On the Design of Optimally Secure Double Block Length Hash Functions with Rate 1*

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Abstract

In this paper, the security of double block length hash functions with rate 1 which based on a block cipher with a block length of n-bit and a key length of $2n$-bit is reconsidered. First, two concrete attacks are designed to break Hirose's two examples which were left as an open problem. Secondly, examples and attacks are presented on a general class of double block length hash functions with rate 1, which disclose there exist uncovered flaws in the former analysis by Satoh *et al.* and Hirose. Some refined conditions are proposed for ensuring this general class of the rate-1 hash functions to be optimally secure against the preimage, the second preimage and the collision attacks. Finally, the security results are extended to a new class of double block length hash functions with rate 1, where one block cipher used in the compression function has the key length is equal to the block length, while the other is doubled..

Key words. Cryptanalysis, Block cipher, Double block length Hash function.

1 Introduction

Cryptographic hash function $H : \{0,1\}^* \rightarrow \{0,1\}^\ell$ is defined as an easily computable algorithm which uniformly maps an arbitrary length message to a fixed length output hash value. The design of today's cryptographic hash functions still follows the Merkle-Damgard (MD) structure[17, 7], by iterating a compression function on the input message. The hash function will be collision resistant if the underlying compression function is. In practice, most of hash functions are either explicitly or implicitly composed from block ciphers. The advantage of the schemes from block ciphers are that one can conveniently choose a well-designed block cipher (e.g., DES, IDEA, AES, etc) to construct the underlying compression function, and also the latest cryptanalysis results on such a block cipher can be used to avoid the potential weakness in the scheme. Discussions of hash functions constructed from n-bit block ciphers are divided into *single block length* (SBL) and *double block length* (DBL) hash functions, where single and

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double are related to the output range of the block cipher that used in the hash function. Assume that greater than or equal to 2^{64} operations(encryption or decryption) are infeasible, the objective of SBL hash functions is to just provide *one-wayness* for cipher of block length near $n = 64$, while fail to *collision resistance* since a doubled 128-bit length range is required to resist the birthday paradox attack. The motivation of double block length is to combine two n-bit block ciphers to obtain a sufficient output range for collision resistance. One such algorithm is MDC-2, which was developed by Brachtl *et al.*[3] for use in combination with DES. It is believed that the complexities of (second) preimage and collision attacks on MDC-2 are about $2^{3n/2}$ and 2^n, respectively. A DBL hash function H is said to be *optimally secure*, if any adversary with non-negligible successful probability must spend the computation costs *no less than* brute-force attacks, which requires the complexities of (second) preimage and collision attacks are no less than 2^{2n} and 2^n, respectively.

Although double block length can realize collision resistance, an obvious disadvantage of DBL hash functions is a decrease in speed. The *rate* of a block-cipher-based hash function is defined as the number of n-bit message blocks processed per encryption or decryption for the measurement of the efficiency. The rate of MDC-2 is only 1/2, which implies that MDC-2 is at least twice as slow as the underlying block cipher. To improve the efficiency, many DBL hash functions with rate 1 have been proposed, such as [4, 11, 20, 27]. Unfortunately, some critical results showed that those proposed schemes unlikely achieve optimally secure. In [13], Knudsen *et al.* presented the attacks on a large class of DBL hash functions with rate 1 such that the key length is equal to the block length n-bit. In particular, the attacks break the proposed schemes in [4, 11, 20]. Still, many advanced block ciphers (e.g., AES, RC5, Blowfish, etc) support variants of key length motivates renewed interest in finding good ways to construct a fast DBL hash function with optimal security. Many instructive examples were proposed recently, e.g., [9, 15, 18, 19]. But all these schemes are less than rate 1, which means they are not efficient enough. In [23], Satoh *et al.* presented the attacks on a general class of DBL hash functions with rate 1 where the key length as twice as the block length, which break the proposed scheme in [27]. In particular, Satoh *et al.* described a necessary condition for this general class of the rate-1 hash functions to be optimally secure. Recently, Hirose[10] gave a comment on Satoh *et al.*'s result[23] and showed that there exists a missed case in their analysis. Based on this comment, two necessary conditions for optimal collision resistance are given by Hirose in [10]. In particular, two examples are left in this paper as an open problem to make it clear whether they are optimally secure.

Our Contributions. Consider the security of the rate-1 double block length hash functions where the key length is double to the block length, our contributions are three-folds. First, we present two concrete attacks on Hirose's two examples which are left as an open problem in [10]. The attacks show the fact that the two schemes are not optimally secure against the preimage and second preimage attacks. Moreover, two examples are presented for disclosing that Hirose's necessary conditions[10] for optimal collision resistance are not precise enough. Secondly, based on these attacks and negative examples, we formally analyze the security of a general class of the rate-1 DBL hash functions, which is defined by Satoh *et al.* in [23], to find whether there exists an optimally secure DBL hash function with rate 1. The necessary conditions for this class of DBL hash functions to be optimally secure are revised by the analysis. Finally, the security results are extended to a new class of DBL hash functions with rate 1, where one block cipher used in the compression function has the key length is equal to

the block length, while the other is doubled. Prior to this paper, there is no rigorous analysis on the half-baked cases proposed by Satoh *et al.*[23] and Hirose[10] to decide whether they are really optimally secure against collision, preimage and the second preimage attacks.

Organization. The remainder of this paper is organized as follows. In Section 2, some definitions and the former results on DBL hash functions with rate 1 are reviewed. In Section 3, first, two concrete attacks are presented on Hirose's two examples, then three examples are given to show the fact that Hirose's two necessary conditions[10] for optimal collision resistance are not precise. Furthermore, attacks are described on this general class of the rate-1 DBL hash functions. Section 4 describes an extended result on a new class of the rate-1 DBL hash functions. The conclusion is given in the last section.

2 Preliminaries

In this section, the notions and definitions are reviewed for the following analysis. Let the symbol $\oplus$ be the bitwise exclusive OR. For binary sequences a and b, $a||b$ denotes their concatenation. Let IV be the initial value. For double block length hash functions, an arbitrary input message M can be looked as a concatenation of the $2n$-bit length blocks such that $M = m_1||m_2||\cdots||m_t$, where $t = \lceil |M|/2n \rceil$ and $m_i = m_{i,1}||m_{i,2}, i \in \{0, t\}$. The function $Rank(\cdot)$ returns the rank of an input matrix. In this paper, length-padding on the last block of input message is implicitly used to avoid some trivial attacks. The same terminology and abbreviations in different definitions are the same meaning, except there are special claims in the context.

2.1 Block-Cipher-Based Hash Functions

Let κ, n, ℓ be numbers. A *block cipher* is a keyed function $E : \{0,1\}^\kappa \times \{0,1\}^n \to \{0,1\}^n$. For each $k \in \{0,1\}^\kappa$, the function $E_k(\cdot) = E(k, \cdot)$ denotes a permutation on $\{0,1\}^n$. If E is a block cipher then E^{-1} is its inverse, where $E_k^{-1}(y) = x$ such that $E_k(x) = y$. Let $\texttt{Bloc}(\kappa, n)$ be the family of all block ciphers $E : \{0,1\}^\kappa \times \{0,1\}^n \to \{0,1\}^n$. To avoid trivial extension attacks, we assume that any block cipher $E \in \texttt{Bloc}$ has no *fixed-point* such that $E_k(x) = k$ or x or $E_k^{-1}(y) = y$ or k and length strengthening technique[7, 17] is explicitly implemented in the constructions. A *block-cipher-based* hash function is a hash function $H : \{0,1\}^* \to \{0,1\}^\ell$ by implementing $E \in \texttt{Bloc}(\kappa, n)$ in the compression function of H. If $\ell = n$, then H is called a single block length(SBL) hash function, e.g., the PGV hash functions[21]. If $\ell = 2n$, then H is called a double block length(DBL) hash function, e.g., MDC-2[3], Parallel-DM[4], QG-I, and LOKI-DBH[13]. The *rate* is widely accepted to measure the efficiency of a block-cipher-based hash function, which is defined as follows.

Definition 1 *Let $H : \{0,1\}^* \to \{0,1\}^\ell$ be a hash function and $E \in \texttt{Bloc}(\kappa, n)$ is a block cipher used in the compression function of H. If the compression function performs T times encryption or decryption of E to process totally ℓ bits long message block, the* rate *of the hash function H equals $\frac{\ell}{T \cdot n}$.*

Ideal Cipher Model. Ideal cipher model is a well-known model for the security analysis of block-cipher-based hash functions, which is dating back to Shannon [24] and has been frequently used for the security analysis of various hash functions[1, 14, 21]. Let $H : \{0,1\}^* \to \{0,1\}^\ell$ be a hash function and $E \in \mathtt{Bloc}(\kappa, n)$ be a block cipher used in the round function of H. An adversary is given access to the encryption oracle E and the decryption oracle E^{-1}. The i-th query-response is defined as a four-tuple $(\sigma_i, k_i, x_i, y_i)$ where $k_i \in \{0,1\}^\kappa, x_i, y_i \in \{0,1\}^n$. If $\sigma_i = 1$ then the adversary queries (k_i, x_i) and gets response $y_i = E_{k_i}(x_i)$, otherwise he queries (k_i, y_i) and gets response $x_i = E_{k_i}^{-1}(y_i)$. Since $E_k(\cdot)$ is a permutation on $\{0,1\}^n$, it holds that

$$Pr[E_{k_i}(x_i) = y_i] = Pr[E_{k_i}^{-1}(y_i) = x_i] = \frac{1}{n}.$$

In the ideal cipher model, one measures the complexity of an attack, on which finding a collision, preimage or second preimage, is based on the total number of encryptions and decryptions the adversary queries. Generally, all repetition queries will be ignored, namely, if adversary asks a query $E_k(x)$ and this returns y, then he does not repeat the query or ask the inverse $E_k^{-1}(y)$. Such trivial queries does not help anything at the view of adversary. The block cipher in this model is variously named "Shannon oracle model", "Black-box model", or "Ideal cipher model". Since the last name is more often called, it will be used throughout the paper.

Recently, Black[2] exhibited a negative result on the ideal cipher model that there exists a block-cipher-based hash function that is provably secure in the ideal cipher model but trivially insecure when instantiated by any block cipher. The scheme is quite artificial and unnatural. Thus far, as in the ideal cipher model analog, no block cipher based hash function is proven secure but was broken after instantiation. Like schemes in the random oracle model, a hash function is proven secure in the ideal cipher model is still reliable, unless one uses the unnatural design for the goal from the beginning.

2.2 Security Definitions

Since cryptographic hash function is a fundamental component for the real-life cryptographic applications (e.g., data or entity authentication, public-key encryption and digital signature), a *secure* hash function must optimally secure against the following attacks for the security of the applications.

Attacks on hash functions. For block-cipher-based hash functions, there are three standard attacks which are called the collision attack, the preimage attack and the second preimage attack. A limitation is that the standard attacks only consider the situation that initial value IV is fixed.

Definition 2 *Let $H : \mathcal{K} \times \mathcal{M} \to \mathcal{Y}$ be a family of hash functions where $\mathcal{K} \in \{0,1\}^\kappa, \mathcal{Y} \in \{0,1\}^\ell$. Let M be a message belongs to message space $\mathcal{M} \in \{0,1\}^*$. By considering whether IV is fixed or not, three standard attacks and three extended attacks are defined as follows.*

1. *The preimage attack (Pre) is that given IV and h, find a message M such that $h = H(IV, M)$.*

2. *The free-start preimage attack ($fPre$) is that given IV and h, find IV' and M such that $h = H(IV', M)$.*

3. *The second preimage attack (Sec) is that given IV and a message M, find another message $M' \neq M$ such that $H(IV, M) = H(IV, M')$.*

4. *The free-start second preimage attack ($fSec$) is that given IV and a message M, find IV' and another message $M' \neq M$ such that $H(IV, M) = H(IV', M')$.*

5. *The collision attack ($Coll$) is that given an initial value IV, find $M \neq M'$ such that $H(IV, M) = H(IV, M')$.*

6. *The free-start collision attack ($fColl$) is that find $IV \neq IV'$ and messages M, M' such that $H(IV, M) = H(IV', M')$.*

The above attacks are from [12]. Similar definitions can be found in [14]. Compare with the standard attacks, the extended attacks are also meaningful since they would be a complete examination on minimizing potential flaws in a class of hash function. To rigorously analyze the security of a hash function at the presents of adversary, a widely accepted security model will be reviewed before the analysis.

Indifferentiability Model. In [16], Maurer *et al.* first introduced the notion of *indifferentiability*, which is formalized to "distinguish" whether a given construction exists any different from a heuristic random oracle. The indifferentiability has been focussed on the question: what conditions should be imposed on the round function $\mathcal{F}$ to ensure that the hash function $\mathcal{C}^{\mathcal{F}}$ satisfies the certain conditions of the random oracle. This approach is based on the fact that one of the problems in assessing the security of a hash function is caused by the arbitrary length of input. It is clear that the weakness of $\mathcal{F}$ will generally result in weakness of $\mathcal{C}^{\mathcal{F}}$, but the converse does not hold in general. The indifferentiability between a hash function and a random oracle is a more rigorous *white-box* analysis which requires the examination of the internal structure of the hash function, while the indistinguishability just requires a *black-box* analysis.

Definition 3 *A Turing machine $\mathcal{C}$ with oracle access to an ideal primitive $\mathcal{F}$ is said to be (t_D, t_S, q, ϵ)-indifferentiable from an ideal primitive Rand if there exists a simulator $\mathcal{S}$, such that for any distinguisher $\mathcal{D}$ it holds the advantage of indifferentiability that:*

$$Adv(D) = |Pr[\mathcal{D}^{\mathcal{C},\mathcal{F}} = 1] - Pr[\mathcal{D}^{Rand,\mathcal{S}} = 1]| < \epsilon,$$

where $\mathcal{S}$ has oracle access to Rand and runs in polynomial time at most t_S, and $\mathcal{D}$ runs in polynomial time at most t_D and makes at most q queries. $\mathcal{C}^{\mathcal{F}}$ is said to be (computationally) indifferentiable from Rand if ϵ is a negligible function of the security parameter k (in polynomial time t_D and t_S).

It is shown in [16] if $\mathcal{C}^{\mathcal{F}}$ is indifferentiable from *Rand*, then $\mathcal{C}^{\mathcal{F}}$ can instantiate *Rand* in any cryptosystems and the resulting cryptosystems is at least as secure in the $\mathcal{F}$ model as in the *Rand* model. In the rest of the paper, the Turing Machine $\mathcal{C}$ will denote the construction

of an iterated hash function and the ideal primitive $\mathcal{F}$ will represent the compression function of $\mathcal{C}$.

For block-cipher-based hash functions, the above definition needs to be slightly modified due to the underlying compression function should be analyzed in the ideal cipher model. Let E be the block cipher used in the compression function and E^{-1} is its inverse. Simulator $\mathcal{S}$ has to simulate both E and E^{-1} because every distinguisher $\mathcal{D}$ can access encryption and decryption oracles in the ideal cipher model. Therefore, distinguisher $\mathcal{D}$ obtain the following rules: either the block-cipher E, E^{-1} is chosen at random and the hash function H is constructed from it, or the hash function H is chosen at random and the block-cipher E, E^{-1} is implemented by a simulator $\mathcal{S}$ with oracle access to H. Those two ways to build up a hash function should be indifferentiable.

Similarly, Hirose proposed the notion of *indistinguishability* on iterated hash functions in [9], which is weaker than the notion of indifferentiability. It is easily to prove if a hash function $C^{E,E^{-1}}$ is indifferentiable from a random oracle in a polynomial time bound t_S, t_D with a probability bound ϵ, it is also indistinguishable in the same bound. For simplicity, one needs only to prove the indifferentiability of the construction.

2.3 Results on Fast DBL Hash Functions

By assuming the key length κ of block cipher $E \in \mathtt{Bloc}(\kappa, n)$ used in compression function is equal to the block length n-bit, Knudsen *et al.* [13] presented attacks on a class of DBL hash functions with rate 1. The general form of this class is described as follows.

$$\begin{cases} h_i = E_A(B) \oplus C, \\ g_i = E_X(Y) \oplus Z. \end{cases} \tag{1}$$

For all rate-1 hash functions defined by (1)(denoted by FDBL-I), (A, B, C) are linear combinations of the n-bit vectors $(h_{i-1}, g_{i-1}, m_{i,1}, m_{i,2})$, (X, Y, Z) are linear combinations of the n-bit vectors $(h_i, h_{i-1}, g_{i-1}, m_{i,1}, m_{i,2})$.

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \underbrace{\begin{pmatrix} L_l & L_r \end{pmatrix}}_{L} \cdot \begin{pmatrix} h_{i-1} \\ g_{i-1} \\ m_i^1 \\ m_i^2 \end{pmatrix}, \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \underbrace{\begin{pmatrix} R_l & R_r \end{pmatrix}}_{R} \cdot \begin{pmatrix} h_i \\ h_{i-1} \\ g_{i-1} \\ m_i^1 \\ m_i^2 \end{pmatrix}. \tag{2}$$

If h_i and g_i can be computed independently, the hash function is called *parallel*, otherwise is called *serial*. Knudsen *et al.*[13] proved that all rate-1 hash functions in FDBL-I are failed to be optimally secure.

Theorem 1 *For the rate-1 iterated hash function with the form* (1)*(FDBL-I), where (at least) one of* $h_i \in \{0,1\}^n$ *and* $g_i \in \{0,1\}^n$ *in the hash function has the form of a (secure) single block length hash function, there exist second preimage attacks with complexities of about* 3×2^n*, primage attacks with complexities of about* 4×2^n*, and collision attacks with complexities of about* $3 \times 2^{n/2}$*.*

In AES algorithm, the key length can be 128,196,256-bit while the block length is 128-bit. This property motivates interest in finding good ways to turn a block cipher into an optimally secure fast DBL hash function whose block length and key length are not limited to the same n-bit. By considering the block cipher $E \in \texttt{Bloc}(\kappa, n)$ where $\kappa = 2n$, Satoh *et al.*[23] proposed a new family of the rate-1 DBL hash functions, which is defined by the general form as follows.

$$\begin{cases} h_i = E_{A||B}(C) \oplus D, \\ g_i = E_{W||X}(Y) \oplus Z. \end{cases} \tag{3}$$

For all rate-1 hash functions defined by (3) (denoted by FDBL-II), both (A, B, C, D) and (W, X, Y, Z) are linear combinations of the n-bit vectors $(h_{i-1}, g_{i-1}, m_{i,1}, m_{i,2})$. Those linear combinations can be represented as

$$\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \underbrace{\begin{pmatrix} L_l & L_r \end{pmatrix}}_{L} \cdot \begin{pmatrix} h_{i-1} \\ g_{i-1} \\ m_i^1 \\ m_i^2 \end{pmatrix}, \begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \underbrace{\begin{pmatrix} R_l & R_r \end{pmatrix}}_{R} \cdot \begin{pmatrix} h_{i-1} \\ g_{i-1} \\ m_i^1 \\ m_i^2 \end{pmatrix}, \tag{4}$$

where L_l and L_r denote 4×2 binary submatrices of L. Let L_r^i denote the 3×2 submatrices of L_r such that the i-th row of L_r are deleted, respectively. Similarly, L_l^i denote the 3×2 submatrices of L_l such that the i-th row of L_l are deleted, respectively. Matrix L is said to be *exceptional* if $Rank(L) = 4$ and $Rank(L_r^3) = Rank(L_r^4) = 2$[23].

In [23], Satoh *et al.* stated attacks on this kind of DBL hash functions whose compression functions do not satisfy the property "exceptional".

Theorem 2 *For the rate-1 iterated hash function with the form* (3)*(FDBL-II), if L or R is not exceptional, there exist the preimage, the second preimage and the collision attacks with complexities of about 4×2^n, 3×2^n and $3 \times 2^{n/2}$, respectively.*

In particular, Satoh *et al.*[23] showed attacks on a subclass of the rate-1 DBL hash functions in FDBL-II. We stress that the scheme in [27] is a paradigm that belongs to this subclass.

Theorem 3 *For a subclass of the rate-1 double block length hash functions in FDBL-II with the compression function h:*

$$\begin{cases} h_i = E_{A||B}(C) \oplus D, \\ g_i = E_{A||B}(C) \oplus F. \end{cases} \tag{5}$$

where (A, B, C, D, F) is linear combinations of $(h_{i-1}, g_{i-1}, m_{i,1}, m_{i,2})$ and $E \in \texttt{Bloc}(2n, n)$. Then, there exist (second) preimage attacks with complexities of about 2×2^n, and collision attacks with complexities of about $2 \times 2^{n/2}$.

In [10], Hirose gave a comment on the analysis by Satoh *et al.*[23]. The comment shows there exist the rate-1 DBL hash functions whose compression functions do not satisfy the property "exceptional" but still no meaningful collision attacks can be found. For convincing

of this result, an example (denoted by HDBL-1) without the property "exceptional" was proposed in [10] as follows.

HDBL-1: Let HDBL-1:$\{0,1\}^* \to \{0,1\}^{2n}$ be a double block length hash function and $E \in \texttt{Bloc}(2n,n)$ is the block cipher used in the compression function of H. The compression function has the following:

$$\begin{cases} h_i = E_{m_{i,1}||m_{i,2}}(h_{i-1} \oplus g_{i-1}) \oplus h_{i-1} \oplus g_{i-1}, \\ g_i = E_{m_{i,1}||m_{i,2}}(h_{i-1}) \oplus h_{i-1}. \end{cases} \tag{6}$$

$$\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}}_{L} \cdot \begin{pmatrix} h_{i-1} \\ g_{i-1} \\ m_{i,1} \\ m_{i,2} \end{pmatrix}, \begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{R} \cdot \begin{pmatrix} h_{i-1} \\ g_{i-1} \\ m_{i,1} \\ m_{i,2} \end{pmatrix} \tag{7}$$

Furthermore, an example (denoted by HDBL-2) with the property "exceptional" was also proposed in [10].

HDBL-2: Let HDBL-2:$\{0,1\}^* \to \{0,1\}^{2n}$ be a double block length hash function and $E \in \texttt{Bloc}(2n,n)$ is the block cipher used in the compression function of H. The compression function has the following:

$$\begin{cases} h_i = E_{m_{i,1}||m_{i,2}}(h_{i-1}) \oplus g_{i-1}, \\ g_i = E_{m_{i,1}||m_{i,2}}(g_{i-1}) \oplus h_{i-1}. \end{cases} \tag{8}$$

$$\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}_{L} \cdot \begin{pmatrix} h_{i-1} \\ g_{i-1} \\ m_{i,1} \\ m_{i,2} \end{pmatrix}, \begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{R} \cdot \begin{pmatrix} h_{i-1} \\ g_{i-1} \\ m_{i,1} \\ m_{i,2} \end{pmatrix} \tag{9}$$

Both HDBL-1 and HDBL-2 are the instances of FDBL-II. Based on the results given by Knudsen *et al.*[13] and Satoh *et al.*[23], Hirose[10] presented two necessary conditions for the rate-1 hash functions in FDBL-II to be optimally collision resistant.

Definition 4 *For any rate-1 iterated hash function in FDBL-II, if it is optimally collision resistant, then it must be in one of the two types:*

1. *Both L and R are exceptional,*
2. *$Rank(L) = Rank(R) = 3$, $c \oplus d = \lambda_1 a \oplus \lambda_2 b$ and $y \oplus z = \lambda_3 w \oplus \lambda_4 x$, for some $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \{0,1\}$, and the upper right 2×2 submatrices of L and R are both non-singular.*

In [10], Hirose claimed that the above conditions are not sufficient but just necessary for the property of optimal collision resistance. It was left as an open problem if the two plausible examples(HDBL-1 and HDBL-2) are really optimally secure.

3 Security Analysis of FDBL-II

In this section, the security of the rate-1 hash functions in FDBL-II is reconsidered. A synthetic analysis is presented which exploits the fact that the former results[10, 23] on the security of FDBL-II are not precise. First, two concrete attacks are presented to disclose that both HDBL-1 and HDBL-2 are failed to be optimally preimage and second preimage resistant. Next, three examples are presented, which disclose Hirose's conditions for optimally collision resistant are failed in some uncovered cases. Finally, some formal proofs are given for the security analysis of FDBL-II.

3.1 Attacks on Hirose's Two Examples

In [23], Satoh *et al.* suggested that any rate-1 hash function in FDBL-II will not to be optimally secure if its compression function does not satisfy the *exceptional* property. Towards this approach, Hirose[10] gave a comment on Satoh *et al.*'s result, and said there exist optimally secure hash functions in FDBL-II whose compression functions do not satisfy the exceptional property. Moreover, Hirose proposed two two rate-1 hash functions in FDBL-II (HDBL-1 and HDBL-2, described in Section 2.4) which are *plausible* secure. HDBL-1 satisfies the exceptional property while HDBL-2 does not(Both of them satisfy Hirose's necessary conditions in Definition 4). In this section, two (second) preimage attacks are presented on these two examples which shows they are both failed to optimally (second) preimage resistant. First, some definitions are given for the analysis. Let $E(\cdot) \in \texttt{Bloc}(2n, n)$ be an encryption function and $E^{-1}(\cdot)$ is its inverse. Let $M^i = m_1^i||m_2^i||\cdots||m_t^i$ be the i-th input message where the $2n$-bit length block $m_j^i = m_{j,1}^i||m_{j,2}^i, j \in \{1, t\}$. Let IV be the initial value and $h_0||g_0 = IV$. $\mathcal{A}$ denotes the adversary in the ideal cipher model.

Theorem 4 *Let HDBL-1 be a hash function defined by the form* (6)*,*

$$\begin{cases} h_i = E_{m_{i,1}||m_{i,2}}(h_{i-1} \oplus g_{i-1}) \oplus h_{i-1} \oplus g_{i-1}, \\ g_i = E_{m_{i,1}||m_{i,2}}(h_{i-1}) \oplus h_{i-1}, \end{cases}$$

then there exists a (second) preimage attack on the hash function with complexity about $4 \times 2^{3n/2}$.

Proof. By using the idea of the *meet-in-the-middle* attack, a preimage attack on the HDBL-1 hash function proceeds as follows.

1. For the preimage attack on (h_i, g_i), $\mathcal{A}$ chooses arbitrary message $M = m_1||m_2||\cdots||m_{i-2}$, and by computing the values of (h_{i-2}, g_{i-2}) iteratively from the initial value $IV = h_0||g_0$.
2. Forward step:
   (a) $\mathcal{A}$ tries 2^n operations to find a pair (m_i, c) where $h_i = E_{m_{i,1}||m_{i,2}}(c) \oplus c = E_{m_i'}(c) \oplus c$.
   (b) $\mathcal{A}$ chooses 2^n values of h_{i-1} where $c = h_{i-1} \oplus g_{i-1}$. Due to the pigeonhole principle, $\mathcal{A}$ can find a value of h_{i-1} satisfies $g_i = E_{m_{i,1}||m_{i,2}}(h_{i-1}) \oplus h_{i-1}$.

(c) $\mathcal{A}$ repeats q_1 times of the forward step to obtain q_1 values of $(m_{i,1}, m_{i,2}, h_{i-1}, g_{i-1})$.

3. Backward step: $\mathcal{A}$ chooses q_2 values of m_{i-1}, then computes q_2 values of (h'_{i-1}, g'_{i-1}) from $(m_{i-1}, h_{i-2}, g_{i-2})$.

The attack succeeds if some (h_{i-1}, g_{i-1}) and some (h'_{i-1}, g'_{i-1}) are matched. Since the quantities in the meet-in-the-middle attack are $2n$-bit long, the successful probability $\Pr(Pre)$ equals

$$\begin{aligned}\Pr(Pre) &= (1-\frac{q_1}{2^{2n}})\cdot(1-\frac{q_1}{2^{2n}-1})\cdots(1-\frac{q_1}{2^{2n}-q_2}) \\ &\geq (1-\frac{q_1}{2^{2n}-q_2})^{q_2}.\end{aligned} \tag{10}$$

Since the complexity of the above attack is the larger value of $2^n \times q_1$ and q_2. For non-negligible probability in the lowest complexity, it follows that

$$\begin{cases} 2^n \times q_1 = q_2, \\ q_1 \times q_2 = 2^{2n} - q_2. \end{cases} \tag{11}$$

Consequently, it holds that $q_1 \approx 2^{n/2}$ and $q_2 \approx 2^{3n/2}$, then the probability

$$\begin{aligned}\Pr(Pre) &\geq (1-\frac{2^{n/2}}{2^{2n}-2^{3n/2}})^{2^{3n/2}} \\ &\approx 1-e^{-1} \approx 0.39.\end{aligned} \tag{12}$$

It is easy to see that the forward and the backward steps require $2 \times 2^{3n/2}$ operations, respectively. Thus the total complexity of the attack is $4 \times 2^{3n/2}$. We stress that a second preimage attack can be constructed by using the same method. So the theorem holds. □

Similar to HDBL-1, a (second) preimage attack can also be found in the HDBL-2 hash function. The attack is described in the following theorem.

Theorem 5 *Let HDBL-2 be a hash function defined by the form* (8)*,*

$$\begin{cases} h_i = E_{m_{i,1}||m_{i,2}}(h_{i-1}) \oplus g_{i-1}, \\ g_i = E_{m_{i,1}||m_{i,2}}(g_{i-1}) \oplus h_{i-1}. \end{cases}$$

then there exists a (second) preimage attack on the hash function with complexity about $4 \times 2^{3n/2}$.

Proof. By using the method of the meet-in-the-middle-attack, a (second) preimage attack on the HDBL-2 hash function proceeds as follows.

1. For the preimage attack on (h_i, g_i), $\mathcal{A}$ chooses arbitrary message $M = m_1||m_2||\cdots||m_{i-2}$, and by computing the values of (h_{i-2}, g_{i-2}) iteratively from the initial value $IV = h_0||g_0$.

2. Forward step:

   (a) $\mathcal{A}$ randomly chooses 2^n values of $(m_{i,1}, m_{i,2}, h_{i-1})$, then computes 2^n values of g_{i-1} where $g_{i-1} = E_{m_{i,1}||m_{i,2}}(h_{i-1}) \oplus h_{i-1}$.

   (b) $\mathcal{A}$ repeats the above step $2^{n/2}$ times. Due to the pigeonhole principle, $\mathcal{A}$ obtains $2^{n/2}$ values of (m_i, h_{i-1}, g_{i-1}) yield the fixed value (h_i, g_i).

3. Backward step: $\mathcal{A}$ chooses $2^{3n/2}$ values of m_{i-1}, then computes $2^{3n/2}$ values of (h'_{i-1}, g'_{i-1}) from $(m_{i-1}, h_{i-2}, g_{i-2})$.

The attack succeeds if some (h_{i-1}, g_{i-1}) and some (h'_{i-1}, g'_{i-1}) are matched. Since the quantities in the meet-in-the-middle attack are $2n$-bit long, same to the equations (10),(11) and (12) in the attack of HDBL-1, the successful probability $\Pr(Pre)$ equals 0.39. Consequently, the complexity of the (second) preimage attack is also $4 \times 2^{3n/2}$. So the theorem holds. □

Since HDBL-1 and HDBL-2 satisfy Type-II and Type-I in Definition 4, respectively, which are the two necessary conditions that defined by Hirose in [10]. The above attacks disclose the point that there exist uncovered flaws in the former security results on the rate-1 hash functions in FDBL-II which are given by Satoh *et al.*[23] and Hirose[10]. Heuristically, we show three plausible examples, which do not satisfy Hirose's necessary conditions but still no efficient collision attack can be found, to support this considerable point.

First we give an examples of the rate-1 hash functions in FDBL-II, which do not satisfy the condition $c \oplus d = \lambda_1 a \oplus \lambda_2 b$ and $y \oplus z = \lambda_3 w \oplus \lambda_4 x$, for some $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \{0, 1\}$.

Example 1:

$$\begin{cases} h_i = E_{m_{i,1} \oplus m_{i,2} \oplus h_{i-1} || m_{i,2} \oplus g_{i-1}}(m_{i,1} \oplus h_{i-1}) \oplus m_{i,2} \oplus g_{i-1} \\ g_i = E_{m_{i,1}||m_{i,2}}(h_{i-1}) \oplus h_{i-1} \end{cases} \tag{13}$$

The second example does not satisfy the upper right 2×2 submatrices of L and R are both non-singular.

Example 2:

$$\begin{cases} h_i = E_{m_{i,1}||h_{i-1}}(m_{i,2} \oplus g_{i-1}) \oplus m_{i,2} \oplus g_{i-1} \\ g_i = E_{m_{i,1}||m_{i,2}}(h_{i-1}) \oplus h_{i-1} \end{cases} \tag{14}$$

For completeness, the reader can refer to Appendix A.1 for the proof in the indifferentiability model. From the above attacks and the examples, it is easy to see that Hirose's two necessary conditions (at least) are not precise enough for the rate-1 hash functions in FDBL-II to be optimally secure against the preimage, the second preimage and the collision attacks. A more rigorous analysis is required to discover more precise conditions which should be imposed on FDBL-II for the property of the optimal security.

3.2 The Exact Security of FDBL-II

In this section, necessary conditions for the rate-1 hash functions in FDBL-II to be optimally secure against the preimage, the second preimage and the collision attacks are analyzed. By using the similar methods for the general attacks on the rate-1 hash functions in FDBL-I[13], some general attacks on FDBL-II are described in the following theorems. For ease of the reader, the general form of FDBL-II is recalled below.

$$\begin{cases} h_i = E_{A||B}(C) \oplus D, \\ g_i = E_{W||X}(Y) \oplus Z. \end{cases}$$

$$\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \underbrace{\begin{pmatrix} L_l & L_r \end{pmatrix}}_{L} \cdot \begin{pmatrix} h_{i-1} \\ g_{i-1} \\ m_i^1 \\ m_i^2 \end{pmatrix}, \begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \underbrace{\begin{pmatrix} R_l & R_r \end{pmatrix}}_{R} \cdot \begin{pmatrix} h_{i-1} \\ g_{i-1} \\ m_i^1 \\ m_i^2 \end{pmatrix}.$$

Theorem 6 *For the rate-1 hash functions H in FDBL-II with the form (3), if T operations are required to find a block $m_i = m_{i,1}||m_{i,2}$ for any given value of (h_{i-1}, g_{i-1}), such that the resulting four-tuple $(h_{i-1}, g_{i-1}, m_{i,1}, m_{i,2})$ yields the fixed value for h_i(or g_i or $h_i \oplus g_i$), then there exist collision, preimage, and second preimage attacks on the hash function with complexities $(T+3) \times 2^{n/2}$, $(T+3) \times 2^n$, and $(T+3) \times 2^n$, respectively.*

Proof. An attacker $\mathcal{A}$ starts the attacks by choosing arbitrary message $M = m_1||m_2||\cdots||m_{i-2}$, and by computing the values of (h_{i-2}, g_{i-2}) iteratively from the initial value $IV = h_0||g_0$. The initial operations for the values of (h_{i-2}, g_{i-2}) can be ignored if $i \ll 2^{n/2}$.

For (second) preimage attacks, $\mathcal{A}$ searches for two blocks m_{i-1} and m_i such that the fixed hash value (h_i, g_i) is hit. First, $\mathcal{A}$ computes the pair (h_{i-1}, g_{i-1}) from the given values (h_{i-2}, g_{i-2}) and $(m_{i-1,1}, m_{i-1,2})$. Next, $\mathcal{A}$ finds a block $(m_{i,1}, m_{i,2})$ such that the resulting four-tuple $(h_{i-1}, g_{i-1}, m_{i,1}, m_{i,2})$ yields the fixed value for h_i(or g_i or $h_i \oplus g_i$). This step costs T times of encryption or decryption. Finally, $\mathcal{A}$ computes the value of g_i(or h_i) from the tuple $(h_{i-1}, g_{i-1}, m_{i,1}, m_{i,2})$. If the value does not hit, $\mathcal{A}$ will repeat the above steps at most 2^n times. Due to the pigeonhole principle, the probability of finding the preimage in the above procedure is non-negligible. The total complexity of these (second) preimage attacks is about $(T+3) \times 2^n$.

For collision attacks, $\mathcal{A}$ searches for a pair of the blocks (m_{i-1}, m_i) and (m'_{i-1}, m'_i) yields the same hash value (h_i, g_i). First, $\mathcal{A}$ chooses a value of h_i. Then $\mathcal{A}$ proceeds $2^{n/2}$ times in the same way as the preimage attack. Due to the birthday paradox, the probability of finding the collision in the above procedure is non-negligible. The total complexity of these collision attacks is about $(T+3) \times 2^{n/2}$. □

In [10], a comment is proved that the attacks given by Satoh *et al.*[23] do not work on some hash functions in FDBL-II, as is expected even the underlying compression function does not satisfy the exceptional property. E.g., HDBL-1 is a counter-example that supports this comment. Let (a, b, c, d) be the values of (A, B, C, D) used in the computations of h_i. In [23], the attacker chooses random triple (a, b, c) such that $c = \alpha \cdot a \oplus \beta \cdot b$, then computes

$d = E_{a||b}(c) \oplus h_i$. Hirose found if $c = \alpha \cdot a \oplus \beta b \oplus d$, the attacker cannot compute d by $E_{a||b}(c) \oplus h_i$. Therefore, besides both L and R are exceptional, a new condition for the rate-1 hash functions in FDBL-II to be optimally secure is defined by Hirose [10] as the second condition described in Definition 4. Due to the two concrete attacks which was described in Section 3.1, HDBL-1 and HDBL-2 become two counter-examples on Hirose's necessary conditions[10]. Moreover, Since HDBL-2 is an instance of FDBL-II with the exceptional property, it means that the given conditions do not directly imply the optimal security. Thus, the result given by Satoh *et al.*[23] is not precise enough as well. To ensure what conditions should be imposed on a hash function to achieve the optimal security, the security of the rate-1 hash functions in FDBL-II is reconsidered through the following attacks.

First, the attacks that simultaneously break the optimal collision and the (second) preimage resistances are described as follows.

Lemma 1 *For a rate-1 hash function H in FDBL-II with the form (3), if the rank of L(or R) is less than three, then there exist collision, preimage, and second preimage attacks on the hash function with complexities of about $4 \times 2^{n/2}$, 3×2^n, and 3×2^n, respectively.*

Proof. Consider the general form of FDBL-II. Since the rank of L(or R) is at most two and h_i depends on a subspace of $(m_{i,1}, m_{i,2}, h_{i-1}, g_{i-1})$, it follows that an attacker can has at least one dimensional of freedom to find the values of $m_{i,1}$(or $m_{i,2}$ or $m_{i,1} \oplus m_{i,2}$) yields the given hash value (h_i, g_i). Based on the attacks defined by Theorem 6, it is easily to prove that $T \simeq 0$ in the (second) preimage attack, and $T \simeq 1$ in the collision attack. So the theorem holds. □

Lemma 2 *For a rate-1 hash function H in FDBL-II with the form (3), if the rank of L_r^3(or L_r^4 or R_r^3 or R_r^4) is less than two, then there exist collision, preimage, and second preimage attacks on the hash function with complexities of about $4 \times 2^{n/2}$, 3×2^n, and 3×2^n, respectively.*

Proof. Consider the general form of FDBL-II. If either L_r^3 or L_r^4 is less than two, then the key $A||B$ of $E_{A||B}(C)$ depends on one dimensional of $(m_{i,1}, m_{i,2})$(or $m_{i,1} \oplus m_{i,2}$). Let (a, b, c, d) be the values of (A, B, C, D) used in the computations of h_i. By computing $d = E_{a||b}(c) \oplus h_i$(in case of L_r^4 is less than two) or $c = E_{a||b}^{-1}(d \oplus h_i)$(in case of L_r^3 is less than two), an attacker can decide the value of $m_{i,1}$(or $m_{i,2}$) from the hash values of $(h_{i-1}, g_{i-1}, h_i, g_i)$. Based on the attacks defined by Theorem 6, it is easily to prove that $T \simeq 0$ in the (second) preimage attack, and $T \simeq 1$ in the collision attack. Same result holds if either R_r^3 or R_r^4 is less than two. □

Subsequently, the attacks that only break the property of the optimal (second) preimage resistance were described as follows.

Theorem 7 *For a rate-1 hash function H in FDBL-II with the form (3), if the rank of L_l^3(or L_l^4 or R_l^3 or R_l^4) is less than two, then there exists a (second) preimage attack on the hash function with complexity of about $4 \times 2^{3n/2}$.*

Proof. Consider the general form of FDBL-II. If either L_l^3 or L_l^4 is less than two, then the key $A||B$ of $E_{A||B}(C)$ depends on one dimensional of (h_{i-1}, g_{i-1})(or $h_{i-1} \oplus g_{i-1}$). Let (a, b, c, d) be

the values of (A, B, C, D) used in the computations of h_i. By computing $d = E_{a||b}(c) \oplus h_i$(in case of L_l^4 is less than two) or $c = E_{a||b}^{-1}(d \oplus h_i)$(in case of L_l^3 is less than two).

An attacker $\mathcal{A}$ start the attacks by choosing arbitrary messages $M = m_1||m_2||\cdots||m_{i-2}$, and by computing the values of (h_{i-2}, g_{i-2}) iteratively from the given initial value $IV = h_0||g_0$.

1. Forward step: $\mathcal{A}$ randomly chooses 2^n values of $(m_{i,1}, m_{i,2}, h_{i-1})$, then computes 2^n values of g_{i-1} where $g_{i-1} = E_{m_{i,1}||m_{i,2}}(h_{i-1}) \oplus h_{i-1}$.

2. $\mathcal{A}$ repeats the above step $2^{n/2}$ times. Due to the pigeonhole principle, $\mathcal{A}$ obtains $2^{n/2}$ values of (m_i, h_{i-1}, g_{i-1}) yield the fixed value (h_i, g_i).

3. Backward step: $\mathcal{A}$ chooses $2^{3n/2}$ values of m_{i-1}, then computes $2^{3n/2}$ values of (h'_{i-1}, g'_{i-1}) from $(m_{i-1}, h_{i-2}, g_{i-2})$.

It is easy to see the attack will succeed with a non-negligible probability due to the pigeonhole principle holds in Step 1. The total complexity is about $4 \times 2^{3n/2}$. Same result holds if either R_l^3 or R_l^4 is less than two. □

We stress that both HDBL-1 and HDBL-2 are failed to be optimally (second) preimage resistance due to Theorem 9.

Finally, the complexities of free-start attacks on the rate-1 hash functions in FDBL-II can be easily deduced from the above results.

Lemma 3 *For a rate-1 hash function H in FDBL-II with the form (3), if one of the ranks of L and R is less than four, then there exist free-start collision and free-start (second) preimage attacks on the hash function with complexities of about $2 \times 2^{n/2}$ and 2×2^n, respectively.*

Based on the above results, the necessary conditions for the rate-1 hash functions in FDBL-II to be optimally secure are listed as follows.

Corollary 1 *For a rate-1 hash functions in FDBL-II, if the compression function matches one of the following two conditions:*

1. *The ranks of L and R are less than three;*

2. *The rank of L_r^3(or L_r^4 or R_r^3 or R_r^4) is less than two,*

then there exist collision , preimage and second preimage preimage attacks with a non-negligible successful probability must spend the complexities of about $O(2^{n/2})$, $O(2^n)$ and $O(2^n)$, respectively. Furthermore, if the compression function matches one of the following two conditions:

1. *The ranks of L and R are less than four;*

2. *The rank of L_l^3(or L_l^4 or R_l^3 or R_l^4) is less than two,*

then there exist preimage and second attacks with a non-negligible successful probability must spend the complexities of about $O(2^{3n/2})$ and $O(2^{3n/2})$, respectively.

4 A New Class of Fast DBL Hash Functions

Based on FDBL-I and FDBL-II, a new class of fast DBL hash functions named FDBL-III can be defined as follows. Hash functions in FDBL-III can be constructed on a block cipher $E \in \mathtt{Bloc}(\kappa, n)$ with variants of key length where $\kappa = n$ or $\kappa = 2n$.

Definition 5 *Let $E \in \mathtt{Bloc}(\kappa, n)$ be a block cipher with variants of key length where $\kappa = n$ or $\kappa = 2n$. A new class of DBL hash functions with rate 1 (denoted by FDBL-III) can be constructed as follows.*

$$\begin{cases} h_i = E_A(B) \oplus C, \\ g_i = E_{W||X}(Y) \oplus Z. \end{cases} \tag{15}$$

Both (A, B, C) and (W, X, Y, Z) are linear combinations of the n-bit vectors $(h_{i-1}, g_{i-1}, m_{i,1}, m_{i,2})$. Those linear combinations can be represented as

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \underbrace{\begin{pmatrix} L_l & L_r \end{pmatrix}}_{L} \cdot \begin{pmatrix} h_{i-1} \\ g_{i-1} \\ m_i^1 \\ m_i^2 \end{pmatrix}, \begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \underbrace{\begin{pmatrix} R_l & R_r \end{pmatrix}}_{R} \cdot \begin{pmatrix} h_{i-1} \\ g_{i-1} \\ m_i^1 \\ m_i^2 \end{pmatrix}, \tag{16}$$

By implementing the similar attacks on FDBL-II, one can easily derive the following lemmas on FDBL-III.

Lemma 4 *Consider a hash function H in FDBL-III with the form (16), if the rank of L (or R) is less than three, then there exist collision, preimage, and second preimage attacks on the hash function with complexities of about $4 \times 2^{n/2}$, 3×2^n, and 3×2^n, respectively.*

Lemma 5 *Consider a hash function in FDBL-III with the form (16), if the rank of L_r^2 (or L_r^3 or R_r^3 or R_r^4) is less than two, then there exist collision, preimage, and second preimage attacks on the hash function with complexities of about $4 \times 2^{n/2}$, 3×2^n, and 3×2^n, respectively.*

Lemma 6 *Consider hash function in FDBL-III with the form (16), if the rank of L_l^2 (or L_l^3 or R_l^3 or R_l^4) is less than two, then there exists a (second) preimage attack on the hash function with complexity of about $4 \times 2^{3n/2}$.*

Lemma 7 *Consider a rate-1 hash function H in FDBL-III with the form (16), there exist free-start collision and free-start (second) preimage attacks on the hash function with complexities of about $2 \times 2^{n/2}$ and 2×2^n, respectively.*

An example of the rate-1 DBL hash function in FDBL-III, which is optimally secure against the preimage, the second preimage and the collision attacks, is described as follows. See Appendix A.2 for its proof.

Example 3:

$$\begin{cases} h_i = E_{m_i^1 \oplus h_{i-1}}(m_i^2 \oplus h_{i-1} \oplus g_{i-1}) \oplus m_i^1 \oplus m_i^2 \oplus h_{i-1} \oplus g_{i-1}, \\ g_i = E_{m_i^1 \oplus g_{i-1} || m_i^2 \oplus h_{i-1}}(h_{i-1}) \oplus g_{i-1}. \end{cases} \tag{17}$$

The rate-1 hash functions in FDBL-III can also be constructed from two different block ciphers where $E_1 \in \texttt{Bloc}(n, n)$ and $E_2 \in \texttt{Bloc}(2n, n)$, which enlarges the candidates for the design.

5 Conclusion

In this paper, new attacks have been described on FDBL-II [10, 23]. In particular, the attacks proved Hirose's two examples are not optimally secure against the preimage and second preimage attacks. Based on the former results, the security of FDBL-II has been reconsidered and the conditions for optimally secure are given. Moreover, the security results are extended to a new class of the rate-1 hash functions (FDBL-III) based on FDBL-I and FDBL-II. These cryptanalysis results are practical and helpful to find the rate-1 DBL hash functions to be optimally secure in FDBL-II and FDBL-III. By considering the security conditions on FDBL-II, Hirose's two examples can be improved as follows.

1. Improved HDBL-1

$$\begin{cases} h_i = E_{m_{i,1} \oplus g_{i-1} || m_{i,2}}(h_{i-1}) \oplus g_{i-1} \\ g_i = E_{m_{i,1} || m_{i,2} \oplus h_{i-1}}(g_{i-1}) \oplus h_{i-1} \oplus g_{i-1} \end{cases}$$

2. Improved HDBL-2

$$\begin{cases} h_i = E_{m_{i,1} || m_{i,2} \oplus g_{i-1}}(h_{i-1}) \oplus h_{i-1} \oplus g_{i-1} \\ g_i = E_{m_{i,1} \oplus g_{i-1} || m_{i,2}}(h_{i-1} \oplus g_{i-1}) \oplus h_{i-1} \end{cases}$$

Future work is to make it clear whether there exists a subclass of the rate-1 hash functions in FDBL-II or FDBL-III which can be formally proved optimally secure against collision, preimage and second preimage attacks. Another interesting work is to compare the performances with the certain applications in between FDBL-II and FDBL-III.

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A. Indifferentiability Analysis on FDBL-II and FDBL-III Examples

Based on the indifferentiability analysis of block-cipher-based hash functions[5, 8], here we present an indifferentiability analysis on FDBL-II and FDBL-III examples. Let distinguisher $\mathcal{D}$ can access to oracles $(\mathcal{O}_1, \mathcal{O}_2)$ where $\mathcal{O}_1 = (H, E, E^{-1})$ and $\mathcal{O}_2 = (Rand, S, S^{-1})$. Let $r_i \leftarrow ((h_{i-1}, g_{i-1}) \xrightarrow{m_i} (h_i, g_i))$ be the i-th query-response to the oracles $\{E, E^{-1}, S, S^{-1}\}$ where $m_i \in \{0,1\}^{2n}$. $\mathcal{R}_i = (r_1, \cdots, r_i)$ denotes the query-response set on the oracles $\{E, E^{-1}, S, S^{-1}\}$ after the i-th query. Let $r'_i \leftarrow (IV \xrightarrow{M} (h_i, g_i))$ be the i-th query-response to the oracles $\{H, Rand\}$ where $M \in \mathcal{M}$. $\mathcal{R}'_i = (r'_1, \cdots, r'_i)$ denotes the query-response set on the oracles $\{H, Rand\}$ after the i-th query. $Pad(\cdot)$ denotes the indifferentiable padding rules, e.g., the prefix-free padding, HMAC/NMAC and the chop construction, which were analyzed in [6]. For simplicity, we stress that all of the examples are implicitly implemented one of those padding rules in the indifferentiability analysis.

A.1 Proof of Example 1

First we give a simulation to prove that Example 1 (defined in Section 3.1) is indifferentiable from a random oracle.

- *Rand*-**Query.** For the i-th query $M_i \in \mathcal{M}$ on *Rand*, if M_i is a repetition query, the simulator $\mathcal{S}$ retrieves $r'_j \leftarrow (IV \xrightarrow{M_i} (h_j, g_j))$ where $r_j \in \mathcal{R}'_{i-1}, j \leq i-1$, then returns $Rand(M_i) = (h_j, g_j)$. Else $\mathcal{S}$ randomly selects a hash value $(h_i, g_i) \in \mathcal{Y}$ and updates $\mathcal{R}'_i = \mathcal{R}'_{i-1} \cup \{IV \xrightarrow{M_i} (h_i, g_i)\}$, then returns $Rand(M_i) = (h_i, g_i)$.
- $\{S, S^{-1}\}$-**Query.** To answer the distinguisher $\mathcal{D}$'s encryption and decryption queries, the simulator $\mathcal{S}$ proceeds as follows.

  1. For the i-th query $(1, k_i, x_i)$ on S:
     (a) If $\exists IV \xrightarrow{M} (h_{i-1}, g_{i-1}) \in \mathcal{R}'_{i-1}$, $\mathcal{S}$ computes $Pad(M) = m_i = m_{i,1}||m_{i,2}$. And then,
        i. if $k_i = m_{i,1} \oplus m_{i,2} \oplus h_{i-1}||m_{i,2} \oplus g_{i-1}$ and $x_i = m_{i,1} \oplus h_{i-1}$, $\mathcal{S}$ runs $Rand(M)$ and obtains the response (h_i, g_i), updates $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{(h_{i-1}, g_{i-1}) \xrightarrow{m_i} (h_i, g_i)\}$, then returns $y_i = h_i \oplus m_{i,2} \oplus g_{i-1}$;
        ii. if $k_i = m_{i,1}||m_{i,2}$ and $x_i = h_{i-1}$, $\mathcal{S}$ runs $Rand(M)$ and obtains the response (h_i, g_i), and updates $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{(h_{i-1}, g_{i-1}) \xrightarrow{m_i} (h_i, g_i)\}$, then returns $y_i = g_i \oplus x_i$.
     (b) Else $\mathcal{S}$ randomly selects $(h_i, g_i, h_{i-1}, g_{i-1})$, computes $m_{i,2} = k_{i,2} \oplus g_{i-1}$ and $m_{i,1} = k_{i,1} \oplus m_{i,2} \oplus h_{i-1}$, and updates $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{(h_{i-1}, g_{i-1}) \xrightarrow{m_i} (h_i, g_i)\}$, then returns $y_i = h_i \oplus m_{i,2} \oplus g_{i-1}$.
  2. For the i-th query $(-1, k_i, y_i)$ on S^{-1}:
     (a) If $\exists IV \xrightarrow{M} (h_{i-1}, g_{i-1}) \in \mathcal{R}'_{i-1}$, the simulator $\mathcal{S}$ computes $Pad(M) = m_i = m_{i,1}||m_{i,2}$. And then,
        i. if $k_i = m_{i,1} \oplus m_{i,2} \oplus h_{i-1}||m_{i,2} \oplus g_{i-1}$, $\mathcal{S}$ runs $Rand(M)$ and obtains the response (h_i, g_i). And then, if $y_i = h_i \oplus m_{i,2} \oplus g_{i-1}$, $\mathcal{S}$ updates $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{(h_{i-1}, g_{i-1}) \xrightarrow{m_i} (h_i, g_i)\}$ and returns $x_i = m_{i,1} \oplus h_{i-1}$;
        ii. if $k_i = m_{i,1}||m_{i,2}$, $\mathcal{S}$ runs $Rand(M)$ and obtains the response (h_i, g_i). And then, if $y_i = g_i \oplus h_{i-1}$, $\mathcal{S}$ updates $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{(h_{i-1}, g_{i-1}) \xrightarrow{m_i} (h_i, g_i)\}$ and returns $x_i = h_{i-1}$.
     (b) Else $\mathcal{S}$ randomly selects (g_i, h_{i-1}, g_{i-1}), computes $h_i = y_i \oplus k_{i,2}$, $m_{i,1} = k_{i,1} \oplus m_{i,2} \oplus h_{i-1}$ and $m_{i,2} = k_{i,2} \oplus g_{i-1}$, updates $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{(h_{i-1}, g_{i-1}) \xrightarrow{m_i} (h_i, g_i)\}$, then returns $x_i = h_i \oplus m_{i,2} \oplus g_{i-1}$.

Before stating the indifferentiability result of Example 1, a simple lemma is proved from the above simulation.

Lemma 8 *In double block length hash functions defined by (13), it holds that* $\Pr(Pre) = 2^{-2n-2} \cdot l \cdot O(q)$ *and* $\Pr(Coll) = 2^{-n-1} \cdot l \cdot O(q)$*, where l is the maximum number of length in a hash query.*

Proof. In case of $\mathcal{O}_2 = (Rand, S, S^{-1})$, the total number of choices is $l \cdot q$, where l is the maximum number of length in a hash query. For every $2 \leq j \leq l \cdot q$, let $Coll_j$ be the collision

event that a pair of inputs yield a same output after the j-th queries. Namely, for some $j' < j$, it follows that

$$(h_j, g_j) = (h_{j'}, g_{j'}) \; or \; h_j = g_j,$$

which is equivalent to

$$(y_j \oplus k_{j,2}, y_j \oplus x_{j-1}) = (y_{j'} \oplus k_{j',2}, y_{j'} \oplus x_{j'-1}) \; or \; (y_j \oplus k_{j,2} = y_j \oplus x_{j-1}).$$

Since $g_i, i \in \{1, 2, \cdots, l \cdot q\}$ is randomly and uniformly selected by the simulator $\mathcal{S}$ in the range $\{0,1\}^n$, the probability that the above event happens after the j-th queries is as follows.

$$\Pr(Coll_j) \leq \frac{(j-1)}{(2^n - (j-1)) \cdot (2^n - (j-1))} + \frac{l}{2^n}.$$

Let $Coll$ be the collision event that a pair of inputs yield a same output after the maximum q times queries. Thus, if $l \cdot q \leq 2^{n-1}$,

$$\begin{aligned}
\Pr(Coll) &= \Pr(Coll_1 \vee Coll_2 \vee \cdots \vee Coll_{l \cdot q}) \leq \sum_{j=2}^{l \cdot q} \Pr(Coll_j) \\
&\leq \sum_{j=2}^{l \cdot q} (\frac{j-1}{(2^n - (j-1)) \cdot (2^n - (j-1))} + \frac{1}{2^n}) \\
&\leq \frac{\sum_{j=2}^{l \cdot q} (j-1)}{(2^n - 2^{n-1}) \cdot (2^n - 2^{n-1})} + \frac{l \cdot q}{2^n} \\
&\leq \frac{(l \cdot q)^2}{2^{2n-1}} + \frac{l \cdot q}{2^n} \leq \frac{2^{n-1}(l \cdot q) + 2^{n-1}(l \cdot q)}{2^{2n-1}} \leq \frac{l \cdot q}{2^{n-1}}
\end{aligned} \tag{18}$$

From the preimage attack on HDBL-1, it is easy to see the probability of the preimage events Pre is

$$\begin{aligned}
\Pr(Pre) &= \Pr(Pre_1 \vee Pre_2 \vee \cdots \vee Pre_{l \cdot q}) \leq \sum_{j=2}^{l \cdot q} \Pr(Pre_j) \\
&\leq \sum_{j=2}^{l \cdot q} (\frac{1}{4 \times 2^{3n/2}}) \leq \frac{l \cdot q}{4 \times 2^{3n/2}}
\end{aligned} \tag{19}$$

Consequently, the probability of the indifferentiable events Bad is

$$\Pr[Bad] = 2 \times Max(\Pr(Coll), \Pr(Pre)) = 2 \times \Pr(Coll) = 2^{-n} \cdot l \cdot O(q).$$

By implementing the advantage of indifferentiability in keyed hash function[8], similar results can be easily deduced in keyed mode. □

From the above analysis, one can easily obtain the following theorem on Example 1. We stress that the above analysis includes a formal proof in the ideal cipher model. By using the above method, one can extend the same result on Example 2.

Theorem 8 *The rate-1 hash function defined by (13)is (t_D, t_S, q, ϵ)-indifferentiable from a random oracle in the ideal cipher model with the prefix-free padding, the NMAC/HMAC, and the chop construction, for any distinguisher $\mathcal{D}$ in polynomial time bound t_D, with $t_S = 2l \cdot O(q)$ and the advantage $\epsilon = 2^{-n} \cdot l \cdot O(q)$, where l is the maximum length of a query made by $\mathcal{D}$.*

A.1 Proof of Example 3

Here we give an indifferentiability analysis on Example 4, which is a typical rate-1 hash functions in FDBL-III.

- *Rand*-**Query.** For the i-th *Rand*-query $M_i \in \mathcal{M}$, if M_i is a repetition query, the simulator $\mathcal{S}$ retrieves $r'_j \leftarrow (IV \xrightarrow{M_i} (h_j, g_j))$ where $r_j \in \mathcal{R}'_{i-1}, j \leq i-1$, then returns $Rand(M_i) = (h_j, g_j)$. Else $\mathcal{S}$ randomly selects a hash value $(h_i, g_i) \in \mathcal{Y}$ and updates $\mathcal{R}'_i = \mathcal{R}'_{i-1} \cup \{IV \xrightarrow{M_i} (h_i, g_i)\}$, then returns $Rand(M_i) = (h_i, g_i)$.
- $\{S, S^{-1}\}$-**Query.** To answer the distinguisher $\mathcal{D}$'s encryption and decryption queries, the simulator $\mathcal{S}$ proceeds as follows.
  1. For the i-th query $(1, k_i, x_i)$ on S:
     (a) If $\exists IV \xrightarrow{M} (h_{i-1}, g_{i-1}) \in \mathcal{R}'_{i-1}$, $\mathcal{S}$ computes $Pad(M) = m_i = m_{i,1}||m_{i,2}$, then,
        i. if $k_i = m_{i,1} \oplus h_{i-1}$ and $x_i = m_{i,2} \oplus h_{i-1} \oplus g_{i-1}$, $\mathcal{S}$ runs $Rand(M)$ and obtains the response (h_i, g_i), updates $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{(h_{i-1}, g_{i-1}) \xrightarrow{m_i} (h_i, g_i)\}$, then returns $y_i = h_i \oplus m_{i,1} \oplus x_i$;
        ii. if $x_i = m_{i,1} \oplus g_{i-1}||m_{i,2} \oplus h_{i-1}$ and $x_i = h_{i-1}$, $\mathcal{S}$ runs $Rand(M)$ and obtains the response (h_i, g_i), and updates $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{(h_{i-1}, g_{i-1}) \xrightarrow{m_i} (h_i, g_i)\}$, then returns $y_i = g_i \oplus g_{i-1}$.
     (b) Else,
        i. if $|k_i| = n$, then $\mathcal{S}$ randomly selects $(h_i, g_i, h_{i-1}, g_{i-1})$, computes $m_{i,1} = k_i \oplus h_{i-1}$ and $m_{i,2} = x_i \oplus h_{i-1} \oplus g_{i-1}$, and updates $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{(h_{i-1}, g_{i-1}) \xrightarrow{m_i} (h_i, g_i)\}$, then returns $y_i = h_i \oplus m_{i,1} \oplus x_i$;
        ii. if $|k_i| = 2n$, then $\mathcal{S}$ randomly selects (h_i, g_i, g_{i-1}), computes $m_{i,1} = k_{i,1} \oplus g_{i-1}$ and $m_{i,2} = k_{i,2} \oplus x_i$, and updates $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{(x_i, g_{i-1}) \xrightarrow{m_i} (h_i, g_i)\}$, then returns $y_i = g_i \oplus g_{i-1}$.
  2. For the i-th query $(-1, k_i, y_i)$ on S^{-1}:
     (a) If $\exists IV \xrightarrow{M} (h_{i-1}, g_{i-1}) \in \mathcal{R}'_{i-1}$, the simulator $\mathcal{S}$ computes $Pad(M) = m_i = m_{i,1}||m_{i,2}$, then,
        i. if $k_i = m_{i,1} \oplus g_{i-1}$, $\mathcal{S}$ runs $Rand(M)$ and obtains the response (h_i, g_i). And then, if $y_i = h_i \oplus k_i \oplus m_{i,2} \oplus g_{i-1}$, $\mathcal{S}$ updates $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{(h_{i-1}, g_{i-1}) \xrightarrow{m_i} (h_i, g_i)\}$ and returns $x_i = m_{i,2} \oplus h_{i-1} \oplus g_{i-1}$;
        ii. if $k_i = m_{i,1} \oplus g_{i-1}||m_{i,2} \oplus h_{i-1}$, $\mathcal{S}$ runs $Rand(M)$ and obtains the response (h_i, g_i). And then, if $y_i = g_i \oplus g_{i-1}$, $\mathcal{S}$ updates $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{(h_{i-1}, g_{i-1}) \xrightarrow{m_i} (h_i, g_i)\}$ and returns $x_i = h_{i-1}$.

(b) Else,

i. if $|k_i| = n$, then $\mathcal{S}$ randomly selects $(h_i, g_i, h_{i-1}, g_{i-1})$, computes $m_{i,1} = k_i \oplus h_{i-1}$ and $m_{i,2} = y_i \oplus h_i \oplus k_i \oplus g_{i-1}$, and updates $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{(h_{i-1}, g_{i-1}) \xrightarrow{m_i} (h_i, g_i)\}$, then returns $x_i = y_i \oplus h_i \oplus m_{i,1}$;

ii. if $|k_i| = 2n$, then $\mathcal{S}$ randomly selects (h_i, g_i, h_{i-1}), computes $m_{i,1} = k_{i,1} \oplus y_i \oplus g_i$ and $m_{i,2} = k_{i,2} \oplus h_{i-1}$, and updates $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{(x_i, g_{i-1}) \xrightarrow{m_i} (h_i, g_i)\}$, then returns $x_i = h_{i-1}$.

Before stating the indifferentiability result of Example 4, a simple lemma is proved from the above simulation.

Lemma 9 *In double block length hash functions defined by (18), it holds that* $\Pr(Pre) = \Pr(Sec) = 2^{-2n-2} \cdot l \cdot O(q)$ *and* $\Pr(Coll) = 2^{-n-1} \cdot l \cdot O(q)$*, where l is the maximum number of length in a hash query.*

Proof. In case of $\mathcal{O}_2 = (Rand, S, S^{-1})$, the total number of choices is $l \cdot q$, where l is the maximum number of length in a hash query. For every $2 \le j \le l \cdot q$, let $Coll_j$ be the collision event that a pair of inputs yield a same output after the j-th queries. Namely, for some $j' < j$, it follows that

$$(h_j, g_j) = (h_{j'}, g_{j'}) \text{ or } h_j = g_j,$$

which is equivalent to

$$(y_j \oplus k_j \oplus h_{j-1}, g_j \oplus g_{j-1}) = (y_{j'} \oplus k_{j'} \oplus h_{j'-1}, g_{j'} \oplus g_{j'-1}) \text{ or } (y_j \oplus y_{j'} \oplus k_{j'} \oplus h_{j'-1} = g_j \oplus g_{j-1}).$$

Since $h_i, g_i, i \in \{1, 2, \cdots, l \cdot q\}$ is randomly and uniformly selected by the simulator $\mathcal{S}$ in the range $\{0,1\}^n$, the probability that the above event happens after the j-th queries is as follows.

$$\Pr(Coll_j) \le \frac{(j-1)}{(2^n - (j-1)) \cdot (2^n - (j-1))} + \frac{l}{2^n}.$$

Let $Coll$ be the collision event that a pair of inputs yield a same output after the maximum q times queries. By implementing the same idea on the proof of Example 1, if $l \cdot q \le 2^{n-1}$, it is easy to find that $\Pr(Coll) \le \frac{l \cdot q}{2^{n-1}}$. Similarly, the probability of the preimage event Pre is $\Pr(Pre) \le \frac{l \cdot q}{4 \times 2^{3n/2}}$.

Consequently, the probability of the indifferentiable events Bad is

$$\Pr[Bad] = 2 \times Max(\Pr(Coll), \Pr(Pre)) = 2 \times \Pr(Coll) = 2^{-n} \cdot l \cdot O(q).$$

By implementing the advantage of indifferentiability in keyed hash function[8], similar results can be easily deduced in keyed mode. □

From the above analysis, one can easily obtain the following theorem on Example 3.

Theorem 9 *The rate-1 hash function defined by (18) is (t_D, t_S, q, ϵ)-indifferentiable from a random oracle in the ideal cipher model with the prefix-free padding, the NMAC/HMAC, and the chop construction, for any distinguisher $\mathcal{D}$ in polynomial time bound t_D, with $t_S = 2l \cdot O(q)$ and the advantage $\epsilon = 2^{-n} \cdot l \cdot O(q)$, where l is the maximum length of a query made by $\mathcal{D}$.*