All Pairings are in a Group

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Abstract. In this paper, we suggest that all pairings be in a group from an abstract angle. It is possible that our observation can be applied into other aspects of pairing-based cryptosystems.

Keywords: Pairing-based cryptosystems, Tate pairing, Ate pairing, Elliptic curves, Group.

1 Introduction

A bilinear pairing is defined as follows:

$$e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$$

where $\mathbb{G}_1$, $\mathbb{G}_2$ are additive groups and $\mathbb{G}_T$ is a multiplicative commutative group. Also, for any $P_1,\ P_2 \in G_1$and $Q_1,\ Q_2 \in G_2$, we require

$$e(P_1 + P_2, Q_1) = e(P_1, Q_1)e(P_2, Q_1),$$

$$e(P_1, Q_1 + Q_2) = e(P_1, Q_1)e(P_1, Q_2).$$

In cryptographical applications, non-degeneracy and compatibility are often required for pairings. Since pairings can be constructed from elliptic curves, pairing-based cryptosystems have been widely studied in elliptic curve cryptography in recent years. Some detailed summaries on this subject can be found in [15] and [10].

An elementary problem in the implementation of pairing-based cryptosystems is to compute the pairings. Pairings on elliptic curves can be evaluated in polynomial time by Miller's algorithm [14]. Many efficient techniques have been suggested for optimizing the computation of the pairings [2, 4, 1, 9, 13]. Some excellent summaries about pairing computations are recommended (see [8, 17]). One of the most elegant techniques for computing the pairings efficiently is to shorten the iteration loop in Miller's algorithm. Inspired by the Duursma-Lee method for some special supersingular curves in [4], Barreto *et al*. introduce the η_T pairing which has a half length of the Miller loop compared to the original Tate pairing on supersingular Abelian varieties [1]. Later, Hess *et al*. suggest the Ate pairing which shortens the length of the Miller loop on ordinary elliptic curves [9]. Matsuda *et al*. optimize the Ate pairing and the twisted Ate pairing and show that both of them are always at least as fast as the Tate pairing [13]. Inspired by the main results of [13], the authors of [19] give more choices on the Ate pairing. Based on the results of [9, 13, 19], Lee *et al*. gives a significant improvement on pairing computations [12]. It should be pointed out that the main results of [12] is very useful on the practical implementations of the pairings.

We now give another look at the techniques of shortening the Miller loop. Using the fact that a fix power of the pairing is still a bilinear pairing, the Eta pairing and the Ate pairing are introduced. Factually, the new derivation in [20] for Scott's algorithm [16] also takes advantage of this fact. Recently, the authors of [12] give an improvement on the Ate pairing using the fact that the Combination of two pairings is also a pairing. Inspired by the above ideas, we first show that all pairings forms a group from an abstract angle, then we apply it into shortening the Miller loop of the Ate pairing. It is possible that our observation can be applied into other aspects of pairing-based cryptosystems.

The rest of this paper is organized as follows. Section 2 introduces basic mathematical concepts of the pairings on elliptic curves. Section 3 gives our main results and Section 4 gives some applications. We draw our conclusion in Section 4.

2 Mathematical Preliminaries

2.1 Tate Pairing

Let $\mathbb{F}_q$ be a finite field with $q = p^m$ elements, where p is a prime. Let E be an elliptic curve defined over $\mathbb{F}_q$ and $\mathcal{O}$ be the point at infinity. $\#E(\mathbb{F}_q)$ is denoted as the order of the rational points group $E(\mathbb{F}_q)$ and r is a large prime satisfying $r|\#E(\mathbb{F}_q)$. Let k be the embedding degree, i.e., the smallest positive integer such that $r|q^k - 1$.

Let $P \in E[r]$ and $Q \in E(\mathbb{F}_{q^k})$. For each integer i and point P, let $f_{i,P}$ be a rational function on E such that

$$(f_{i,P}) = i(P) - (iP) - (i-1)(\mathcal{O}).$$

Let D be a divisor [18] which is linearly equivalent to $(Q) - (\mathcal{O})$ with its support disjoint from $(f_{r,P})$. The Tate pairing [6] is a bilinear map

$$\hat{e} : E[r] \times E(\mathbb{F}_{q^k})/rE(\mathbb{F}_{q^k}) \rightarrow \mathbb{F}^*_{q^k}/(\mathbb{F}^*_{q^k})^r,$$

$$\hat{e}(P, Q) = f_{r,P}(D).$$

If P is restricted in $E(\mathbb{F}_q)$, one can define the reduced Tate pairing as

$$e(P, Q) = f_{r,P}(Q)^{\frac{q^k-1}{r}}$$

according to Theorem 1 in [2]. The above definition is convenient since a unique element of $\mathbb{F}^*_{q^k}$ is often required in many cryptographic protocols.

2.2 Ate Pairing and Twisted Ate Pairing

We recall the definition of the Ate pairing and twisted Ate pairing from [9, 13] in this subsection. The Ate pairing extends the η_T pairing on the ordinary elliptic curves.

Let $\mathbb{F}_q$ be a finite field with $q = p^m$ elements, where p is a prime. Let E be an ordinary elliptic curve over $\mathbb{F}_q$, r a large prime satisfying $r \mid \#E(\mathbb{F}_q)$ and let t denote the trace of Frobenius, i.e., $\#E(\mathbb{F}_q) = q+1-t$. Let $T = t-1$ and then $T \equiv q \bmod r$. Let π_q be the Frobenius endomorphism, $\pi_q : E \rightarrow E : (x, y) \mapsto (x^q, y^q)$. Denote $Q \in \mathbb{G}_2 = E[r] \cap Ker(\pi_q - [q])$ and $P \in \mathbb{G}_1 = E[r] \cap Ker(\pi_q - [1])$. Let $N = gcd(T^k - 1, q^k - 1) > 0$ and $T^k - 1 = LN$, where k is its embedding degree.

Denote the normalized function $f_{T,Q}^{norm} = f_{T,Q}/(z^r f_{T,Q})(\mathcal{O})$, where $Q \in \mathbb{G}_2$ and z is a local parameter for the infinity point $\mathcal{O}$. Then the Ate pairing is defined as $f_{T,Q}^{norm}(P)$ and

$$e(Q,P)^L = f_{T,Q}^{norm}(P)^{c(q^k-1)/N},$$

where $c = \sum_{i=0}^{k-1} T^{k-1-i}q^i \bmod N$.

Let E' over $\mathbb{F}_q$ be a twist of degree d of E, i.e., E' and E are isomorphic over $\mathbb{F}_{q^d}$ and d is minimal with this property. Let $m = gcd(k,d)$ and $e = k/m$. Then the twisted Ate pairing is defined as $f_{T^e,P}(Q)$ and

$$e(P,Q)^L = f_{T^e,P}(Q)^{c_t(q^k-1)/N},$$

where $c_t = \sum_{i=0}^{m-1} T^{e(m-1-i)}q^{ei} \bmod N$.

The Ate pairing and twisted Ate pairing are both non-degenerate provided that $r \nmid L$. The length of the Miller loop of computing the reduced Ate pairing and the reduced twisted Ate pairing depend on the bit length of T and T^e respectively. Replacing T and T^e with $T \bmod r$ and $T^e \bmod r$ respectively, Matsuda *et al.* give the definition of the optimized Ate pairing and the twisted Ate pairing [13]. This also shows that computing the optimized versions of the Ate pairing and twisted Ate pairing is always at least as efficient as computing the Tate pairing. The authors of [19] suggest that $T^i(T^{ei}) \bmod r (1 \leq i \leq k)$ induce to some variants of the (twisted) Ate pairings.

3 Main Results

The follow results are very easy to the experts on pairings, but we have not found a location in the literature. Therefore, we present these facts and then shorten the Miller loop of the pairing on elliptic curves using them.

Let e be a bilinear pairing defined as follows:

$$e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$$

where $\mathbb{G}_1$, $\mathbb{G}_2$ are additive groups and $\mathbb{G}_T$ is a multiplicative commutative group. Also, for any $P_1,\ P_2 \in G_1$and $Q_1,\ Q_2 \in G_2$, we have

$$e(P_1 + P_2, Q_1) = e(P_1, Q_1)e(P_2, Q_1),$$

$$e(P_1, Q_1 + Q_2) = e(P_1, Q_1)e(P_1, Q_2).$$

Now we consider all pairings from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$ and show that they forms a multiplicative group.

Lemma 1. *Let $\mathbb{G}_1,\ \mathbb{G}_2$ and $\mathbb{G}_T$ be defined as above. Both e_1 and e_2 are the pairings from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$. Then $f = e_1/e_2$ and $h = e_1e_2$ are also the pairings from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$. In particular, a fix power of a pairing still defines a bilinear pairing.*

Proof: For any $P_1,\ P_2 \in G_1$ and $Q_1,\ Q_2 \in G_2$, we obtain

$$f(P_1+P_2,Q_1) = \frac{e_1(P_1+P_2,Q_1)}{e_2(P_1+P_2,Q_1)} = \frac{e_1(P_1,Q_1)}{e_2(P_1,Q_1)} \cdot \frac{e_1(P_2,Q_1)}{e_2(P_2,Q_1)} = f(P_1,Q_1) \cdot f(P_2,Q_1).$$

Similarly, we see that

$$f(P_1,Q_1+Q_2) = f(P_1,Q_1)f(P_1,Q_2).$$

This shows that f is a new bilinear pairing from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$.

For $h = e_1e_2$, we have

$$\begin{aligned} h(P_1+P_2,Q_1) =& e_1(P_1+P_2,Q_1) \cdot e_2(P_1+P_2,Q_1) \\ =& e_1(P_1,Q_1)e_1(P_2,Q_1) \cdot e_2(P_1,Q_1)e_2(P_2,Q_1) \\ =& h(P_1,Q_1) \cdot h(P_2,Q_1). \end{aligned}$$

Similarly, we see that

$$h(P_1,Q_1+Q_2) = h(P_1,Q_1)h(P_1,Q_2).$$

This also shows that $h = e_1e_2$ is a new bilinear pairing from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$.

Finally, Let e be a pairing from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$ and n be an integer. From

$$e(P_1+P_2,Q_1)^n = (e(P_1,Q_1) \cdot e(P_2,Q_1))^n = e(P_1,Q_1)^n \cdot e(P_2,Q_1)^n$$

and

$$e(P_1,Q_1+Q_2)^n = (e(P_1,Q_1) \cdot e(P_1,Q_2))^n = e(P_1,Q_1)^n \cdot e(P_1,Q_2)^n,$$

we conclude that e^n is also a new pairing. □

From the above lemma, we can easily obtain the following theorem.

Theorem 1. *Let I be a pairing from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$ satisfying $I(P,Q) = 1_{G_T}$ where $P \in G_1,\ Q \in G_2$ and $1_{\mathbb{G}_T}$ is the identity in $\mathbb{G}_T$. Then the set of all pairings from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$ is a multiplicative group with identity I.*

Proof: From Lemma 1, we easily obtain that the product of two pairings is still a pairing. Also, every pairing e from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$ has its inverse element I/e. This completes the whole proof of Theorem 1. □

Applying Theorem 1 into the bilinear pairing on elliptic curves, we can easily obtain the following useful corollary.

Corollary 1. *Let $e_1 \cdots e_n$ be the pairings from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$ corresponding to their Miller loops $\lambda_1 \cdots \lambda_n$. Then*

$$e = \prod_{i=1}^{n} e_i^{s_i},\ s_i \in \mathbb{Z},\ 1 \leq i \leq n$$

is also a pairing from $\mathbb{G}_1 \times \mathbb{G}_2$ to $\mathbb{G}_T$ with its Miller loop $\lambda = \sum_{i=1}^{n} s_i \lambda_i$.

In the implementations of pairings on elliptic curves, the short Miller loop are often required. Therefore, we can choose the suitable $s_i \in \mathbb{Z}$ which enable λ as small as possible. Note that λ can not be equal to 0 or ± 1 since they will define a trivial pairing. A second remark is that s_i should be not too large which will yield a complicated product of the line equations.

4 Applications

We now apply Corollary 1 into constructing some new pairings from the generalized Ate pairing. Denote $Q \in \mathbb{G}_2 = E[r] \cap Ker(\pi_q - [q])$ and $P \in \mathbb{G}_1 = E[r] \cap Ker(\pi_q - [1])$. Then $f_{T_i,Q}^{norm}(P)$ define the generalized Ate pairing [19]. We gives two examples where the Miller loop can also reach $r^{1/\varphi(k)}$. In these examples, the new defined pairings have the Miller loop as short as in [12]. It should be pointed out that our idea is inspired by [12]. Using Theorem 1, we can define more pairings than before.

Example 1. Let E be B-N curves over $\mathbb{F}_p$ in [3] with $k = 12$. Also $p = 36u^4 + 36u^3 + 24u^2 + 6u + 1$ and $r = 36u^4 + 36u^3 + 18u^2 + 6u + 1$.

According to the main result of [19], we have

- $T_1 = 6u^2$

- $T_{10} = 36u^3 + 18u^2 + 6u + 2$
- $T_{11} = 36u^3 + 30u^2 + 12u + 3$

Also, note that all $f_{T_i,Q}^{norm}$ give bilinear pairings, which is called the Ate$_i$ pairing. Let $e_1,\ e_2$ and e_3 be the pairing $f_{T_1,Q}^{norm}$, $f_{T_{10},Q}^{norm}$ and $f_{T_{11},Q}^{norm}$ respectively. Using Theorem 1, we can define a new pairing $e = e_1^{-2}e_2^{-1}e_3$.

Since $T_{11} = T_{10} + 2T_1 + \lambda$ where $\lambda = (-2)T_1 + (-1)T_{10} + T_{11} = 6u + 1$, we easily have

$$\begin{aligned}(f_{T_{11},Q}^{norm}) =&(f_{T_{10},Q}^{norm} f_{2T_1+\lambda,Q}^{norm} \cdot \frac{l_{T_{10}Q,T_1Q}^{norm}}{v_{T_{11}Q}^{norm}})\\ =&(f_{T_{10},Q}^{norm} \cdot (f_{T_1,Q}^{norm})^2 \cdot f_{\lambda,Q}^{norm} \frac{l_{T_1Q,T_1Q}^{norm}}{v_{2T_1Q}^{norm}} \cdot \frac{l_{2T_1Q,\lambda Q}^{norm}}{v_{(2T_1+\lambda)Q}^{norm}} \frac{l_{T_{10}Q,T_1Q}^{norm}}{v_{T_{11}Q}^{norm}}).\end{aligned}$$

This shows that

$$(\frac{f_{T_{11},Q}^{norm}}{(f_{T_{10},Q}^{norm} \cdot (f_{T_1,Q}^{norm})^2)}) = (f_{\lambda,Q}^{norm} \frac{l_{T_1Q,T_1Q}^{norm}}{v_{2T_1Q}^{norm}} \cdot \frac{l_{2T_1Q,\lambda Q}^{norm}}{v_{(2T_1+\lambda)Q}^{norm}} \frac{l_{T_{10}Q,T_1Q}^{norm}}{v_{T_{11}Q}^{norm}})$$

Therefore, we can see that e indeed defines a new pairing and also e has its explicit expression

$$f_{\lambda,Q}^{norm} \frac{l_{T_1Q,T_1Q}^{norm}}{v_{2T_1Q}^{norm}} \cdot \frac{l_{2T_1Q,\lambda Q}^{norm}}{v_{(2T_1+\lambda)Q}^{norm}} \frac{l_{T_{10}Q,T_1Q}^{norm}}{v_{T_{11}Q}^{norm}}.$$

So the Miller loop of the new pairing e is $\lambda = 6u + 1$. Since $\lambda = 6u + 1$, we also enable that the Miller loop of the new pairing e reaches $r^{1/\varphi(k)}$ similar to [12].

Example 2. The pairing-friendly curves from [11] for $k = 16$ with a $\rho-$value of $5/4$ have the following parametrization. $r = u^8 + 48u^4 + 625$ and $t = \frac{1}{35}(2u^5 + 41u + 35)$. Note that

- $T_1 = \frac{1}{35}(2u^5 + 41u) \pmod r$
- $T_5 = \frac{1}{35}(u^5 + 38u) \pmod r$

Then for $P \in G_1$ and $Q \in G_2$ in the generalized Ate pairing, we see that $e_1(Q,P) = f_{T_1,Q}^{norm}(P)^{(q^k-1)/r}$ and $e_2(Q,P) = f_{T_5,Q}^{norm}(P)^{(q^k-1)/r}$ gives two bilinear pairings. Therefore, $e = e_1^{-1}e_2^2$ defines a new pairing with the Miller loop $\lambda = (-)T_1 + 2T_5 = u \bmod r$ according to Corollary 1. Also e has its explicit expression $e = (f_{u,Q}^{norm} \frac{l_{T_1Q,uQ}}{v_{2T_5Q}})^{(q^k-1)/r}$. Note that the Miller loop u also reaches $r^{1/\varphi(k)}$.

5 Conclusions

In this paper, we suggest that the set of all pairings be a multiplicative group. It is possible that our observation can be applied into other aspects of pairing-based cryptosystems.

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