Constructing new APN functions from known ones

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Abstract

We present a method for constructing new quadratic APN functions from known ones. Applying this method to the Gold power functions we construct an APN function $x^3+\mathrm{tr}(x^9)$ over $\mathbb{F}_{2^n}$. It is proven that in general this function is CCZ-inequivalent to the Gold functions (and therefore EA-inequivalent to power functions), to the inverse and Dobbertin mappings, and in the case $n = 7$ it is CCZ-inequivalent to all power mappings.

Keywords: Affine equivalence, Almost bent, Almost perfect nonlinear, CCZ-equivalence, Differential uniformity, Nonlinearity, S-box, Vectorial Boolean function

1 Introduction

A function $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is called *almost perfect nonlinear* (APN) if, for every $a \neq 0$ and every b in $\mathbb{F}_2^n$, the equation $F(x)+F(x+a) = b$ admits at most two solutions (it is also called *differentially* 2*-uniform*). Vectorial Boolean functions used as S-boxes in block ciphers must have low differential uniformity to allow high resistance to the differential cryptanalysis (see [3, 31]). In this sense APN functions are optimal. The notion of APN function is closely connected to the notion of almost bent (AB) function. A function $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is called AB if the minimum Hamming distance between all the Boolean functions $v \cdot F$, $v \in \mathbb{F}_2^n \setminus \{0\}$ (called the *component* functions of F), and all affine Boolean functions on $\mathbb{F}_2^n$ is maximal. Here, "$\cdot$" denotes the usual inner product in $\mathbb{F}_2^n$. Any other choice of an inner product would lead to the same notion. For instance, the vector space $\mathbb{F}_2^n$ can be identified to the field $\mathbb{F}_{2^n}$ and we can then take for inner product $x \cdot y = \mathrm{tr}(xy)$ where tr is the absolute trace function. The minimum Hamming distance between all component functions of F and all affine Boolean functions on $\mathbb{F}_2^n$ is called the *nonlinearity* of F and its maximum equals $2^{n-1} - 2^{\frac{n-1}{2}}$ (see [16]). AB functions exist for n odd only and oppose an

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optimum resistance to the linear cryptanalysis (see [29, 16]). Besides, every AB function is APN [16], and in the n odd case, any quadratic function is APN if and only if it is AB [15].

The APN and AB properties are preserved by some transformations of functions [15, 31]. If F is an APN function, A_1, A_2 are affine permutations and A is affine then the function $F' = A_1 \circ F \circ A_2 + A$ is also APN (the functions F and F' are called extended affine equivalent (*EA-equivalent*)). Besides, the inverse of any APN permutation is APN too. Until recently, the only known constructions of APN and AB functions were EA-equivalent to power functions $F(x) = x^d$ over finite fields ($\mathbb{F}_{2^n}$ being identified with $\mathbb{F}_2^n$). Table 1 gives all known values of exponents d (up to multiplication by a power of 2 modulo $2^n - 1$, and up to taking the inverse when a function is a permutation) such that the power function x^d over $\mathbb{F}_{2^n}$ is APN. For n odd the Gold, Kasami, Welch and Niho APN functions from Table 1 are also AB (for the proofs of AB property see [12, 13, 24, 25, 27, 31]).

Table 1
Known APN power functions x^d on $\mathbb{F}_{2^n}$.

Functions	Exponents d	Conditions	Proven in
Gold	$2^i + 1$	$\gcd(i, n) = 1$	[24, 31]
Kasami	$2^{2i} - 2^i + 1$	$\gcd(i, n) = 1$	[26, 27]
Welch	$2^t + 3$	$n = 2t + 1$	[21]
Niho	$2^t + 2^{\frac{t}{2}} - 1$, t even $2^t + 2^{\frac{3t+1}{2}} - 1$, t odd	$n = 2t + 1$	[20]
Inverse	$2^{2t} - 1$	$n = 2t + 1$	[2, 31]
Dobbertin	$2^{4t} + 2^{3t} + 2^{2t} + 2^t - 1$	$n = 5t$	[22]

Table 2
Known APN functions EA-inequivalent to power functions on $\mathbb{F}_{2^n}$.

Functions	Conditions	Alg. degree
$x^{2^i+1} + (x^{2^i} + x + \mathrm{tr}(1) + 1)\,\mathrm{tr}(x^{2^i+1} + x\,\mathrm{tr}(1))$	$n \geq 4$ $\gcd(i, n) = 1$	3
$\left(x + \mathrm{tr}_{n/3}(x^{2(2^i+1)} + x^{4(2^i+1)}) + \mathrm{tr}(x)\,\mathrm{tr}_{n/3}(x^{2^i+1} + x^{2^{2i}(2^i+1)})\right)^{2^i+1}$	n divisible by 6 $\gcd(i, n) = 1$	4
$\left(x^{\frac{1}{2^i+1}} + \mathrm{tr}_{m/3}(x + x^{2^{2i}})\right)^{-1}$	n divisible by 3 $\gcd(2i, n) = 1$	4
$x^{2^i+1} + \mathrm{tr}_{n/m}(x^{2^i+1}) + x^{2^i}\,\mathrm{tr}_{n/m}(x) + x\ \mathrm{tr}_{n/m}(x)^{2^i}$ $+ \left(\mathrm{tr}_{n/m}(x)^{2^i+1} + \mathrm{tr}_{n/m}(x^{2^i+1}) + \mathrm{tr}_{n/m}(x)\right)^{\frac{1}{2^i+1}} (x^{2^i} + \mathrm{tr}_{n/m}(x)^{2^i} + 1)$ $+ \left(\mathrm{tr}_{n/m}(x)^{2^i+1} + \mathrm{tr}_{n/m}(x^{2^i+1}) + \mathrm{tr}_{n/m}(x)\right)^{\frac{2^i}{2^i+1}} (x + \mathrm{tr}_{n/m}(x))$	$m \neq n$ n odd n divisible by m $\gcd(i, n) = 1$	$m + 2$

In [15], Carlet, Charpin and Zinoviev introduced an equivalence relation of functions, more recently called CCZ-equivalence, which corresponds to the affine equivalence of the graphs of functions and preserves APN and AB properties. EA-equivalence is a particular

case of CCZ-equivalence and any permutation is CCZ-equivalent to its inverse [15]. In [9, 10], it is proven that CCZ-equivalence is more general, and applying CCZ-equivalence to the Gold mappings classes of APN functions EA-inequivalent to power functions are constructed in [5, 9, 10]. These classes are presented in Table 2. When n is odd, these functions are also AB.

These new results on CCZ-equivalence have raised several interesting questions. One of them is whether the known classes of APN power functions are CCZ-inequivalent. Partly the answer is given in [7]: it is proven that in general the Gold functions are CCZ-inequivalent to the Kasami and Welch functions, and that for different parameters $1 \leq i, j \leq \frac{n-1}{2}$ the Gold functions x^{2^i+1} and x^{2^j+1} are CCZ-inequivalent. Another interesting question is the existence of APN polynomials CCZ-inequivalent to power functions. In [23] it is shown that one of the ways to construct such polynomials is to consider linear combinations of two different Gold power functions. Using this approach they have introduced two quadratic APN binomials on $\mathbb{F}_{2^{10}}$ and $\mathbb{F}_{2^{12}}$ which are CCZ-inequivalent to power maps. After that, two infinite classes of quadratic APN binomials CCZ-inequivalent to power functions have been constructed in [6, 7, 8]. These classes are presented in Table 3 (this table gives all known classes of APN functions CCZ-inequivalent to power functions) for the cases n divisible by 3 and 4. Another approach for constructing quadratic APN polynomials CCZ-inequivalent to power functions is introduced in [18]: the idea is to consider quadratic hexanomials of a certain type over $\mathbb{F}_{2^{2m}}$ as good candidates for being differentially 4-uniform. This approach gives new examples of quadratic APN functions over $\mathbb{F}_{2^6}$ and $\mathbb{F}_{2^8}$ which are CCZ-inequivalent to power functions [18]. Similar approach was used to construct new quadratic APN quadrinomials over $\mathbb{F}_{2^6}$ in [30]. Also it is proven in [4] that for $n \leq 5$ there exist no APN functions CCZ-inequivalent to power mappings.

Table 3
Known APN functions CCZ-inequivalent to power functions on $\mathbb{F}_{2^n}$.

	Functions	Conditions	Proven in
The case n divisible by 3	$x^{2^s+1} + wx^{2^{ik}+2^{mk+s}}$	$n = 3k$, $\gcd(k,3) = \gcd(s,3k) = 1$ $k \geq 4$, $i = sk \mod 3$, $m = 3 - i$ w has the order $2^{2k} + 2^k + 1$	[7, 8]
The case n divisible by 4	$x^{2^s+1} + wx^{2^{ik}+2^{mk+s}}$	$n = 4k$, $\gcd(k,2) = \gcd(s,2k) = 1$ $k \geq 3$, $i = sk \mod 4$, $m = 4 - i$ w has the order $2^{3k} + 2^{2k} + 2^k + 1$	[6]
The new case	$x^3 + \mathrm{tr}(x^9)$	$n \geq 7$ $n > 2p$ for the smallest possible $p > 1$ such that $p \neq 3$, $\gcd(p,n) = 1$	Corollary 1 of the present paper

All constructions of APN polynomials CCZ-inequivalent to power functions mentioned above have not given new APN polynomials with coefficients in $\mathbb{F}_2$. A natural question is whether all APN polynomials with coefficients in $\mathbb{F}_2$ are CCZ-equivalent to power functions. In the present paper we show that the answer to this question is negative. We give a new approach for constructing quadratic APN functions and using it we construct a class of quadratic APN polynomials with coefficients in $\mathbb{F}_2$. We prove that the function $F(x) =$

$x^3 + \mathrm{tr}(x^9)$ is APN over $\mathbb{F}_{2^n}$ for any n, and that for almost all $n \geq 7$ it is CCZ-inequivalent to the Gold functions (and therefore EA-inequivalent to power functions), to the inverse and Dobbertin functions. Obviously, this function is AB for all odd n. We conjecture that for $n \geq 7$ the function F is CCZ-inequivalent to any power function. This conjecture is confirmed for the case $n = 7$. Further we show that applying CCZ-equivalence to quadratic APN functions, it is possible to construct classes of nonquadratic APN mappings CCZ-inequivalent to power functions. Note that the existence of APN functions CCZ-inequivalent to power functions and to quadratic functions is still an open problem.

2 Preliminaries

Let $\mathbb{F}_2^n$ be the n-dimensional vector space over the field $\mathbb{F}_2$. Any function F from $\mathbb{F}_2^n$ to itself can be uniquely represented as a polynomial on n variables with coefficients in $\mathbb{F}_2^n$, whose degree with respect to each coordinate is at most one:

$$F(x_1, ..., x_n) = \sum_{u \in \mathbb{F}_2^n} c(u)\big(\prod_{i=1}^{n} x_i^{u_i}\big), \qquad c(u) \in \mathbb{F}_2^n.$$

This representation is called the *algebraic normal form* of F and its degree $d^\circ(F)$ the *algebraic degree* of the function F.

Besides, the field $\mathbb{F}_{2^n}$ can be identified with $\mathbb{F}_2^n$ as a vector space. Then, viewed as a function from this field to itself, F has a unique representation as a univariate polynomial over $\mathbb{F}_{2^n}$ of degree smaller than 2^n:

$$F(x) = \sum_{i=0}^{2^n-1} c_i x^i, \quad c_i \in \mathbb{F}_{2^n}.$$

For any k, $0 \leq k \leq 2^n - 1$, the number $w_2(k)$ of the nonzero coefficients $k_s \in \{0, 1\}$ in the binary expansion $\sum_{s=0}^{n-1} 2^s k_s$ of k is called the 2-*weight* of k. The algebraic degree of F is equal to the maximum 2-weight of the exponents i of the polynomial $F(x)$ such that $c_i \neq 0$, that is, $d^\circ(F) = \max_{0 \leq i \leq n-1, c_i \neq 0} w_2(i)$ (see [15]).

A function $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is *linear* if and only if $F(x)$ is a linearized polynomial over $\mathbb{F}_{2^n}$, that is,

$$\sum_{i=0}^{n-1} c_i x^{2^i}, \quad c_i \in \mathbb{F}_{2^n}.$$

The sum of a linear function and a constant is called an *affine function*.

Let F be a function from $\mathbb{F}_{2^n}$ to itself and A_1, $A_2 : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ be affine permutations. The functions F and $A_1 \circ F \circ A_2$ are then called *affine equivalent*. Affine equivalent functions have the same algebraic degree (i.e. the algebraic degree is *affine invariant*).

As recalled in the Introduction, we say that the functions F and F' are *extended affine equivalent* if $F' = A_1 \circ F \circ A_2 + A$ for some affine permutations A_1, A_2 and an affine function A. If F is not affine, then F and F' have again the same algebraic degree.

Two mappings F and F' from $\mathbb{F}_{2^n}$ to itself are called Carlet-Charpin-Zinoviev equivalent (*CCZ-equivalent*) if the graphs of F and F', that is, the subsets $G_F = \{(x, F(x)) \mid x \in \mathbb{F}_{2^n}\}$ and $G_{F'} = \{(x, F'(x)) \mid x \in \mathbb{F}_{2^n}\}$ of $\mathbb{F}_{2^n} \times \mathbb{F}_{2^n}$, are affine equivalent. Hence, F and F' are CCZ-equivalent if and only if there exists an affine automorphism $\mathcal{L} = (L_1, L_2)$ of $\mathbb{F}_{2^n} \times \mathbb{F}_{2^n}$ such that

$$y = F(x) \Leftrightarrow L_2(x, y) = F'(L_1(x, y)).$$

Note that since $\mathcal{L}$ is a permutation then the function $L_1(x, F(x))$ has to be a permutation too (see [7]). As shown in [15], EA-equivalence is a particular case of CCZ-equivalence and any permutation is CCZ-equivalent to its inverse.

For a function $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ and any elements $a, b \in \mathbb{F}_{2^n}$ we denote

$$\delta_F(a, b) = |\{x \in \mathbb{F}_2^n : F(x + a) + F(x) = b\}|.$$

F is called a *differentially δ-uniform* function if $\max_{a \in \mathbb{F}_{2^n}^*, b \in \mathbb{F}_{2^n}} \delta_F(a, b) \leq \delta$. Note that $\delta \geq 2$ for any function over $\mathbb{F}_{2^n}$. Differentially 2-uniform mappings are called *almost perfect nonlinear*.

For any function $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ we denote

$$\lambda_F(a, b) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{\mathrm{tr}(bF(x) + ax)}, \qquad a, b \in \mathbb{F}_{2^n},$$

where $\mathrm{tr}(x) = x + x^2 + x^4 + ... + x^{2^{n-1}}$ is the trace function from $\mathbb{F}_{2^n}$ into $\mathbb{F}_2$. The set $\Lambda_F = \{\lambda_F(a, b) : a, b \in \mathbb{F}_{2^n}, b \neq 0\}$ is called the *Walsh spectrum* of the function F and the multiset $\{|\lambda_F(a, b)| : a, b \in \mathbb{F}_{2^n}, b \neq 0\}$ is called the *extended Walsh spectrum* of F. The value

$$\mathcal{NL}(F) = 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_{2^n}, b \in \mathbb{F}_{2^n}^*} |\lambda_F(a, b)|$$

equals the *nonlinearity* of the function F. The nonlinearity of any function F satisfies the inequality

$$\mathcal{NL}(F) \leq 2^{n-1} - 2^{\frac{n-1}{2}}$$

([16, 33]) and in case of equality F is called *almost bent* or *maximum nonlinear*.

Obviously, AB functions exist only for n odd. It is proven in [16] that every AB function is APN and its Walsh spectrum equals $\{0, \pm 2^{\frac{n+1}{2}}\}$. If n is odd, every APN mapping which is quadratic (that is, whose algebraic degree equals 2) is AB [15], but this is not true for nonquadratic cases: the Dobbertin and the inverse APN functions are not AB (see [13, 15]). When n is even, the inverse function x^{2^n-2} is a differentially 4-uniform permutation [31] and has the best known nonlinearity [28], that is $2^{n-1} - 2^{\frac{n}{2}}$ (see [13, 19]). This function has been chosen as the basic S-box, with $n = 8$, in the Advanced Encryption Standard (AES), see [17]. A comprehensive survey on APN and AB functions can be found in [14].

It is shown in [15] that, if F and G are CCZ-equivalent, then F is APN (resp. AB) if and only if G is APN (resp. AB). More generally, CCZ-equivalent functions have the same differential uniformity and the same extended Walsh spectrum (see [9]). Further

invariants for CCZ-equivalence are given in [23] (see also [18]) in terms of group algebras. Let $G = \mathbb{F}_2[\mathbb{F}_{2^n} \times \mathbb{F}_{2^n}]$ be the group algebra of $\mathbb{F}_{2^n} \times \mathbb{F}_{2^n}$ over $\mathbb{F}_2$. It consists of the formal sums

$$\sum_{g \in G} a_g g$$

where $a_g \in \mathbb{F}_2$. If S is a subset of $\mathbb{F}_{2^n} \times \mathbb{F}_{2^n}$ then it can be identified with the element $\sum_{s \in S} s$ of G. For any APN mapping F we denote

$$\Delta_F = \{(a, b) : F(x) + F(x + a) = b \text{ has 2 solutions}\} \subset \mathbb{F}_{2^n} \times \mathbb{F}_{2^n} .$$

The dimensions of the ideals of G generated by Δ_F and by the graph G_F of F are called Δ- and Γ-ranks, respectively. According to [23] (and also [18]), Δ- and Γ-ranks of a function are CCZ-invariant.

3 Construction of new quadratic APN functions

In the theorem below we give a general approach for constructing new quadratic APN functions from known ones.

Theorem 1 *Let F be a quadratic APN function from $\mathbb{F}_2^n$ to itself, let f be a quadratic Boolean function on $\mathbb{F}_2^n$ and*

$$\varphi_F(x, a) = F(x) + F(x + a) + F(a) + F(0),$$

$$\varphi_f(x, a) = f(x) + f(x + a) + f(a) + f(0).$$

Then the function $F(x) + f(x)$ is APN if for every nonzero $a \in \mathbb{F}_2^n$ there exists a linear Boolean function ℓ_a satisfying the conditions

1) $\varphi_f(x, a) = \ell_a(\varphi_F(x, a))$,

2) *if $\varphi_F(x, a) = 1$ for some $x \in \mathbb{F}_2^n$ then $\ell_a(1) = 0$.*

Proof. Since the function $F(x) + f(x)$ is quadratic, it is APN if and only if, for every nonzero $a \in \mathbb{F}_2^n$, the equation $\varphi_F(x, a) + \varphi_f(x, a) = 0$ admits at most two solutions (see e.g. [14]). According to the hypothesis on ℓ_a, a solution to this equation must be such that $\varphi_f(x, a) = 0$ and therefore such that $\varphi_F(x, a) = 0$. Then, F being quadratic APN, this equation admits at most two solutions. □

Remark 1 Note that, in the situation of Theorem 1, a linear function l_a satisfying $\varphi_f(x, a) = \ell_a(\varphi_F(x, a))$ always exists. This is due to the fact that, by the assumption F is APN and then the kernel of $\varphi_F(x, a)$ equals $\{0, a\}$. This set is always a subset of the kernel of $\varphi_f(x, a)$, which is indeed the necessary and sufficient condition for the existence of l_a.

A direct consequence of Theorem 1 is that, if F is APN and if ℓ is a linear form such that $\ell(1) = 0$, then the function $F(x) + \ell(F(x))$ is APN. But this function is affine equivalent to F since it is equal to $L \circ F$ where $L(x) = x + \ell(x)$, and the condition that $\ell(1) = 0$ is equivalent to saying that L is a permutation.

We give now an example where Theorem 1 leads to a function which is CCZ-inequivalent to the original function F.

Corollary 1 *Let n be any positive integer. Then the function $x^3 + \mathrm{tr}(x^9)$ is APN on $\mathbb{F}_{2^n}$.*

Proof. We can apply Theorem 1 with $F(x) = x^3$, $\varphi_F(x, a) = a^2x + ax^2$, $f(x) = \mathrm{tr}(x^9)$, $\varphi_f(x, a) = \mathrm{tr}(a^8x + ax^8)$ and $\ell_a(y) = \mathrm{tr}(a^6y + a^3y^2 + a^{-3}y^4)$. Indeed, we have then $\ell_a(\varphi_F(x, a)) = \mathrm{tr}\left(a^6(a^2x + ax^2) + a^3(a^4x^2 + a^2x^4) + a^{-3}(a^8x^4 + a^4x^8)\right) = \varphi_f(x, a)$ and if there exists $x \in \mathbb{F}_2^n$ such that $\varphi_F(x, a) = 1$ then $\ell_a(1) = \mathrm{tr}\left(a^{-3}\right) = \mathrm{tr}\left(\frac{x}{a} + \left(\frac{x}{a}\right)^2\right) = 0$. □

Remark 2 Note that the same principle as in Theorem 1 allows generating a large variety of differentially 4-uniform functions from APN functions. For example, for any APN function F the following functions are differentially 4-uniform

- $F(x) + \mathrm{tr}(G(x))$ for any function G;
- $F \circ A$ and $A \circ F$ for any affine function A which is 2-to-1. □

4 CCZ-inequivalence of the new APN function to power mappings

Theorem 2 *The function of Corollary 1 is CCZ-inequivalent to any Gold function on $\mathbb{F}_{2^n}$ if $n \geq 7$ and $n > 2p$ where p is the smallest positive integer different from 1 and 3 and coprime with n.*

Proof. Let $F(x) = x^3 + \mathrm{tr}(x^9)$ and $G(x) = x^{2^r+1}$ be APN functions on $\mathbb{F}_{2^n}$, $n \geq 7$, $r \leq (n-1)/2$.

Suppose the functions F and G are EA-equivalent. Then, there exist affine permutations L_1, L_2 and an affine function L' such that

$$L_1(x^3) + L_1(\mathrm{tr}(x^9)) = (L_2(x))^{2^t+1} + L'(x).$$

That is,

$$L_1(x^3) + L_1(1)\,\mathrm{tr}(x^9) = (L_2(x))^{2^t+1} + L'(x).$$

Since the functions are quadratic, we can assume without loss of generality that L_1 and L_2 are linear: $L_1(x) = \sum_{m\in\mathbb{Z}/n\mathbb{Z}} b_m x^{2^m}$, $L_2(x) = \sum_{p\in\mathbb{Z}/n\mathbb{Z}} c_p x^{2^p}$. Then we get

$$\sum_{m\in\mathbb{Z}/n\mathbb{Z}} b_m x^{3\cdot 2^m} + \mathrm{tr}(x^9) \sum_{m\in\mathbb{Z}/n\mathbb{Z}} b_m = \sum_{l,p\in\mathbb{Z}/n\mathbb{Z}} c_p c_l^{2^t} x^{2^{l+t}+2^p} + L'(x). \tag{1}$$

On the left hand side of the identity (1) we have only items of the type $x^{3\cdot 2^m}$, $x^{9\cdot 2^m}$, with some coefficients. Therefore this must be true also for the right hand side of the identity.

Let p be the smallest positive integer different from 1 and 3 such that $\gcd(n,p)=1$ (for example, if n is odd then $p=2$, if n is even and not divisible by 5 then $p=5$). If $n>2p$ then 2^p+1 is not in the same cyclotomic coset with 3 or 9. Therefore, the items of the type $x^{2^k(2^p+1)}$ must cancel. That is, for any k

$$c_k c_{k-t+p}^{2^t} = c_{k+p} c_{k-t}^{2^t}. \tag{2}$$

Since $n \geq 7$ then 3 and 9 are in different cyclotomic cosets and we have for any k

$$L_1(1) = c_k c_{k-t+3}^{2^t} + c_{k+3} c_{k-t}^{2^t}.$$

If $L_1(1) \neq 0$ then

$$c_k c_{k-t+3}^{2^t} \neq c_{k+3} c_{k-t}^{2^t}. \tag{3}$$

If $c_k \neq 0$ for all k then from (2) and (3) we get

$$c_k c_{k-t}^{-2^t} = c_{k+p} c_{k-t+p}^{-2^t}, \tag{4}$$

$$c_k c_{k-t}^{-2^t} \neq c_{k+3} c_{k-t+3}^{-2^t}. \tag{5}$$

Since $\gcd(n,p)=1$ and from (4)

$$c_k c_{k-t}^{-2^t} = c_m c_{m-t}^{-2^t}$$

for any m. It contradicts (5). Thus, $c_k = 0$ for some k. Then from (2) and (3) we get that $c_{k+p}=0$. Repeating this step for $c_{k+p}, c_{k+2p}, \ldots$ we get $c_{k+ps}=0$ and since $\gcd(n,p)=1$ then $c_k=0$ for all k. A contradiction. If $L_1(1)=0$ then the equation $L(x)=0$ has at least 2 solutions $0,1$ and therefore L_1 is not a permutation. Thus, F and G are EA-inequivalent.

Suppose that $F(x)$ and $G(x)$ are CCZ-equivalent, that is, there exists an affine automorphism $L=(L_1,L_2)$ of $\mathbb{F}_{2^n}\times\mathbb{F}_{2^n}$ such that $y=F(x) \Leftrightarrow L_2(x,y)=G(L_1(x,y))$ and $L_1(x,F(x))$ is a permutation. This implies then $L_2(x,F(x))=G(L_1(x,F(x)))$. Writing $L_1(x,y)=L(x)+L'(y)$ and $L_2(x,y)=L''(x)+L'''(y)$ gives

$$L''(x)+L'''(F(x)) = G\left(L(x)+L'(F(x))\right). \tag{6}$$

We can write

$$\begin{aligned}
L(x) &= b+\sum_{m\in\mathbb{Z}/n\mathbb{Z}} b_m x^{2^m},\\
L'(x) &= b'+\sum_{m\in\mathbb{Z}/n\mathbb{Z}} b'_m x^{2^m},\\
L''(x) &= b''+\sum_{m\in\mathbb{Z}/n\mathbb{Z}} b''_m x^{2^m},\\
L'''(x) &= b'''+\sum_{m\in\mathbb{Z}/n\mathbb{Z}} b'''_m x^{2^m},\\
b+b' &= c.
\end{aligned}$$

Then we get

$$\begin{aligned}
G(L(x) &+ L'(F(x))) = \left(L(x) + L'(x^3 + \mathrm{tr}(x^9))\right)\left(L(x) + L'(x^3 + \mathrm{tr}(x^9))\right)^{2^r} \\
&= \left(c + \sum_{m\in\mathbb{Z}/n\mathbb{Z}} b_m x^{2^m} + \sum_{m\in\mathbb{Z}/n\mathbb{Z}} b'_m x^{2^m(2+1)} + \mathrm{tr}(x^9)\sum_{m\in\mathbb{Z}/n\mathbb{Z}} b'_m\right) \\
&\times \left(c^{2^r} + \sum_{m\in\mathbb{Z}/n\mathbb{Z}} b_m^{2^r} x^{2^{m+r}} + \sum_{m\in\mathbb{Z}/n\mathbb{Z}} b_m'^{2^r} x^{2^{m+r}(2+1)} + \mathrm{tr}(x^9)\sum_{m\in\mathbb{Z}/n\mathbb{Z}} b_m'^{2^r}\right) \\
&= Q(x) + [\sum_{m,k\in\mathbb{Z}/n\mathbb{Z}} b_m b_k'^{2^r} x^{2^m+2^{k+r}+2^{k+r+1}} \\
&+ L'(1)^{2^r}\sum_{m,k\in\mathbb{Z}/n\mathbb{Z}} b_m x^{2^m+2^{k+3}+2^k} + \sum_{m,k\in\mathbb{Z}/n\mathbb{Z}} b'_m b_k^{2^r} x^{2^{m+1}+2^m+2^{k+r}} \\
&+ L'(1)\sum_{m,k\in\mathbb{Z}/n\mathbb{Z}} b_m^{2^r} x^{2^{m+r}+2^{k+3}+2^k)}] + [\sum_{m,k\in\mathbb{Z}/n\mathbb{Z}} b'_m b_k'^{2^r} x^{2^{m+1}+2^m+2^{k+r+1}+2^{k+r}} \\
&+ L'(1)^{2^r}\sum_{m,k\in\mathbb{Z}/n\mathbb{Z}} b'_m x^{2^{m+1}+2^m+2^{k+3}+2^k} + L'(1)\sum_{m,k\in\mathbb{Z}/n\mathbb{Z}} b_m'^{2^r} x^{2^{m+r+1}+2^{m+r}+2^{k+3}+2^k}],
\end{aligned}$$

where $Q(x)$ is a quadratic polynomial. Obviously, all terms in the expression above whose exponents have 2-weight strictly greater than 2 must cancel.
Since F and G are EA-inequivalent then L' is not a constant. Then there exists $m \in \mathbb{Z}/n\mathbb{Z}$ such that $b'_m \neq 0$.

Let $L'(1) \neq 0$. Since the items with the exponet $2^{m+1}+2^m+2^{m+2}+2^{m+5}$ have to vanish then we get $L'(1)^{2^r} b'_m = L'(1) b_{m-r}'^{2^r}$ and since $L'(1) \neq 0, b'_m \neq 0$ and r is coprime with n then $b'_k \neq 0$ and $b'_k b_{k-r}'^{-2^r} = L'(1)^{1-2^r}$ for all k. Now we can deduce that $b'_{k+r} = L'(1)^{1-2^r} b_k'^{2^r}$ for all k. Then, introducing μ such that $L'(1)^{1-2^r} = \mu^{2^r-1}$, we deduce that $\mu b'_{k+r} = (\mu b'_k)^{2^r}$ for all k and then that $\mu b'_{k+1} = (\mu b'_k)^2$ (using that $\gcd(r,n) = 1$) and then $\mu b'_k = (\mu b'_0)^{2^k}$. This means that $\mu L'(x) = \mu b' + \mathrm{tr}(\mu b'_0 x)$. It implies that all nonquadratic items in the last bracket vanish and $L'(x) = d + \mathrm{tr}(d'x)$ for some d, d'.
The function L is not 0 because L' is not a permutation, then $b_m \neq 0$ for some m. Since the items with the exponent $2^m + 2^{m+2} + 2^{m+5}$ have to vanish then $L'(1)^{2^r} b_m = L'(1) b_{m-r}'^{2^r}$. Like above we get $L(x) = d + \mathrm{tr}(d'x)$. Thus, $L_1(x, F(x)) = d'' + \mathrm{tr}(F'(x))$ for some d' and $F'(x)$ and $L_1(x, F(x))$ is not a permutation. A contradiction.

Let $L'(1) = 0$ and $r \neq 1$. Then $2^{m+1} + 2^m + 2^{m+r+1} + 2^{m+r}$ has 2-weight 4 and since the items with this exponent should cancel then we get $b_m'^{2^r+1} = b'_{m+r} b_{m-r}'^{2^r}$. Since $b'_m \neq 0$ then $b'_{m+r}, b'_{m-r} \neq 0$ and $b'_m b_{m-r}'^{-2^r} = b'_{m+r} b_m'^{-2^r}$. Since $\gcd(n,r) = 1$ then $b'_k \neq 0$, $b'_k b_{k-r}'^{-2^r} = b'_m b_{m-r}'^{-2^r}$ for all k and this implies $L'(x) = d + \mathrm{tr}(d'x)$ for some d, d'. Since $L_1(x, F(x))$ is a permutation then $L \neq 0$ and $b_m \neq 0$ for some m. The items with the exponent $2^m + 2^{m+r} + 2^{m+r+1}$ should vanish. Therefore, $b_m b_m'^{2^r} = b'_{m+r} b_{m-r}'^{2^r}$ and $b_m b_{m-r}'^{-2^r} = b'_{m+r} b_m'^{-2^r}$. As above it leads to the equality $L(x) = d + \mathrm{tr}(d'x)$ which is in contradiction with $L_1(x, F(x))$ being a permutation.

Let $L'(1) = 0$ and $r = 1$. Since $L'(1) = 0$ and $b'_m \neq 0$ then there exists t such that $b'_{m+t} \neq 0$. If $t \neq -1, -2$ then $2^{m+1} + 2^m + 2^{m+t+2} + 2^{m+t+1}$ has 2-weight 4 and we get $b'_m b'^{2^r}_{m+t} = b'_{m+t+1} b'^{2^r}_{m-1}$ and $b'_m b'^{-2^r}_{m-1} = b'_{m+t+1} b'^{-2^r}_{m+t}$. Therefore, $L'(x) = d + \mathrm{tr}(d'x)$ for some d, d'. If $t \neq 1, 2$ then $2^{m+t+1} + 2^{m+t} + 2^{m+2} + 2^{m+1}$ has 2-weight 4 and we get $b'_{m+t} b'^{2^r}_m = b'_{m+1} b'^{2^r}_{m+t-1}$ and again $L'(x) = d + \mathrm{tr}(d'x)$ for some d, d'. Thus, $L \neq 0$ and then $b_m \neq 0$ for some m. Since the items with the exponent $2^m + 2^{m+2} + 2^{m+3}$ cancel then $b_m b'^{2^r}_{m+1} = b'_{m+2} b^{2^r}_{m-1}$ and $b_m b^{-2^r}_{m-1} = b'_{m+2} b'^{-2^r}_{m+1}$. This implies $L(x) = d + \mathrm{tr}(d'x)$ and, thus, $L_1(x, F(x))$ is not a permutation. Therefore, F and G are not CCZ-equivalent. □

Corollary 2 *The function of Corollary 1 is EA-inequivalent to any power function on $\mathbb{F}_{2^n}$ if $n \geq 7$ and $n > 2p$, where p is the smallest positive integer different from 1 and 3 and coprime with n.*

Proof. The function $F(x) = x^3 + \mathrm{tr}(x^9)$ is quadratic and by Theorem 2 it is EA-inequivalent to any quadratic power function. Since the algebraic degree is EA-invariant then F is EA-inequivalent to any power mapping. □

Dobbertin and inverse APN functions have unique Walsh spectra (except the case $n = 3$ when the inverse function is EA-equivalent to x^3) which are different from the Walsh spectra of quadratic APN functions (see [12, 15, 32]). Since the extended Walsh spectrum of a function is invariant under CCZ-equivalence then we can make the following conclusion.

Proposition 1 *The function of Corollary 1 is CCZ-inequivalent to the inverse and Dobbertin APN functions for $n \geq 7$.*

For $n = 7$ the Δ-rank of the function $F(x) = x^3 + \mathrm{tr}(x^9)$ equals 212 and differs from the Δ-ranks of the Kasami functions x^{13} and x^{23} (which equal 338 and 436, respectively). Thus, for $n = 7$ the function F is CCZ-inequivalent to Kasami functions, and by Theorem 2 to the Gold functions. Since in this field the Welch and Niho cases coincide with the Kasami cases then F is CCZ-inequivalent to all power maps on $\mathbb{F}_{2^7}$.

Corollary 3 *The function $F(x) = x^3 + \mathrm{tr}(x^9)$ is CCZ-inequivalent to power functions on $\mathbb{F}_{2^7}$.*

Conjecture 1 *The function $F(x) = x^3 + \mathrm{tr}(x^9)$ is CCZ-inequivalent to any power function on $\mathbb{F}_{2^n}$ if $n \geq 7$ and $n > 2p$, where p is the smallest positive integer different from 1 and 3 and coprime with n.*

Remark 3 Applying CCZ-equivalence to the quadratic APN function $F(x) = x^3 + \mathrm{tr}(x^9)$, it is possible to construct classes of *nonquadratic* APN mappings which are CCZ-inequivalent to power functions. For example,

- for n odd the function

$$x^3 + \mathrm{tr}(x^9) + (x^2 + x)\,\mathrm{tr}(x^3 + x^9)$$

and for n even the function

$$x^3 + \mathrm{tr}(x^9) + (x^2 + x + 1)\,\mathrm{tr}(x^3)$$

are CCZ-equivalent to F (using the affine permutation $\mathcal{L}(x, y) = (x + \mathrm{tr}(y), y)$) and have the algebraic degree 3;

- for n divisible by 6 the function

$$[x+\mathrm{tr}_{n/3}(x^6+x^{12})+\mathrm{tr}(x)\,\mathrm{tr}_{n/3}(x^3+x^{12})]^3+\mathrm{tr}([x+\mathrm{tr}_{n/3}(x^6+x^{12})+\mathrm{tr}(x)\,\mathrm{tr}_{n/3}(x^3+x^{12})]^9)$$

is CCZ-equivalent to F (using the affine permutation $\mathcal{L}(x, y) = (x+\mathrm{tr}_{n/3}(y^2+y^4), y)$) and have the algebraic degree 4.

The proof is the same as for the cases from [9, 10]. Note that for n even both functions $F'(x) = x^3 + (x^2 + x + 1)\,\mathrm{tr}(x^3)$ and $F'(x) + \mathrm{tr}(x^9)$ are APN like in the case of the functions x^3 and $x^3 + \mathrm{tr}(x^9)$. □

5 Further quadratic APN constructions?

There is a straightforward generalization of Theorem 1:

Theorem 3 *Let F be a quadratic APN function from $\mathbb{F}_{2^n}$ to itself, let f be a quadratic function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^m}$ where m is a divisor of n, and*

$$\varphi_F(x, a) = F(x) + F(x + a) + F(a) + F(0),$$

$$\varphi_f(x, a) = f(x) + f(x + a) + f(a) + f(0).$$

Then the function $F(x) + f(x)$ is APN if for every nonzero $a \in \mathbb{F}_{2^n}$ there exists a linear function ℓ_a from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^m}$ which satisfies the conditions

1) $\varphi_f(x, a) = \ell_a(\varphi_F(x, a))$,

2) *for every $u \in \mathbb{F}_{2^m}^*$, if $\varphi_F(x, a) = u$ for some $x \in \mathbb{F}_{2^n}$ then $\ell_a(u) \neq u$.*

We could find an application of Theorem 3:

Corollary 4 *Let $n = 2m$ where m is an even positive integer. Let us denote by $\mathrm{tr}_{n/m}$ the trace function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^m}$: $\mathrm{tr}_{n/m}(x) = x + x^{2^m}$. The functions $F(x) = x^3 + \mathrm{tr}_{n/m}(x^{2^m+2}) = x^3 + x^{2^m+2} + x^{2^{m+1}+1}$ and $F'(x) = x^3 + (\mathrm{tr}_{n/m}(x))^3$ are APN.*

But unfortunately, these functions are not EA-inequivalent to power functions. Indeed, let $G(x)$ be the Gold function $G(x) = x^{2^{m-1}+1}$. Let γ be any element of $\mathbb{F}_4 \setminus \mathbb{F}_2$ and L_1, L_2 be the linear mappings $L_1(x) = \gamma^2 x^{2^{m+1}} + \gamma\, x^2$, $L_2(x) = \gamma\, x^{2^m} + \gamma^2 x$. Then $\mathcal{L} = (L_1, L_2)$ is

an isomorphism since the system $\begin{cases} \gamma^2 x^{2^{m+1}} + \gamma\, x^2 = 0 \\ \gamma\, x^{2^m} + \gamma^2 x = 0 \end{cases}$ clearly admits 0 as only solution. And since $\gamma^{2^m} = \gamma$, $\gamma^{2^{m-1}} = \gamma^2$ and $\gamma + \gamma^2 = 1$, we have

$$\begin{aligned} G \circ L_1(x) &= \left(\gamma^2 x^{2^{m+1}} + \gamma\, x^2\right)^{2^{m-1}+1} = \left(\gamma\, x + \gamma^2 x^{2^m}\right)\left(\gamma^2 x^{2^{m+1}} + \gamma\, x^2\right) \\ &= \gamma\left(x^3 + x^{2^m+2} + x^{2^{m+1}+1}\right)^{2^m} + \gamma^2\left(x^3 + x^{2^m+2} + x^{2^{m+1}+1}\right) \\ &= L_2 \circ F(x). \end{aligned}$$

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