Overlap-free Karatsuba-Ofman Polynomial Multiplication Algorithm

Haining Fan, Jiaguang Sun, Ming Gu and Kwok-Yan Lam

Abstract

We describe how a new way to split input operands allows for fast VLSI implementations of the polynomial Karatsuba-Ofman algorithm. Compared to previous Karatsuba-Ofman multipliers, the XOR gate delay of the proposed $GF(2)[x]$ multiplier is reduced by about 33% for $n = 2^i$ $(i > 1)$.

Index Terms

Karatsuba algorithm, Karatsuba-Ofman algorithm, polynomial multiplication, subquadratic space complexity multiplier, finite field.

I. Introduction

Published in 1962 [1], Karatsuba-Ofman algorithm (KOA) was the first integer multiplication method broke the quadratic complexity barrier in positional number systems. Due to its simplicity, its polynomial version is widely adopted to design VLSI $GF(2^n)$ parallel multipliers in $GF(2^n)$-based cryptosystems [7]-[25]. Two parameters are often used to measure the performance of a $GF(2^n)$ parallel multiplier, namely, the space and time complexities. The space complexity is often represented in terms of the total number of 2-input XOR and AND gates used. The corresponding time complexity is given in terms of the maximum delay faced by a signal due to these XOR and AND gates. Symbols "T_A" and "T_X" are often used to represent the delay of one 2-input AND and XOR gates, respectively.

E-mails: fan_haining@yahoo.com, {sunjg, guming, lamky}@tsinghua.edu.cn October 7, 2007

The existing bit parallel $GF(2^n)$ multipliers may be simply classified into the following three categories according to the asymptotic space complexity of the multiplication algorithm : subquadratic, quadratic and hybrid multipliers. KOA has been used in many subquadratic and hybrid multipliers. These multipliers first perform a KOA-based multiplication of two input binary polynomials $A = \sum_{i=0}^{n-1} a_i x^i$ and $B = \sum_{i=0}^{n-1} b_i x^i$, and then a modulo reduction operation using the field generating irreducible polynomial. To explain the general idea of KOA easily, we assume that $n = 2m = 2^i$ $(i > 1)$ in the following.

First, previous KOA implementations split polynomials A and B into the "most significant half" and the "least significant half" as follows:

$$A = \sum_{i=0}^{n-1} a_i x^i = x^m \sum_{i=0}^{m-1} a_{m+i} x^i + \sum_{i=0}^{m-1} a_i x^i = x^m A_H + A_L,$$

$$B = \sum_{i=0}^{n-1} b_i x^i = x^m \sum_{i=0}^{m-1} b_{m+i} x^i + \sum_{i=0}^{m-1} b_i x^i = x^m B_H + B_L,$$

where $A_H = \sum_{i=0}^{m-1} a_{m+i} x^i$, $A_L = \sum_{i=0}^{m-1} a_i x^i$, B_H and B_L are defined similarly.

Then the product AB is computed recursively using

$$AB = A_H B_H x^{2m} + \{[(A_H + A_L)(B_H + B_L)] - [A_H B_H + A_L B_L]\} x^m + A_L B_L. \qquad (1)$$

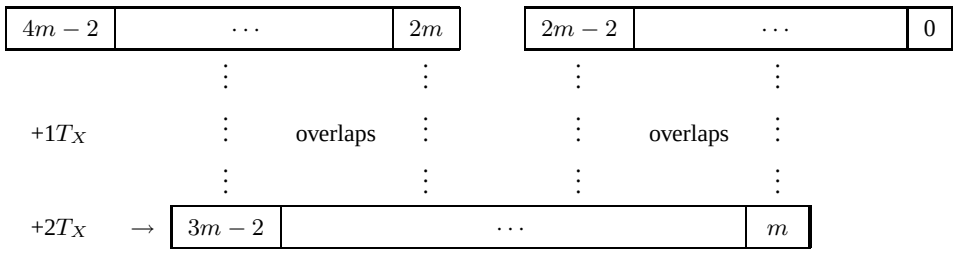
Please note that "$-$" is the same as "+" in $GF(2)[x]$. For the VLSI implementation of (1), terms in square brackets are calculated concurrently, and therefore two XOR gate delays, i.e., $2T_X$, are required to compute the expression in curly brackets besides the cost to compute the half sized product $A_L B_L$ and $A_H B_H$.

Finally, the three polynomials $A_H B_H x^{2m}$, $[(A_H + A_L)(B_H + B_L) - A_H B_H - A_L B_L] x^m$ and $A_L B_L$ in (1) are XORed in an overlap module by adding coefficients of common exponents of x together. In order to explain overlaps of common exponents of x clearly, we present the following table which shows ranges of x's exponents in these three polynomials.

Please note that overlaps occur only when $n \geq 4$ (or $m \geq 2$), and there is no overlap when $n = 2$ (or $m = 1$). Because of these overlaps, one XOR gate delay T_X is required in the overlap

TABLE I

RANGES OF x'S EXPONENTS IN THREE POLYNOMIALS OF (1)



module. Therefore, a total of 3 XOR gate delays, i.e., $3T_X$, are required in (1) besides the cost of the recursive computation of three half sized products.

Let $n = kd$, previous generalizations of KOA split the two input operands into k successive block each with d coefficients. Since the product of two degree-($d-1$) polynomials is a polynomial of degree-($2d-2$), overlaps always exist if $d > 1$.

In order to obtain exact complexities of the above binary polynomial KOA (1), we introduce some symbols of [3]. Let $\mathcal{S}$ and $\mathcal{D}$ stand for "Space" and "Delay", respectively. We use $\mathcal{S}^{\otimes}(n)$, $\mathcal{S}^{\oplus}(n), \mathcal{D}^{\otimes}(n)$ and $\mathcal{D}^{\oplus}(n)$ to denote the number of multiplication (AND) and addition (XOR) operations, the time delays introduced by multiplication and addition operations, respectively. Then the following recurrence relations, which describe the complexities of KOA, can be established.

$$\begin{cases} \mathcal{S}^{\otimes}(2) = 3, \\ \mathcal{S}^{\otimes}(n) = 3\mathcal{S}^{\otimes}(n/2); \end{cases} \qquad \begin{cases} \mathcal{D}^{\otimes}(2) = 1, \\ \mathcal{D}^{\otimes}(n) = \mathcal{D}^{\otimes}(n/2); \end{cases}$$

$$\begin{cases} \mathcal{S}^{\oplus}(2) = 4, \\ \mathcal{S}^{\oplus}(n) = 3\mathcal{S}^{\oplus}(n/2) + 4n - 4; \end{cases} \text{ and } \begin{cases} \mathcal{D}^{\oplus}(2) = 2, \\ \mathcal{D}^{\oplus}(n) = \mathcal{D}^{\oplus}(n/2) + 3. \end{cases}$$

After solving the above recurrence relations using formulae derived in [3], we have the

following complexity results for the binary polynomial KOA [7].

$$\begin{cases} \mathcal{S}^{\otimes}(n) = n^{\log_2 3}, \\ \mathcal{S}^{\oplus}(n) = 6n^{\log_2 3} - 8n + 2, \\ \mathcal{D}^{\otimes}(n) = 1, \\ \mathcal{D}^{\oplus}(n) = 3\log_2 n - 1. \end{cases}$$

Besides KOA, a Toeplitz matrix-vector product approach was presented recently to construct subquadratic $GF(2^n)$ multipliers [3]. It takes advantage of a shifted polynomial basis [4] and applies the coordinate transformation technique of [5] and [6]. Both the space and time complexities of the resulting multiplier are better than those of the best KOA-based subquadratic multipliers. For example, with $n = 2^i$ ($i > 0$), the space complexity is about 8% better, while the time complexity is about 33% better, respectively.

Since these Toeplitz matrix-vector product formulae are obtained by transposing [2, Th6, p.17] corresponding polynomial KOA-like formulae, the following question arises naturally: is it possible to reduce the time or space complexity of KOA further? We answer this question positively in the next section. We will propose a new implementation of the polynomial KOA. When it is used recursively to design binary polynomial multipliers, the asymptotic time complexity is also about 33% better than the previous KOA for $n = 2^i$ ($i > 0$). The key point of the proposed implementation is a new method to split input operands. The splitting method is simple and straightforward, but it does not seem to have been reported in the open literature.

II. New Polynomial KOA Implementation

Instead of splitting input operands into the "most significant half" and the "least significant half", we split them according to the parity of x's exponent. That is to say, we may rewrite A and B as follows

$$A = \sum_{i=0}^{n-1} a_i x^i = \sum_{i=0}^{m-1} a_{2i} x^{2i} + \sum_{i=0}^{m-1} a_{2i+1} x^{2i+1} = \sum_{i=0}^{m-1} a_{2i} x^{2i} + x \sum_{i=0}^{m-1} a_{2i+1} x^{2i},$$

$$B = \sum_{i=0}^{n-1} b_i x^i = \sum_{i=0}^{m-1} b_{2i} x^{2i} + \sum_{i=0}^{m-1} b_{2i+1} x^{2i+1} = \sum_{i=0}^{m-1} b_{2i} x^{2i} + x \sum_{i=0}^{m-1} b_{2i+1} x^{2i}.$$

Now let $y = x^2$, $A_e(y) = \sum_{i=0}^{m-1} a_{2i} y^i$ and $A_o(y) = \sum_{i=0}^{m-1} a_{2i+1} y^i$. $B_e(y)$ and $B_o(y)$ are defined similarly. Operands A and B can be rewritten as $A = A_e(y) + xA_o(y)$ and $B = B_e(y) + xB_o(y)$. Please note that terms $A_e(y)$, $A_o(y)$, $B_e(y)$ and $B_o(y)$ are polynomials in y of degrees less than m. Therefore multiplication operations between them may also be computed recursively. Using the above splitting method of A and B, we have the following KOA-like formula

$$\begin{aligned} AB &= (A_e(y) + xA_o(y))(B_e(y) + xB_o(y)) \\ &= \{A_e(y)B_e(y) + x^2 A_o(y)B_o(y)\} + x\{A_e(y)B_o(y) + A_o(y)B_e(y))\} \\ &= \{A_e(y)B_e(y) + yA_o(y)B_o(y)\} + \\ &\quad x\{[(A_e(y) + A_o(y))(B_e(y) + B_o(y))] + [A_e(y)B_e(y) + A_o(y)B_o(y)]\}. \end{aligned} \tag{2}$$

For the VLSI implementation of (2), multiplying a polynomial by x or $y = x^2$ is equivalent to shifting its coefficients, and no gate is required. It is easy to see that the expansion of $\{A_e(y)B_e(y) + yA_o(y)B_o(y)\}$ in (2) contains only terms with even exponents of x since $y = x^2$, and the expansion of $x\{[(A_e(y)+A_o(y))(B_e(y)+B_o(y))]+[A_e(y)B_e(y)+A_o(y)B_o(y)]\}$ contains only terms with odd exponents of x. Thus, there is no overlap exists when computing their summation, and no gate is required either. Moreover, terms in square brackets can be computed concurrently, and the addition operation requires 1 XOR gate delay T_X. Therefore, we know that computing AB via (2) needs $only$ a total of $2T_X$ besides the cost of the recursive computation of three half sized products. Please recall that the corresponding XOR gate delay is $3T_X$ in (1). Consequently, the following recurrence relations, which describe the algorithm complexities, can be established.

$$\begin{cases} \mathcal{S}^{\otimes}(2) = 3, \\ \mathcal{S}^{\otimes}(n) = 3\mathcal{S}^{\otimes}(n/2); \end{cases} \qquad \begin{cases} \mathcal{D}^{\otimes}(2) = 1, \\ \mathcal{D}^{\otimes}(n) = \mathcal{D}^{\otimes}(n/2); \end{cases}$$

$$\begin{cases} \mathcal{S}^{\oplus}(2) = 4, \\ \mathcal{S}^{\oplus}(n) = 3\mathcal{S}^{\oplus}(n/2) + 4n - 4; \end{cases} \text{ and } \begin{cases} \mathcal{D}^{\oplus}(2) = 2, \\ \mathcal{D}^{\oplus}(n) = \mathcal{D}^{\oplus}(n/2) + 2. \end{cases}$$

Their solutions are as follows:

$$\begin{cases} \mathcal{S}^{\otimes}(n) = n^{\log_2 3}, \\ \mathcal{S}^{\oplus}(n) = 6n^{\log_2 3} - 8n + 2, \\ \mathcal{D}^{\otimes}(n) = 1, \\ \mathcal{D}^{\oplus}(n) = 2\log_2 n. \end{cases}$$

Compared to previous implementations of polynomial KOA , the XOR gate delay of (2), i.e., $\mathcal{D}^{\oplus}(n)$, is about 33% better for $n = 2^i$ ($i > 0$).

Similar to the generalizations of KOA, we may also derive some new KOA-like formulae for polynomials of higher degrees. As an example, we show the formula for $n = 3k = 3^i$ ($i > 1$). Let $y = x^3$ and split A as follows

$$\begin{aligned} A &= \sum_{i=0}^{n-1} a_i x^i = \sum_{i=0}^{k-1} a_{3i} x^{3i} + x \sum_{i=0}^{k-1} a_{3i+1} x^{3i} + x^2 \sum_{i=0}^{k-1} a_{3i+2} x^{3i} \\ &= A_0(y) + xA_1(y) + x^2 A_2(y), \end{aligned}$$

where $A_0(y) = \sum_{i=0}^{k-1} a_{3i} y^i$, $A_1(y) = \sum_{i=0}^{k-1} a_{3i+1} y^i$ and $A_2(y) = \sum_{i=0}^{k-1} a_{3i+2} y^i$.

Then we have

$$\begin{aligned} AB &= \{A_0B_0 + y[(A_1 + A_2)(B_1 + B_2) + A_1B_1 + A_2B_2]\} + \\ &\quad x\{[(A_0 + A_1)(B_0 + B_1) + A_0B_0 + A_1B_1] + yA_2B_2\} + \\ &\quad x^2\{(A_0 + A_2)(B_0 + B_2) + A_0B_0 + A_2B_2 + A_1B_1\}, \end{aligned}$$

where "(y)"s in expressions $A_i(y)$ and $B_i(y)$ are omitted.

The following recurrence relations describe the complexities of this formula.

$$\begin{cases} \mathcal{S}^{\otimes}(3) = 6, \\ \mathcal{S}^{\otimes}(n) = 6\mathcal{S}^{\otimes}(n/3); \end{cases} \qquad \begin{cases} \mathcal{D}^{\otimes}(3) = 1, \\ \mathcal{D}^{\otimes}(n) = \mathcal{D}^{\otimes}(n/3); \end{cases}$$

$$\begin{cases} \mathcal{S}^{\oplus}(3) = 12, \\ \mathcal{S}^{\oplus}(n) = 6\mathcal{S}^{\oplus}(n/3) + \frac{22}{3}n - 10; \end{cases} \text{ and } \begin{cases} \mathcal{D}^{\oplus}(3) = 3, \\ \mathcal{D}^{\oplus}(n) = \mathcal{D}^{\oplus}(n/3) + 3. \end{cases}$$

Table II compares asymptotic complexities of proposed formulae with the previous KOA and Toeplitz matrix-vector product (TMVP) formulae over the ground field $GF(2)$. The size of operands is assumed to be $n = 2^t$ or 3^t ($t > 0$).

TABLE II

COMPARISONS OF ASYMPTOTIC COMPLEXITIES FOR $n = b^t$

b	Algorithm	#AND	#XOR	Gate delay
2	KOA [7]	$n^{\log_2 3}$	$6n^{\log_2 3} - 8n + 2$	$(3\log_2 n - 1)T_X + T_A$
	Proposed	$n^{\log_2 3}$	$6n^{\log_2 3} - 8n + 2$	$(2\log_2 n)T_X + T_A$
	TMVP	$n^{\log_2 3}$	$5.5n^{\log_2 3} - 6n + 0.5$	$(2\log_2 n)T_X + T_A$
3	KOA [13] [17]	$n^{\log_3 6}$	$\frac{16}{3}n^{\log_3 6} - \frac{22}{3}n + 2$	$(4\log_3 n - 1)T_X + T_A$
	Proposed	$n^{\log_3 6}$	$\frac{16}{3}n^{\log_3 6} - \frac{22}{3}n + 2$	$(3\log_3 n)T_X + T_A$
	TMVP	$n^{\log_3 6}$	$\frac{24}{5}n^{\log_3 6} - 5n + \frac{1}{5}$	$(3\log_3 n)T_X + T_A$

III. CONCLUSIONS

We have proposed a new implementation of the polynomial KOA. It is based on a different splitting method of input operands. In this paper, the polynomial ring $GF(2)[x]$ has been used as the background to demonstrate the new implementation. It is clear that the method may also be generalized to other polynomial rings that the original polynomial KOA can be applied.

Although both KOA and TMVP can be used to design $GF(2^n)$ parallel multipliers, we must emphasize that these two algorithms are $distinct$. One obvious difference is the size of their inputs and outputs. Take the multiplication operation of two degree-$(n-1)$ binary polynomials A and B for example, KOA computes their $(2n-1)$-bit product while the TMVP multiplier of [3] calculates the residue of AB modulo a degree-n binary polynomial $f(x)$. Or, more precisely, KOA computes the product of two elements in the ring $GF(2)[x]$ while the TMVP multiplier of [3] performs the multiplication operation in the quotient ring $GF(2)[x]/(f(x))$, which becomes the finite field $GF(2^n)$ if $f(x)$ is irreducible over $GF(2)$.

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