[2, 11]$ for good efficiency. Note that the denominator can be omitted in Miller's algorithm since the x-coordinates of P, Q and $\hat{\phi}(Q)$ belong to $\mathbb{F}_p^*$ now. Let $l_{R,T}$ be the equation of the line through points R and T. The proposed algorithm is given in Algorithm 2.

We give some useful remarks on Algorithm 2. The equation of the line $l_{\phi(P),\hat{\phi}(P)}$ is easily obtained since $\phi(P)$ and $\hat{\phi}(P)$ can be computed using efficient endomorphisms. $\hat{\phi}(Q)$ can be computed using only one multiplication since β^2 and x-coordinate of Q are in $\mathbb{F}_p$. Let T be in $E(\mathbb{F}_p)$ with coordinates (x_T, y_T), and let m be the slope of the line $l_{T,T}$. Then the equation of the line $l_{T,T}$ is $(y - y_T) - m(x - x_T) = 0$. Therefore the evaluation of $l_{T,T}(Q)$ and $l_{T,T}(\hat{\phi}(Q))$ only requires two multiplications in $\mathbb{F}_p$. Similarly, computing $l_{T,P}(Q)$ and $l_{T,P}(\hat{\phi}(Q))$ also requires two multiplications in $\mathbb{F}_p$.

Algorithm 2 Computations of $e(P,Q)^a$ using efficient endomorphisms

Input: P, Q and $\lambda = \sum_{i=0}^{n} l_i 2^i$, where $l_i \in \{0, 1\}$.

Output: $e(P,Q)^a$

1. $T \leftarrow P$, $f_1 \leftarrow 1$, $f_2 \leftarrow 1$, $f_3 \leftarrow l_{\phi(P),\hat{\phi}(P)}(Q)$

2. for $i = n-1, n-2, ..., 1, 0$ do

2.1 $f_1 \leftarrow f_1^2 \cdot l_{T,T}(Q)$, $f_2 \leftarrow f_2^2 \cdot l_{T,T}(\hat{\phi}(Q))$, $T \leftarrow 2T$

2.2 if $l_i = 1$ then

2.3 $f_1 \leftarrow f_1 \cdot l_{T,P}(Q)$, $f_2 \leftarrow f_2 \cdot l_{T,P}(\hat{\phi}(Q))$, $T \leftarrow T + P$

3. $f_1 \leftarrow f_1^{\lambda+1}$,

4. return $(f_1 f_2 f_3)^{(p-1)(p+1)/r}$

4 Efficiency Consideration

Now the performance of the proposed algorithm is considered in this section. We neglect the cost of field additions and subtractions, as well as the cost of multiplication by small constants. The computational cost of one multiplication in $\mathbb{F}_p^*$ is denoted as M.

We first point out that $\hat{\phi}(Q)$ can be precalculated using only $1M$ because β^2 and x-coordinate of Q are in $\mathbb{F}_p^*$. The evaluation of $l(Q)$ and $l(\hat{\phi}(Q))$ cost $2M$ since the equation of the line can be reused. One multiplication and one square in $\mathbb{F}_{p^2}^*$ require $3M$ and $2M$, respectively [11]. We assume that the computational cost of an inverse in $\mathbb{F}_p^*$ is $10M$. We also count one square as one multiplication in $\mathbb{F}_p^*$. One point doubling requires $14M$ and one point addition requires $13M$ in $E(\mathbb{F}_p)$, respectively [7].

For good efficiency, we can make that λ has a low Hamming weight. Scott have found such a suitable elliptic curve with $\lambda = 2^{80} + 2^{16}$ using Cocks-Pinch algorithm [10].We compute the Tate pairing on this curve using the proposed algorithm in the following. Note that $r = \lambda^2 + \lambda + 1$ has 161 bits there. Hence a is equal to 1 in this case. Therefore the value of $e(P,Q)^a$ is same as the value of the traditional Tate pairing. Furthermore, the numbers of iteration loop in the new algorithm is a half of the numbers of iteration loop in the traditional Miller's algorithm. However, the new algorithm requires more multiplications than the traditional algorithm in one iteration loop.

Now we give a detailed efficiency consideration on the new algorithm. Let $P = (x_P, y_P) \in E(\mathbb{F}_p)[r]$ and $Q = (x_Q, y_Q) \in E(\mathbb{F}_{p^2})$, where $x_Q \in \mathbb{F}_p$ and $y_Q \in \mathbb{F}_{p^2}$. We first consider the cost of line 1 in Algorithm 2. It is easily checked that $l_{\phi(P),\hat{\phi}(P)}(Q)$ is equal to $y_Q - y_P$ since P, $\phi(P)$ and $\hat{\phi}(P)$ have the same y-coordinate. So line 1 requires no multiplications in Algorithm 2. Now we consider the computational cost of line 2.1 in Algorithm 2. The cost of one point doubling is $14M$. The two line equation evaluations require $2M$ since $\hat{\phi}(Q)$ can be precalculated easily. The remainder in 2.1 requires two squares and two multiplications in $F_{p^2}^*$, which cost $10M$. Therefore line 2.1 requires $26M$. The numbers of the main loop are 79, so the total cost of line 2.1 is $26 \cdot 79 = 2054M$. It is not difficult to show that the total cost of line 2.3 requires $25M$. Line 3 requires $80 \cdot 2 = 160M$ for this exponentiation. By now we cost $2054 + 25 + 160 = 2239M$. There are two multiplications in $\mathbb{F}_{p^2}$ in line 4 of Algorithm 2, which require $6M$. The exponentiation $(p-1)$ requires 5 multiplications and one inverse in $\mathbb{F}_p^*$ since

the Frobenius map can be used here. The exponentiation $(p+1)/r$ requires $(512-161)\cdot 2 = 702M$ using the Lucas laddering algorithm mainly [12]. So the total contribution of line 4 is $6+15+702 = 723M$. Therefore the total computational cost of the new algorithm is $2239+723 = 2962M$.

Finally, we compare the new algorithm with the previous methods at the same levels of security in Table 1. Algorithm 4 in [10] computes the Tate pairing on the same elliptic curve. However, it requires the whole iteration main loop in Miller's algorithm. In [11], Scott analyzes the efficiency of the pairing calculation in IBE scheme, which requires $4070M$. From Table 1, it shows that the proposed algorithm is more efficient than the previous algorithms indeed at the same levels of security .

Table 1. Cost comparisons of the proposed algorithms

Algorithm	Cost of Multiplications in $\mathbb{F}_p^*$
the proposed algorithm	2962M
Algorithm 4 in [10]	3329M
Miller's algorithm in [11]	4070M

5 Conclusion

A new efficient algorithm has been proposed for computing the bilinear pairing on a family of non-supersingular curves, which have efficient endomorphisms with degree 1. The main technique is to shorten the main loop in Miller's algorithm, which is same as the eta pairing and the ate pairing. The proposed method is more efficient than the previous methods on these elliptic curves. It should be pointed out that the new method can be used for the large embedding degrees. It is possible that the new algorithm can be further optimized.

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