All Pairings are in a Group

Chang-An Zhao, Fangguo Zhang and Jiwu Huang

[1] School of Information Science and Technology,
Sun Yat-Sen University, Guangzhou 510275, P.R.China
[2] Guangdong Key Laboratory of Information Security Technology,
Sun Yat-Sen University, Guangzhou 510275, P.R.China
zhcha@mail2.sysu.edu.cn
isszhfg@mail.sysu.edu.cn
isshjw@mail.sysu.edu.cn

Abstract. In this paper, we suggest that all pairings be in a group from an abstract angle. It is possible that our observation can be applied into speeding up the implementations of pairing-based cryptosystems.

Keywords: Pairing-based cryptosystems, Tate pairing, Ate pairing, Elliptic curves.

1 Introduction

A bilinear pairing is defined as follows:

$$e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$$

where $\mathbb{G}_1,\ \mathbb{G}_2$ are additive groups and $\mathbb{G}_T$ is a multiplicative commutative group. Also, for any $P_1,\ P_2 \in G_1$and $Q_1,\ Q_2 \in G_2$, we require

$$e(P_1 + P_2, Q_1) = e(P_1, Q_1)e(P_2, Q_1),$$

$$e(P_1, Q_1 + Q_2) = e(P_1, Q_1)e(P_1, Q_2).$$

In cryptographical applications, non-degeneracy and compatibility are often required for pairings. Since pairings can be constructed from elliptic curves, pairing-based cryptosystems have been widely studied in elliptic curve cryptography in recent years. Some detailed summaries on this subject can be found in [15] and [10].

An elementary problem in the implementation of pairing-based cryptosystems is to compute the pairings. Pairings on elliptic curves can be evaluated in polynomial time by Miller's algorithm [14]. Many efficient techniques have been suggested for optimizing the computation of the pairings [2, 4, 1, 9, 13]. Some excellent summaries about pairing computations are recommended (see [8, 17]). One of the most elegant techniques for computing the pairings efficiently is to shorten the iteration loop in Miller's algorithm. Inspired by the Duursma-Lee method for some special supersingular curves in [4], Barreto *et al*. introduce the η_T pairing which has a half length of the Miller loop compared to the original Tate pairing on supersingular Abelian varieties [1]. Later, Hess *et al*. suggest the Ate pairing which shortens the length of the Miller loop on ordinary elliptic curves [9]. Matsuda *et al*. optimize the Ate pairing and the twisted Ate pairing and show that both of them are always at least as fast as the Tate pairing [13]. Inspired by the main results of [13], the authors of [19] give more choices on the Ate pairing.

We now give another look at the techniques of shortening the Miller loop. Using the fact that a fix power of the pairing is still a bilinear pairing, the Eta pairing and the Ate pairing are introduced. Factually, the new derivation in [20] for Scott's algorithm [16] also takes advantage of this fact. Recently, the authors of [12] give an improvement on the Ate pairing using the fact that the Combination of two pairings is also a pairing. Inspired by the above ideas, we first show that all pairings forms a group from an abstract angle. Then we apply it into shortening the Miller loop of the Ate pairing.

The rest of this paper is organized as follows. Section 2 introduces basic mathematical concepts of the pairings on elliptic curves. Section 3 gives our main results. We draw our conclusion in Section 4.

2 Mathematical Preliminaries

2.1 Tate Pairing

Let $\mathbb{F}_q$ be a finite field with $q = p^m$ elements, where p is a prime. Let E be an elliptic curve defined over $\mathbb{F}_q$ and $\mathcal{O}$ be the point at infinity. $\#E(\mathbb{F}_q)$ is denoted as the order of the rational points group $E(\mathbb{F}_q)$ and r is a large prime satisfying $r|\#E(\mathbb{F}_q)$. Let k be the embedding degree, i.e., the smallest positive integer such that $r|q^k - 1$.

Let $P \in E[r]$ and $Q \in E(\mathbb{F}_{q^k})$. For each integer i and point P, let $f_{i,P}$ be a rational function on E such that

$$(f_{i,P}) = i(P) - (iP) - (i-1)(\mathcal{O}).$$

Let D be a divisor [18] which is linearly equivalent to $(Q) - (\mathcal{O})$ with its support disjoint from $(f_{r,P})$. The Tate pairing [6] is a bilinear map

$$\hat{e} : E[r] \times E(\mathbb{F}_{q^k})/rE(\mathbb{F}_{q^k}) \to \mathbb{F}^*_{q^k}/(\mathbb{F}^*_{q^k})^r,$$

$$\hat{e}(P, Q) = f_{r,P}(D).$$

If P is restricted in $E(\mathbb{F}_q)$, one can define the reduced Tate pairing as

$$e(P, Q) = f_{r,P}(Q)^{\frac{q^k-1}{r}}$$

according to Theorem 1 in [2]. The above definition is convenient since a unique element of $\mathbb{F}^*_{q^k}$ is often required in many cryptographic protocols.

2.2 Ate Pairing and Twisted Ate Pairing

We recall the definition of the Ate pairing and twisted Ate pairing from [9, 13] in this subsection. The Ate pairing extends the η_T pairing on the ordinary elliptic curves.

Let $\mathbb{F}_q$ be a finite field with $q = p^m$ elements, where p is a prime. Let E be an ordinary elliptic curve over $\mathbb{F}_q$, r a large prime satisfying $r \mid \#E(\mathbb{F}_q)$ and let t denote the trace of Frobenius, i.e., $\#E(\mathbb{F}_q) = q+1-t$. Let $T = t-1$ and then $T \equiv q \bmod r$. Let π_q be the Frobenius endomorphism, $\pi_q : E \to E : (x, y) \mapsto (x^q, y^q)$. Denote $Q \in \mathbb{G}_2 = E[r] \cap Ker(\pi_q - [q])$ and $P \in \mathbb{G}_1 = E[r] \cap Ker(\pi_q - [1])$. Let $N = gcd(T^k - 1, q^k - 1) > 0$ and $T^k - 1 = LN$, where k is its embedding degree. Denote the normalized function $f^{norm}_{T,Q} = f_{T,Q}/(z^r f_{T,Q})(\mathcal{O})$, where $Q \in \mathbb{G}_2$ and z is a local parameter for the infinity point $\mathcal{O}$. Then the Ate pairing is defined as $f^{norm}_{T,Q}(P)$ and

$$e(Q, P)^L = f^{norm}_{T,Q}(P)^{c(q^k-1)/N},$$

where $c = \sum_{i=0}^{k-1} T^{k-1-i} q^i \bmod N$.

Let E' over $\mathbb{F}_q$ be a twist of degree d of E, i.e., E' and E are isomorphic over $\mathbb{F}_{q^d}$ and d is minimal with this property. Let $m = gcd(k, d)$ and $e = k/m$. Then the twisted Ate pairing is defined as $f_{T^e,P}(Q)$ and

$$e(P, Q)^L = f_{T^e,P}(Q)^{c_t(q^k-1)/N},$$

where $c_t = \sum_{i=0}^{m-1} T^{e(m-1-i)} q^{ei} \bmod N$.

The Ate pairing and twisted Ate pairing are both non-degenerate provided that $r \nmid L$. The length of the Miller loop of computing the reduced Ate pairing and the reduced twisted Ate pairing depend on the bit length of T and T^e respectively. Replacing T and T^e with $T \bmod r$ and $T^e \bmod r$ respectively, Matsuda *et al*. give the definition of the optimized Ate pairing and the twisted Ate pairing [13]. This also shows that computing the optimized versions of the Ate pairing and twisted Ate pairing is always at least as efficient as computing the Tate pairing. The authors of [19] suggest that $T^i(T^{ei}) \bmod r(1 \leq i \leq k)$ induce to some variants of the (twisted) Ate pairings.

3 Main Results

The follow results are very easy to the experts on pairings, but we have not found a location in the literature. Therefore, we present these facts and then shorten the Miller loop of the pairing on elliptic curves using them.

Let e be a bilinear pairing defined as follows:

$$e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$$

where $\mathbb{G}_1,\ \mathbb{G}_2$ are additive groups and $\mathbb{G}_T$ is a multiplicative commutative group. Also, for any $P_1,\ P_2 \in G_1$and $Q_1,\ Q_2 \in G_2$, we have

$$e(P_1 + P_2, Q_1) = e(P_1, Q_1)e(P_2, Q_1),$$

$$e(P_1, Q_1 + Q_2) = e(P_1, Q_1)e(P_1, Q_2).$$

Now we consider all pairings from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$ and show that they forms a multiplicative group.

Lemma 1. *Let $\mathbb{G}_1,\ \mathbb{G}_2$ and $\mathbb{G}_T$ be defined as above. Both e_1 and e_2 are the pairings from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$. Then $f = e_1/e_2$ and $h = e_1e_2$ are also the pairings from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$. In particular, a fix power of a pairing still defines a bilinear pairing.*

Proof: For any $P_1,\ P_2 \in G_1$ and $Q_1,\ Q_2 \in G_2$, we obtain

$$f(P_1+P_2, Q_1) = \frac{e_1(P_1 + P_2, Q_1)}{e_2(P_1 + P_2, Q_1)} = \frac{e_1(P_1, Q_1)}{e_2(P_1, Q_1)} \cdot \frac{e_1(P_2, Q_1)}{e_2(P_2, Q_1)} = f(P_1, Q_1) \cdot f(P_2, Q_1).$$

Similarly, we see that

$$f(P_1, Q_1 + Q_2) = f(P_1, Q_1)f(P_1, Q_2).$$

This shows that f is a new bilinear pairing from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$.

For $h = e_1e_2$, we have

$$\begin{aligned} h(P_1 + P_2, Q_1) =& e_1(P_1 + P_2, Q_1) \cdot e_2(P_1 + P_2, Q_1) \\ =& e_1(P_1, Q_1)e_1(P_2, Q_1) \cdot e_2(P_1, Q_1)e_2(P_2, Q_1) \\ =& h(P_1, Q_1) \cdot h(P_2, Q_1). \end{aligned}$$

Similarly, we see that

$$h(P_1, Q_1 + Q_2) = h(P_1, Q_1)h(P_1, Q_2).$$

This also shows that $h = e_1e_2$ is a new bilinear pairing from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$.

Finally, Let e be a pairing from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$ and n be an integer. From

$$e(P_1 + P_2, Q_1)^n = (e(P_1, Q_1) \cdot e(P_2, Q_1))^n = e(P_1, Q_1)^n \cdot e(P_2, Q_1)^n$$

and

$$e(P_1, Q_1 + Q_2)^n = (e(P_1, Q_1) \cdot e(P_1, Q_2))^n = e(P_1, Q_1)^n \cdot e(P_1, Q_2)^n,$$

we conclude that e^n is also a new pairing. □

From the above lemma, we can easily obtain the following theorem.

Theorem 1. *Let I be a pairing from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$ satisfying $I(P, Q) = 1_{G_T}$ where $P \in G_1$, $Q \in G_2$ and $1_{\mathbb{G}_T}$ is the identity in $\mathbb{G}_T$. Then the set of all pairings from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$ is a multiplicative group with identity I.*

Proof: From Lemma 1, we easily obtain that the product of two pairings is still a pairing. Also, every pairing e from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$ has its inverse element I/e. This completes the whole proof of Theorem 1. □

Applying Theorem 1 into the bilinear pairing on elliptic curves, we can easily obtain the following useful corollary.

Corollary 1. *Let $e_1 \cdots e_n$ be the pairings from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$ corresponding to their Miller loops $\lambda_1 \cdots \lambda_n$. Then*

$$e = \prod_{i=1}^{n} e_i^{s_i}, \ s_i \in \mathbb{Z}, \ 1 \le i \le n$$

is also a pairing from $\mathbb{G}_1 \times \mathbb{G}_2$ *to* $\mathbb{G}_T$ *with its Miller loop* $\lambda = \sum_{i=1}^{n} s_i\lambda_i$.

In the implementations of pairings on elliptic curves, the short Miller loop are often required. So we can choose the suitable $s_i \in \mathbb{Z}$ making λ as small as possible. Note that λ can not be equal to 0 or ± 1 since they will define a trivial pairing.

4 Applications

We now apply Corollary 1 into constructing some new pairings from the generalized Ate pairing. Denote $Q \in \mathbb{G}_2 = E[r] \cap Ker(\pi_q - [q])$ and $P \in \mathbb{G}_1 = E[r] \cap Ker(\pi_q - [1])$. Then $f_{T_i,Q}^{norm}(P)$ define the generalized Ate pairing [19]. We gives two examples where the Miller loop can also reach $r^{1/\varphi(k)}$. In these examples, the new defined pairings have the Miller loop as short as in [12]. It should be pointed out that our idea is inspired by [12]. Using Theorem 1, we can define more pairings than before.

Example 1. Let E be B-N curves over $\mathbb{F}_p$ in [3] with $k = 12$. Also $p = 36u^4 + 36u^3 + 24u^2 + 6u + 1$ and $r = 36u^4 + 36u^3 + 18u^2 + 6u + 1$.

According to the main result of [19], we have

- $T_1 = 6u^2$
- $T_{10} = 36u^3 + 18u^2 + 6u + 2$
- $T_{11} = 36u^3 + 30u^2 + 12u + 3$

Also, note that all $f_{T_i,Q}^{norm}$ give bilinear pairings, which is called the Ate$_i$ pairing. Let e_1, e_2 and e_3 be the pairing $f_{T_1,Q}^{norm}$, $f_{T_{10},Q}^{norm}$ and $f_{T_{11},Q}^{norm}$ respectively. Using Theorem 1, we can define a new pairing $e = e_1^{-2}e_2^{-1}e_3$.

Since $T_{11} = T_{10} + 2T_1 + \lambda$ where $\lambda = (-2)T_1 + (-1)T_{10} + T_{11} = 6u + 1$, we easily have

$$\begin{aligned}(f_{T_{11},Q}^{norm}) =& (f_{T_{10},Q}^{norm} f_{2T_1+\lambda,Q}^{norm} \cdot \frac{l_{T_{10}Q,T_1Q}^{norm}}{v_{T_{11}Q}^{norm}}) \\ =& (f_{T_{10},Q}^{norm} \cdot (f_{T_1,Q}^{norm})^2 \cdot f_{\lambda,Q}^{norm} \frac{l_{T_1Q,T_1Q}^{norm}}{v_{2T_1Q}^{norm}} \cdot \frac{l_{2T_1Q,\lambda Q}^{norm}}{v_{(2T_1+\lambda)Q}^{norm}} \frac{l_{T_{10}Q,T_1Q}^{norm}}{v_{T_{11}Q}^{norm}}).\end{aligned}$$

This shows that

$$(\frac{f_{T_{11},Q}^{norm}}{(f_{T_{10},Q}^{norm} \cdot (f_{T_1,Q}^{norm})^2)}) = (f_{\lambda,Q}^{norm} \frac{l_{T_1Q,T_1Q}^{norm}}{v_{2T_1Q}^{norm}} \cdot \frac{l_{2T_1Q,\lambda Q}^{norm}}{v_{(2T_1+\lambda)Q}^{norm}} \frac{l_{T_{10}Q,T_1Q}^{norm}}{v_{T_{11}Q}^{norm}})$$

Therefore, we can see that e indeed defines a new pairing and also e has its explicit expression

$$f^{norm}_{\lambda,Q} \frac{l^{norm}_{T_1Q,T_1Q}}{v^{norm}_{2T_1Q}} \cdot \frac{l^{norm}_{2T_1Q,\lambda Q}}{v^{norm}_{(2T_1+\lambda)Q}} \frac{l^{norm}_{T_{10}Q,T_1Q}}{v^{norm}_{T_{11}Q}}.$$

So the Miller loop of the new pairing e is $\lambda = 6u + 1$. Since $\lambda = 6u + 1$, we also enable that the Miller loop of the new pairing e reaches $r^{1/\varphi(k)}$ similar to [12].

Example 2. The pairing-friendly curves from [11] for $k = 16$ with a $\rho-$value of 5/4 have the following parametrization. $r = u^8 + 48u^4 + 625$ and $t = \frac{1}{35}(2u^5 + 41u + 35)$. Note that

- $T_1 = \frac{1}{35}(2u^5 + 41u) \pmod r$
- $T_5 = \frac{1}{35}(u^5 + 38u) \pmod r$

Then for $P \in G_1$ and $Q \in G_2$ in the generalized Ate pairing, we see that $e_1(Q, P) = f^{norm}_{T_1,Q}(P)^{(q^k-1)/r}$ and $e_2(Q, P) = f^{norm}_{T_5,Q}(P)^{(q^k-1)/r}$ gives two bilinear pairings. Therefore, $e = e_1^{-1}e_2^2$ defines a new pairing with the Miller loop $\lambda = (-1)T_1 + T_5 = u$ according to lemma 2. Also e has its explicit expression $e = (f^{norm}_{u,Q}\frac{l_{T_1Q,uQ}}{v_{2T_5Q}})^{(q^k-1)/r}$. Note that the Miller loop u also reaches $r^{1/\varphi(k)}$.

5 Conclusions and Further Work

In this paper, we suggest that the set of all pairings be a multiplicative group. Using this fact, some new pairings are introduced. It is possible that our observation can be applied into other aspects of pairing-based cryptosystems. It is an open problem how to construct a pairing on elliptic curves with the Miller loop shorter than $r^{1/\varphi(k)}$.

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