On the Decomposition of an Element of Jacobian of a Hyperelliptic Curve

Koh-ichi Nagao, nagao@kanto-gakuin.ac.jp,

Dept. of Engineering, Kanto-Gakuin Univ.

Abstract. In this manuscript, if a reduced divisor D_0 of hyperelliptic curve of genus g over an extension field $\mathbb{F}_{q^n}$ is written by a linear sum of ng elements of $\mathbb{F}_{q^n}$-rational points of the hyperelliptic curve whose x-coordinates are in the base field $\mathbb{F}_q$, D_0 is noted by a decomposed divisor and the set of such $\mathbb{F}_{q^n}$-rational points is noted by the decomposed factor of D_0. We propose an algorithm which checks whether a reduced divisor is decomposed or not, and compute the decomposed factor, if it is decomposed. This algorithm needs a process for solving equations system of degree 2, $(n^2-n)g$ variables, and $(n^2-n)g$ equations over $\mathbb{F}_q$. Further, for the cases $(g,n)=(1,3),(2,2)$, and $(3,2)$, the concrete computations of decomposed factors are done by computer experiments.
Keywords Index calculus attack, Jacobian, Hyperelliptic curve, DLP, Weil descent attack

1 Introduction

In [6], Gaudry presents a frame work of the following attack for the DLP of the Jacobian of a curve C over an extension field $\mathbb{F}_{q^n}$. A point of the Jacobian over the extension field $\mathbb{F}_{q^n}$ has some representation of the form $(x_1, x_2, ...)$ with $x_i \in \mathbb{F}_{q^n}$. In this attack, the set of the potentially smooth elements of index calculus is taken as $B_0 = \{(x_1, x_2, ...)|x_1, x_2, .., x_{ng} \in \mathbb{F}_q\}$ where g is the genus of the curve. (The elements of the first ng coordinates are in $\mathbb{F}_q$.) When the curve E is an elliptic curve, B_0 is taken as $B_0 = \{(x,y) \in E(\mathbb{F}_{q^n})|x \in \mathbb{F}_q\}$ and by the use of Semaev's formula [10], the decomposition of a $\mathbb{F}_{q^n}$ rational point of E into n elements of B_0 is checked by solving the equations system of degree 2^{n-1}, n variables and n equations over $\mathbb{F}_q$. However, in the other cases including hyperelliptic curve cases, there is no concrete formula working in the role of Semaev's formula. So the decomposition may be complicated. In this manuscript, we present an alternative attack. In this attack, the set of the potentially smooth elements is taken as

$$B_0 := \{P - \infty \in \mathrm{Div}_0(C)\,|P = (x,y) \in C(\mathbb{F}_{q^n}),\, x \in \mathbb{F}_q\}$$

where ∞ is some fixed point on $C(\mathbb{F}_{q^n})$. Note that when the curve is an elliptic curve, ∞ is taken as the unique point of infinity and B_0 is the same as Gaudry's one. In this manuscript, we will treat the case that the curve is a hyperelliptic curve, since it is very simple case in our situation. In the case that the curve is a hyperelliptic curve, further ∞ will be taken as the unique point at infinity. In

this manuscript, we will show that the decomposition of a reduced divisor into ng elements of B_0 is checked by solving equation systems of degree 2, $(n^2-n)g$ variables, and $(n^2-n)g$ equations over $\mathbb{F}_q$ by the use of Riemann-Roch theorem (not using Semaev's formula). The complexity of the decomposition in the elliptic curve case may be the same as that of Gaudry's method using Semaev's formula.
Further let C be a hyperelliptic curve (including elliptic curve) of genus g of the form

$$C : y^2 = f(x), \text{ where } f(x) = x^{2g+1} + a_{2g}x^{2g} + ... + a_0$$

over $\mathbb{F}_{q^n}$ where the characteristic of $\mathbb{F}_q$ is not 2 and $n \geq 2$. Let D_0 be a $\mathbb{F}_{q^n}$ rational point of the Jacobian of C. Since D_0 has Mumford representation i.e., it is written as follows:

$$D_0 = (\phi_1(x), \phi_2(x)),$$

where $\phi_1(x) \in \mathbb{F}_{q^n}[x]$ is a monic polynomial with $\deg(\phi_1(x)) \leq g$ and $\phi_2(x) \in \mathbb{F}_{q^n}[x]$ satisfies $\deg(\phi_2(x)) < \deg(\phi_1(x))$ and $f(x) - \phi_2(x)^2 \equiv 0 \bmod \phi_1(x)$. Further, we will assume $\deg(\phi_1(x)) = g$. So, put $\phi_{i,j} \in \mathbb{F}_{q^n}$ by

$$\phi_1(x) = x^g + \phi_{1,g-1}x^{g-1} + ... + \phi_{1,1}x + \phi_{1,0}, \ \phi_2(x) = \phi_{2,g-1}x^{g-1} + ... + \phi_{2,0}.$$

Note that there are $Q_1, .., Q_g \in C(\overline{\mathbb{F}}_{q^n})$ such that

$$D_0 = Q_1 + .. + Q_g - (g)\infty. \tag{1}$$

Put

$$B_0 := \{P - \infty \in \mathrm{Div}_0(C) \,|\, P = (x, y) \in C(\mathbb{F}_{q^n}), \, x \in \mathbb{F}_q\}.$$

We see easily that since $|Jac(C/\mathbb{F}_{q^n})| \approx q^{gn}$ and $|B_0| \approx q$, the probability that there are some $P_1, P_2, .., P_{ng} \in B_0$ (exactly ng elements) such that

$$\begin{aligned} &D_0 + P_1 + P_2 + ... + P_{ng} - (ng)\infty \\ &\quad = \textstyle\sum_{i=1}^{g} Q_i + P_1 + P_2 + ... + P_{ng} - (ng+g)\infty \sim 0 \end{aligned} \tag{2}$$

is $1/(gn)!$.
Further in this manuscript, we will fix a reduced divisor $D_0 \in Jac_C(\mathbb{F}_{q^n})$ and exceed the argument. So, from the notations of $\phi_1(x)$, $\phi_2(x)$, and $Q_1, ..., Q_g$, which are depended on D_0, they will be also fixed.

Definition 1 *If a reduced divisor D_0 (also assuming $\deg(\phi_1(x)) = g$) is written by the form (2), D_0 is called the potentially B_0-smooth reduced divisor and in this case, $\{P_i\}_{i=1}^{ng}$ are called decomposed factors.*

In this manuscript, we will show the following theorem.

Theorem 1. *Let $V_1, V_2, ..., V_{(n^2-n)g}$ be variables and let D_0 be a reduced divisor of $C/\mathbb{F}_{q^n}$. Then there are some degree 2 polynomials*

$$C_{i,j} \in \mathbb{F}_q[V_1, V_2, ..., V_{(n^2-n)g}] \quad (0 \leq i \leq ng-1,\, 0 \leq j \leq n-1)$$

satisfying the following.
The condition that D_0 is potentially B_0-smooth is equivalent to the following 1) and 2).
1) The equations system $S = \{C_{i,j} = 0 \,|\, 0 \leq i \leq ng-1,\, 1 \leq j \leq n-1\}$ has some solution $\boldsymbol{v} = (v_1, .., v_{(n^2-n)g}) \in \mathbb{A}^{(n^2-n)g}(\mathbb{F}_q)$.
2) Put $c_i = C_{i,0}(v_1, .., v_{(n^2-n)g})$ for $0 \leq i \leq ng-1$. Then $G(x) = x^{ng} + c_{ng-1}x^{ng-1} + ... + c_0 \in \mathbb{F}_q[x]$ factors completely.
Moreover, if D_0 is potentially B_0-smooth, the x-coordinates of the decomposed factor are the solutions of $G(x) = 0$.

In the next section, we will construct such multivariable polynomials $\{C_{i,j}\}$ and show Theorem 1.

2 Construction of equations system and proof of the theorem

In this section, we will construct the multivariable polynomials $\{C_{i,j}\}$ in Theorem 1 and prove this theorem. Let $D = \sum_{P \in C(\overline{\mathbb{F}}_{q^n})} n_p P$, $n_p \in \mathbb{Z}$ be a divisor of $C/\mathbb{F}_{q^n}$. Put $\deg(D) := \sum_{P \in C(\overline{\mathbb{F}}_{q^n})} n_p$, and $L(D) := \{f \in \overline{\mathbb{F}}_{q^n}(C) \,|\, (f) + D \geq 0\}$. From Riemann-Roch Theorem (cf [7] Corollary A.4.2.3), we have the following lemma.

Lemma 1. *(**Riemann-Roch**) 1) $L(D)$ is a $\overline{\mathbb{F}}_{q^n}$ vector space.*
2) If $\deg(D) \geq 2g-1$, $dim\, L(D) = \deg(D) - g + 1$.

From the equation of C, we see $\mathrm{ord}_\infty x = 2$, and $\mathrm{ord}_\infty y = 2g+1$. Put $N_1 := \lfloor \frac{(n+1)g}{2} \rfloor$ and $N_2 := \lfloor \frac{ng-g-1}{2} \rfloor$.

Lemma 2. *1) $N_1 + N_2 = ng - 1$.*
2) $N_2 + g - 1 < N_1$.

Proof. Trivial.

Lemma 3. *$\{1, x, x^2, .., x^{N_1}, y, xy, ...x^{N_2}y\}$ is a base of $L((ng+g)\infty)$.*

Proof. From $\mathrm{ord}_\infty x = 2$, $\mathrm{ord}_\infty y = 2g+1$, each element in the above list is in $L(\,(ng+g)\infty)$. The independence is from the definition of the hyperelliptic curve. Thus, since the number of the elements of the list $N_1 + N_2 + 2 = ng + 2$ is the same as the dim $L((ng+g)\infty)$ (from Lemma 1), we have this lemma.

Lemma 4. *$\{\phi_1(x), \phi_1(x)x, ..., \phi_1(x)x^{N_1-g}, (y - \phi_2(x)), (y - \phi_2(x))x, ..., (y - \phi_2(x))x^{N_2}\}$ is a base of $L((ng)\infty - D_0) = L((ng+g)\infty - \sum_{i=1}^{g} Q_i)$.*

Proof. From the definition of $\phi_1(x)$ and $\phi_2(x)$, each element in the list has zero at each Q_i. Since $\deg(\phi_1(x)) = g$, $\deg(\phi_2(x)) \leq g-1$, and $N_2 + g - 1 < N_1$(from Lemma 2), each element in the list has at most $(ng+g)$ poles at ∞. Then they are in $L((ng)\infty - D_0)$. Now, we show the independence. Assume they are not independent, out that there are some non zero $f_1(x), f_2(x) \in \overline{\mathbb{F}}_{q^n}[x]$ such that $\phi_1(x)f_1(x) + (y - \phi_2(x))f_2(x) = 0$. However, the relation $\phi_1(x)f_1(x) + (y - \phi_2(x))f_2(x) = 0$ induces $yf_2(x) \in \overline{\mathbb{F}}_{q^n}[x]$ and $f_1(x) = f_2(x) = 0$. As it is a contradiction, they are independent. On the other hand, the number of the elements of the list is $N_1 + N_2 + 2 - g = ng - g + 2$ from Lemma 2, which is the same as the $\dim L((ng)\infty - D_0)$. So we have this lemma.

From this lemma, an element $h \in L((ng)\infty - D_0)$ is written by

$$h(x,y) = \phi_1(x)(A_0 + A_1x + ... + A_{N_1-g}x^{N_1-g}) + (y - \phi_2(x))(B_0 + B_1x + ... + B_{N_2}x^{N_2}) \tag{3}$$

where A_i,B_i are parameters moving in $\overline{\mathbb{F}}_{q^n}$.

Lemma 5. *Let $h(x,y) \in L((ng)\infty - D_0)$. Assume div(h(x,y)) is written by the form $P_1 + P_2 + ... + P_{ng} + \sum_{i=1}^{g} Q_i - (ng+g)\infty$ for $P_i \in C(\mathbb{F}_{q^n})$. Then we have the following:*
1) $A_{N_1-g} \neq 0$ when $ng+g$ is even.
2) $B_{N_2} \neq 0$ when $ng+g$ is odd.

Proof. When $ng+g$ is even, assume $A_{N_1-g} = 0$, thus we have the order of the zero of $h(x,y)$ being truly less than $ng+g$ and $\mathrm{div}(h(x,y))$ is not written by the form of (2). Similarly, when $ng+g$ is odd, assume $B_{N_2} = 0$, thus we have the order of the zero of $h(x,y)$ being truly less than $ng+g$ and $\mathrm{div}(h(x,y))$ is not written by the form of (2). So, we can assume that $A_{N_1-g} \neq 0$ if $ng+g$ is even and that $B_{N_2} \neq 0$ if $ng+g$ is odd.

Further, we compute the cross points of $h(x,y) = 0$ on C. For this purpose, y must be eliminated. From $h(x,y) = 0$, y is written by

$$y = \frac{(A_0 + A_1x + ... + A_{N_1-g}x^{N_1-g}) - \phi_2(x)(B_0 + B_1x + ... + B_{N_2}x^{N_2})}{B_0 + B_1x + ... + B_{N_2}x^{N_2}} \tag{4}$$

By this y's representation, the number of the parameters must be decrease. So, put $A_{N_1-g} = 1$ when $ng+g$ is even and put $B_{N_2} = 1$ when $ng+g$ is odd (this can be done from the above lemma). Also put $M_1 = \begin{cases} N_1 - g & \text{when } ng+g \text{ is even} \\ N_1 - g - 1 & \text{when } ng+g \text{ is odd} \end{cases}$, $M_2 = \begin{cases} N_2 - 1 & \text{when } ng+g \text{ is even} \\ N_2 & \text{when } ng+g \text{ is odd} \end{cases}$. Note that $M_1 + M_2 = ng - g - 1$ form Lemma 2.

Lemma 6. *Assume the assumption of the previous lemma. Then $A_i \in \mathbb{F}_{q^n}$ $(0 \leq i \leq M_1)$, $B_i \in \mathbb{F}_{q^n}$ $(0 \leq i \leq M_2)$. (i.e., The parameters A_i, B_i move in $\mathbb{F}_{q^n}$.)*

Proof. Let $\sigma \in Gal(\overline{\mathbb{F}}_{q^n}/\mathbb{F}_{q^n})$. Since $D_0 \in Jac_C(\mathbb{F}_{q^n})$, $(\sum Q_i)^\sigma = \sum Q_i$, and $P_i^\sigma = P_i$, we have $\mathrm{div}(h(x,y)^\sigma) = \mathrm{div}(h(x,y))^\sigma = \mathrm{div}(h(x,y))$ and $h(x,y)^\sigma = \mathcal{C} \cdot h(x,y)$ where $\mathcal{C}$ is some non-zero constant in $\bar{\mathbb{F}}_{q^n}$. Since the set $\{A_i\}_{i=0}^{N_1-g} \cup \{B_i\}_{i=0}^{N_2}$ is the set of the coefficients of the basis and A_{N_1-g} or $B_{N_2} \in \mathbb{F}_{q^n}$, we have $\mathcal{C} = 1$. Thus we have this lemma.

Put

$$S(x) := \begin{cases} -(\text{denominator of (4)})^2 f(x) + (\text{numerator of (4)})^2, & \text{if } ng+g \text{ is even} \\ (\text{denominator of (4)})^2 f(x) - (\text{numerator of (4)})^2, & \text{if } ng+g \text{ is odd} \end{cases}.$$

(Remember that $y^2 = f(x)$ is the equation of C.) From the construction, $S(x)$ is a monic polynomial of the degree $ng+g$, whose coefficients are degree 2 polynomials in $\mathbb{F}_{q^n}[A_0,..,A_{M_1},B_0,..,B_{M_2}]$, and $\phi_1(x)|S(x)$. Put $g(x) := S(x)/\phi_1(x)$. Since $\phi_1(x)$ is a monic polynomial in $\mathbb{F}_{q^n}[x]$, $g(x)$ is also a monic polynomial of degree ng, whose coefficients are degree 2 polynomials in $\mathbb{F}_{q^n}[A_0,..,A_{M_1},B_0,..,B_{M_2}]$. Put $C_i \in \mathbb{F}_{q^n}[A_0,..,A_{M_1},B_0,..,B_{M_2}]$ by i-th coefficient of $g(x)$, i.e.,

$$g(x) = x^{ng} + C_{ng-1}x^{ng-1} + ... + C_0.$$

Since the coefficients of $g(x)$ are written by the multivariable polynomials of the parameters $\{A_i\}_{i=0}^{M_1} \cup \{B_i\}_{i=0}^{M_2}$ which move in $\mathbb{F}_{q^n}$, $g(x)$ is considered as an element of $\mathbb{F}_{q^n}[x]$, if the parameters are considered as values in $\mathbb{F}_{q^n}$, and it is considered as an element of $\mathbb{F}_{q^n}[A_0,...A_{M_1},B_0,...B_{M_2},x]$, if the parameters are considered as variables.

The zeros of $g(x) = 0$ are the x-coordinate of the cross points of $h(x,y) = 0$ on C except $Q_1,...,Q_g$. Thus, we have the following lemma.

Lemma 7. *The condition that D_0 is a potentially B_0-smooth reduced divisor is equivalent to the following:*
There are some $A_0,..,A_{M_1},B_0,...B_{M_2} \in \mathbb{F}_{q^n}$ such that $g(x) \in \mathbb{F}_q[x]$ and $g(x) \in \mathbb{F}_q[x]$ factors completely in $\mathbb{F}_q[x]$.

Further, we find the $A_i \in \mathbb{F}_{q^n}$ $(0 \le i \le M_1)$ and $B_i \in \mathbb{F}_{q^n}$ $(0 \le i \le M_2)$ such that $g(x) \in \mathbb{F}_q[x]$. Let $[\alpha_0(=1),\alpha_1,..,\alpha_{n-1}]$ be a base of $\mathbb{F}_{q^n}/\mathbb{F}_q$. We will fix this base. Let $A_{i,j}$ $(0 \le i \le M_1, 0 \le j \le n-1)$, $B_{i,j}$ $(0 \le i \le M_2, 0 \le j \le n-1)$, be new parameters moving in $\mathbb{F}_q$ such that

$$A_i = \sum_{j=0}^{n-1} A_{i,j}\alpha_j \quad (0 \le i \le M_1), \quad B_i = \sum_{j=0}^{n-1} B_{i,j}\alpha_j \quad (0 \le i \le M_2).$$

Note that the number of the parameters $\{A_{i,j}\} \cup \{B_{i,j}\}$ is

$$(M_1 + M_2 + 2)n = (N_1 + N_2 - g + 1)n = (n^2 - n)g.$$

For simplicity, put $\{V_1, V_2, ..., V_{(n^2-n)g}\}$ by $\cup_{j=0}^{n-1}((\cup_{i=0}^{M_1}\{A_{i,j}\}) \cup (\cup_{i=0}^{M_2}\{B_{i,j}\}))$. Then C_i is written by

$$C_i = \sum_{j=0}^{n-1} C_{i,j}\alpha_j, \quad C_{i,j} \in \mathbb{F}_q[V_1, V_2, ..., V_{(n^2-n)g}].$$

Thus from Lemma 7, the condition $g(x) \in \mathbb{F}_q[x]$ is equivalent to the condition that there are some $v_1, v_2, ..., v_{(n^2-n)g} \in \mathbb{F}_q$ such that

$$C_{i,j}(v_1, v_2, ..., v_{(n^2-n)g}) = 0 \text{ for } 0 \leq i \leq ng-1,\ 1 \leq j \leq n-1.$$

Moreover, when $g(x) \in \mathbb{F}_q[x]$, $g(x) = x^{ng} + C_{ng-1,0}x^{ng-1} + ... + C_{0,0}$. The condition that $g(x)$ factors completely in $\mathbb{F}_q[x]$ is equivalent to the above condition and $G(x) := x^{ng} + c_{ng-1}x^{ng-1} + ... + c_0$ factors completely in $\mathbb{F}_q[x]$ where $c_i = C_{i,0}(v_1, v_2, ..., v_{(n^2-n)g})$. In this case, the solutions of $G(x) = 0$ are the x-coordinates of the decomposed factor. Then, we finish the proof of Theorem 1 and construct the equations system $\{C_{i,j} = 0\}$.

3 Example

In this section, we state the three computational experiments of the decomposition of elements of Jacobian. The computations are done by using Windows XP preinstalled PC (CPU:Pentium M 2GHz, RAM:1GB). We compute three cases 1) $(g, n) = (1, 3)$, 2) $(g, n) = (2, 2)$, 3) $(g, n) = (3, 2)$, where g and n are the genus and the extension degree of the definition field of the chosen hyperelliptic/elliptic curve, respectively. In all cases, one trial, which means the judge as to whether a given element of Jacobian is decomposed or not and the computation of the decomposed factor, if it is decomposed, is done within 1 second. Since the probability that an element of Jacobian is decomposed is $1/(gn)!$, the times for obtaining one potentially B_0-smooth reduced divisor are within 6 sec, 24 sec, and 720 sec respectively. Further, we will give the following three examples.

Case 1. Let $q = 1073741789$(prime number), $\mathbb{F}_{q^3} := \mathbb{F}_q[t]/(t^3 + 456725524 * t^2 + 251245663 * t + 746495860)$, and let $E/\mathbb{F}_{q^3}$ be an elliptic curve defined by $y^2 = x^3 + (1073741788 * t^2 + t) * x + (126 * t + 3969)$ and $P_0 := (t, t+63) \in E$. We investigate whether $nP_0 : n = 1, 2, ..30$ are decomposed and find the following 7 decompositions. ($24P_0$ is written by 2 forms.)

$$2P_0 = (1050861583, 6509843 * t^2 + 387051565 * t + 920296030)$$

$$+(742900894, 362262801 * t^2 + 6480079 * t + 886701711)$$

$$+(571975376, 938916909 * t^2 + 910769097 * t + 139897863)$$

$$5P_0 = ((806296922, 113931706 * t^2 + 863383473 * t + 133427995)$$

$$+(797256157, 360646567 * t^2 + 663390692 * t + 1012046566)$$

$$+(389333914, 986077188 * t^2 + 829314065 * t + 687783827)$$

$$8P_0 = (1063441336, 113661172 * t^2 + 942865616 * t + 744283566)$$

$$+(894045278, 863335768 * t^2 + 637284565 * t + 937810737)$$

$$+(694935460, 740353309 * t^2 + 505910431 * t + 597402219)$$

$$20P_0 = (996570058, 341336613 * t^2 + 450680674 * t + 72874200)$$

$$+(141768271, 589122734 * t^2 + 930205049 * t + 713557032)$$

$$+(73505168, 432994198 * t^2 + 405986289 * t + 233154172)$$

$$24P_0 = (529735815, 20343700 * t^2 + 780030904 * t + 490121669)$$

$$+(515960254, 269821984 * t^2 + 561547517 * t + 348990487)$$

$$+(207183771, 712543643 * t^2 + 356522343 * t + 895634732)$$

$$= (818683055, 1034251164 * t^2 + 705927333 * t + 1062879754),$$

$$(754504105, 23461217 * t^2 + 961620879 * t + 1015889110)$$

$$+(489159707, 271295793 * t^2 + 600348670 * t + 1022482426)$$

$$26P_0 = (628174301, 138296704 * t^2 + 104824480 * t + 858118320)$$

$$+(371888603, 417445284 * t^2 + 850151153 * t + 126970733)$$

$$(55411433, 560274594 * t^2 + 609956706 * t + 821692494)$$

Case 2. Let $q = 1073741789$(prime number), $\mathbb{F}_{q^2} := \mathbb{F}_q[t]/(t^2 + 746495860 * t + 206240189)$, and let $C/\mathbb{F}_{q^2}$ be a hyperelliptic curve defined by

$$y^2 = x^5 + (673573223 * t + 771820244) * x + 6 * t + 9$$

and

$$D_0 := (x^2 + 1073741787 * t * x + 327245929 * t + 867501600,$$

$$(1023168391{*}t{+}350252228){*}x{+}658555356{*}t{+}446913597) \in Jac(C)$$

(Mumford representation). We investigate whether $nD_0 : n = 1, 2, ..100$ are decomposed and find the following 9 decompositions. ($71D_0$ is written by 2 forms.)

$$6D_0 \sim (1025731975, 776505688{*}t{+}911495013){+}(728060789, 648475468{*}t{+}1067025179)$$

$$+(341799975, 145077925{*}t{+}187604034){+}(61964999, 227570631{*}t{+}639782700){-}4\infty$$

$$19D_0 \sim (1039361498, 15180988{*}t{+}396695374){+}(828360115, 179412594{*}t{+}719919461){-}4\infty$$

$$+(483171045, 677645208{*}t{+}604714840){+}(34566209, 753841024{*}t{+}14375633){-}4\infty$$

$$33D_0 \sim (970690833, 608141084{*}t{+}889165804){+}(260086243, 894605411{*}t{+}261264640)$$

$$+(208957980, 43330622{*}t{+}581461318){+}(190782894, 124873649{*}t{+}510328990){-}4\infty$$

$$35D_0 \sim (699447787, 267523741{*}t{+}562899544){+}(559470007, 197827114{*}t{+}99971197)$$

$$+(472594781, 579187919{*}t{+}266558458){+}(453661772, 449424806{*}t{+}977318920){-}4\infty$$

$$48D_0 \sim (1009979214, 959734525{*}t{+}990871450){+}(995813251, 44186049{*}t{+}288496638)$$

$$+(521299995, 556594200{*}t{+}468424666){+}(17946008, 977064852{*}t{+}1071618742){-}4\infty$$

$71D_0 \sim (1019155056, 573896856{*}t{+}103042116){+}(944470217, 829781939{*}t{+}184620624)$

${+}(727156004, 462612591{*}t{+}582877732){+}(281900623, 553507533{*}t{+}42660552){-}4\infty$

$\sim (502979299, 412632304{*}t{+}1036827718){+}(74527656, 927651409{*}t{+}452588110)$

${+}(50078888, 801072540{*}t{+}888737005){+}(2986754, 556402789{*}t{+}236723678){-}4\infty$

$73D_0 \sim (843747137, 682161676{*}t{+}600252618){+}(829302257, 145878028{*}t{+}853397395)$

${+}(290487906, 645896278{*}t{+}279001181){+}(184873704, 567002729{*}t{+}620354511){-}4\infty$

$80D_0 \sim (907811987, 216534804{*}t{+}936839244){+}(808513243, 873487475{*}t{+}273845273)$

${+}(520893378, 757248670{*}t{+}381150138){+}(486203744, 494475019{*}t{+}791571132){-}4\infty$

Case 3. Let $q = 1073741789$(prime number), $\mathbb{F}_{q^2} := \mathbb{F}_q[t]/(t^2 + 746495860 * t + 206240189)$, and let $C/\mathbb{F}_{q^2}$ be a hyperelliptic curve defined by

$$y^2 = x^7 + (111912375 * t + 1046743132) * x + 6 * t + 9$$

and

$$D_0 := (x^2 + 1073741787 * t * x + 327245929 * t + 867501600,$$

$$(473621736{*}t{+}256126568){*}x{+}145989647{*}t{+}687383736) \in Jac(C)$$

(Mumford representation). We investigate whether $nD_0 : n = 1, 2, ..3000$ are decomposed and find the following 6 decompositions.

$414D_0 \sim (1001437837, 752632260{*}t{+}700158497){+}(747112084, 656073918{*}t{+}400137619)$

${+}(620249588, 127943213{*}t{+}635474623){+}(614180498, 206297635{*}t{+}445250468)$

${+}(515769009, 607297126{*}t{+}554290493){+}(488549466, 627952783{*}t{+}854182612){-}6\infty$

$657D_0 \sim (939617127, 695261735{*}t{+}239531611){+}(933351280, 935312661{*}t{+}961494096)$

${+}(799612924, 341923983{*}t{+}677495100){+}(294787599, 279723229{*}t{+}760003067)$

${+}(273118782053704103{*}t{+}577497766){+}(153381525, 983211238{*}t{+}517037777){-}6\infty$

$921D_0 \sim (1034634787, 400751409{*}t{+}829801342){+}(763888873, 757155774{*}t{+}829936954)$

${+}(619620874, 800641683{*}t{+}200272230){+}(603032615, 115219564{*}t{+}655011145)$

${+}(436423191, 285214454{*}t{+}450812747){+}(125198811, 884750621{*}t{+}123305741){-}6\infty$

$1026D_0 \sim (1024020017, 267457905{*}t{+}41452942){+}(794174628, 615676821{*}t{+}723336407)$

${+}(738567269, 433647609{*}t{+}128304659){+}(629287731, 465842490{*}t{+}789390318)$

${+}(435082408, 878213106{*}t{+}603353206){+}(79621979, 479459622{*}t{+}672937516){-}6\infty$

$1121D_0 \sim (764081031, 812350603{*}t{+}347878564){+}(673426715, 687737442{*}t{+}381588704)$

${+}(610252208200 7139{*}t{+}99219637){+}(467560104, 619342780{*}t{+}228756808)$

${+}(179787786, 333322906{*}t{+}75482151){+}(59221667, 860686653{*}t{+}625301206){-}6\infty$

$2289D_0 \sim (729358563, 482925408{*}t{+}170057124){+}(529840657, 42328987{*}t{+}857983002)$

${+}(514618236, 436901100{*}t{+}416530686){+}(350106356, 183495333{*}t{+}950710579)$

${+}(175898979, 411808870{*}t{+}427518366){+}(96240558, 703780413{*}t{+}461022225){-}6\infty$

4 Estimation of the complexity of solving DLP

In this section, we estimate the complexity of the index calculus using this decomposition for fixed g, n and q going to infinity. Moreover, the estimation is done without concern for the term of $\mathrm{Poly}(q)$. The complexity is essentially the same as that of Gaudry [6]. However, after [6] appears, the new variant of the index calculus by the use of two large primes [9], [5] appears, and a little improvement is done. So we have summarized the results.
For the simplicity, the terms of Poly(q)-part of the complexity must be omitted. For this purpose, we denote the symbol $\tilde{O}$ that the complexity $\tilde{O}(N)$ is estimated by

$$x_1(\log q)^{y_1}N < \tilde{O}(N) < x_2(\log q)^{y_2}N, \quad \text{for some } x_1, x_2 \in \mathbf{R}_{>0}, \text{and } y_1, y_2 \in \mathbf{R}$$

and the symbol $\approx$ that the relation $N_1 \approx N_2$ is defined by

$$x_1(\log q)^{y_1}N_2 < N_1 < x_2(\log q)^{y_2}N_2, \quad \text{for some } x_1, x_2 \in \mathbf{R}_{>0}, \text{and } y_1, y_2 \in \mathbf{R}.$$

It is necessary to once again review the index calculus of the Jacobian of a curve $C/\mathbb{F}_q$. First, Gaudry pointed out that when taking $B_0 = \{P|P \in C(\mathbb{F}_q\}$ as a set of smooth elements, the index calculus works well. The complexity of the part of collecting smooth divisors is $O(q)$ and that of solving linear algebra is $\tilde{O}(q^2)$. Gaudry and Harley (cf.[3]) proposed an improvement of taking B, a subset of B_0 as a set of smooth elements and doing the rebalance between the collecting part and the linear algebra part. The complexity is $\tilde{O}(q^{2g/(g+1)})$. Further, $B_0\backslash B$ is called a set of large primes. Thérialut [11] proposed an improvement using the almost smooth divisor, which is written by one large prime and other smooth elements. The complexity is $\tilde{O}(q^{(4g-2)/(2g+1)})$. Finally, Nagao [9] and Gaudry et al. [5] proposed an improvement using the 2-almost smooth divisor, which is written by 2 large primes and other smooth elements. The complexity is $\tilde{O}(q^{(2g-2)/g)})$.
Now, we give the estimation of the complexity as follows. Let G be a group and let $B_0 \subset G$ be a subset. Also let N be a positive integer. Assume the following i), ii), iii), iv), v):
i) The probability that $g \in G$ is written by $g = g_1 + .. + g_N$ for some $g_i \in B_0$ is $O(1)$.
ii) For a $g \in G$, the cost for checking whether g is written by the form $g = g_1 + .. + g_N$ with $g_i \in B_0$ or not is $O(1)$.
iii) For the $g \in G$ written by $g = g_1 + .. + g_N$ with $g_i \in B_0$, the cost of computing $g_1, .., g_N$ from g is $O(1)$.
iv) For the $g's \in G$ written by $g = g_1 + .. + g_N$ with $g_i \in B_0$, the distribution of $\{g_i\}$ is uniform.
v) $|B_0|^2 \ll |G|$.
Let $B \subset B_0$ be a subset.

Definition 2 *1) An element of $g \in G$ written by $g = g_1 + .. + g_N$ for $g_1, ...g_n \in B$ is called a smooth group element.*

2) An element of $g \in G$ written by $g = g_1 + .. + g_N$ for one $g_i \in B_0 \backslash B$ and other $g_j \in B$ $(1 \leq j \leq N, j \neq i)$ is called an almost smooth group element.
3) An element of $g \in G$ written by $g = g_1 + .. + g_N$ for two $g_{i1}, g_{i2} \in B_0 \backslash B$ and other $g_j \in B$ $(1 \leq j \leq N, j \neq i_1, i_2)$is called a 2-almost smooth group element.

From these definitions, we have the following estimations of the complexities 1), 2), and 3).

Lemma 8. *1) The complexity of the index calculus taking B as a set of smooth elements by the rebalanced method is minimized at $|B| \approx |B_0|^{N/(N-1)}$ and it is $\tilde{O}(|B_0|^{(2N)/(N+1)})$.*
2) The complexity of the index calculus taking B as a set of smooth elements and taking $B_0 \backslash B$ as a set of large primes by the one large prime method is minimized at $|B| \approx |B_0|^{(2N-1)/(2N+1)}$ and it is $\tilde{O}(|B_0|^{(4N-2)/(2N+1)})$.
3) The complexity of the index calculus taking B as a set of smooth elements and taking $B_0 \backslash B$ as a set of large primes by the two large prime method is minimized at $|B| \approx |B_0|^{(N-1)/N}$ and it is $\tilde{O}(|B_0|^{(2N-2)/N})$.

Proof. (Sketch of the proof) In every case, the cost of the part of linear algebra is $\tilde{O}(|B|^2)$ and by the rebalance, which is needed for minimizing the complexity, it is the same as the cost of the collecting divisors. So, we only estimate the optimized size $|B|$.
1)**The case of rebalanced method.** The probability that $g \in G$ is a smooth group element is $O(|B/B_0|^N)$. So, the cost to obtain one smooth group element g is $\tilde{O}(|B_0/B|^N)$. We must have $O(|B|)$ number of such g. So

$$|B_0/B|^N \cdot |B| \approx |B|^2$$

where the left hand side is the cost for collecting enough smooth group elements. Thus we have $|B| \approx |B_0|^{(2N)/(N+1)}$.
2) **The case of one large prime method.** The probability that $g \in G$ is an almost smooth group element is $O(|B/B_0|^{N-1})$. Let V_1 be the set of almost smooth group elements needed for the computation. So the cost of collecting V_1 is $\tilde{O}(|V_1| \cdot |B_0/B|^{N-1})$, which equals to $\tilde{O}(|B|^2)$. So, $|V_1| \cdot |B_0/B|^{N-1} \approx |B|^2$. On the other hand, in this method, the number of the smooth group elements obtained from the elimination of the large prime from V_1 is $|V_1|^2/|B_0|$. So, $|V_1|^2/|B_0| \approx |B|$. Thus, we have $|B| \approx |B_0|^{(2N-1)/(2N+1)}$.
3) **The case of two large primes method.** The probability that $g \in G$ is a 2-almost smooth group element is $O(|B/B_0|^{N-2})$. Let V_2 be the set of 2-almost smooth group elements needed for the computation. So the cost of collecting V_2 is $\tilde{O}(|V_2| \cdot |B_0/B|^{N-2})$,

which equals to $O(|B|^2)$. So, $|V_2| \cdot |B_0/B|^{N-2} \approx |B|^2$. On the other hand, in this method, the relation $|V_2| \geq \mathrm{Const} \cdot |B_0|$ is needed. So, $|B_0| \cdot |B_0/B|^{N-2} \approx |B|^2$. Thus we have $|B| \approx |B_0|^{(N-1)/N}$.

When applying this lemma for the index calculus for the Jacobian of a curve over an extension field, note that $B_0 = \{P - \infty \,|\, x(P) \in \mathbb{F}_q\}$, $|B_0| \approx q$, $N = ng$ and thus, we have the following theorem.

Theorem 2. *1) The complexity of the index calculus by the rebalanced method is $\tilde{O}(q^{(2ng)/(ng+1)})$.*
2) The complexity of the index calculus by the one large prime method is $\tilde{O}(q^{(4ng-2)/(2ng+1)})$.
3) The complexity of the index calculus by the two large prime method is $\tilde{O}(q^{(2ng-1)/(ng)})$.

5 Conclusion

In this manuscript, we propose an algorithm which checks whether a reduced divisor is decomposed or not, and we compute the decomposed factor, if it is decomposed. From this algorithm, the concrete computations of decomposed factors are done by computer experiments when the pairs of the genus of the hyperelliptic curve and the degree of extension field are $(1, 3), (2, 2)$, and $(3, 2)$.

Acknowledgment

The author would like to thank Professor Kazuto Matsuo in the Institute of Information Security for useful comments and fruitful discussions.

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