

# Collisions and other Non-Random Properties for Step-Reduced SHA-256\*

Sebastiaan Indestege<sup>1,2,\*\*</sup>, Florian Mendel<sup>3</sup>, Bart Preneel<sup>1,2</sup>, and Christian Rechberger<sup>3</sup>

<sup>1</sup> Department of Electrical Engineering ESAT/SCD-COSIC, Katholieke Universiteit Leuven. Kasteelpark Arenberg 10, B-3001 Heverlee, Belgium.

`sebastiaan.indestege@esat.kuleuven.be`

<sup>2</sup> Interdisciplinary Institute for BroadBand Technology (IBBT), Belgium.

<sup>3</sup> Institute for Applied Information Processing and Communications Inffeldgasse 16a, A-8010 Graz, Austria.

**Abstract.** We study the security of step-reduced but otherwise unmodified SHA-256. We show the first collision attacks on SHA-256 reduced to 23 and 24 steps with complexities  $2^{18}$  and  $2^{28.5}$ , respectively. We give example colliding message pairs for 23-step and 24-step SHA-256. The best previous, recently obtained result was a collision attack for up to 22 steps. Additionally, we show non-random behaviour of SHA-256 in the form of free-start near collisions for up to 31 steps, which is 6 more steps than the recently obtained non-random behaviour in the form of a semi-free start near-collision. Even though this represents a step forwards in terms of cryptanalytic techniques, the results do not threaten the security of applications using SHA-256.

**Key words:** SHA-256, hash functions, collisions, semi-free start collisions, free start collisions, pseudo-near-collisions.

## 1 Introduction

In the light of previous break-through results on hash functions like MD5 and SHA-1, the security of their successor, SHA-256 and sisters, against all kinds of cryptanalytic attacks deserves special attention. This is even more important as many products and services that used to rely on SHA-1 are now migrating to SHA-256.

---

\* This work was supported in part by the IAP Programme P6/26 BCRYPT of the Belgian State (Belgian Science Policy), in part by the European Commission through the IST Programme under Contract IST-2002-507932 ECRYPT, and in part by the Austrian Science Fund (FWF), project P19863. This work was done during a visit of the first author to the Graz University of Technology.

\*\* F.W.O. Research Assistant, Fund for Scientific Research — Flanders (Belgium).

## 1.1 Previous work on members of the SHA-2 family

Below, we briefly discuss existing work. Results on older variants of the larger MD4 related hash function family including SHA-1 suggest that the concept of local collisions might also be important for the SHA-2 members. The first published analysis on members of the SHA-2 family, by Gilbert and Handschuh [2], goes in this direction. They show that there exists a 9-step local collision with probability  $2^{-66}$ . Later on, the result was improved by Hawkes *et al.* [3]. By considering modular differences, they increased the probability to  $2^{-39}$ . Using XOR differences, local collisions with probability as high as  $2^{-38}$  were used by Hölbl *et al.* [4]. Local collisions with lower probability but with other properties were studied by Sanadhya and Sarkar in [12].

Now we turn our attention to the analysis of simplified variants of SHA-256. In [15], all modular additions are replaced by XOR. For this variant, a search for pseudo-collisions is described, which is faster than brute force search for up to 34 steps. In [7], a variant of SHA-256 is analysed where all  $\Sigma$ - and  $\sigma$ -functions are removed. The conclusion is that for this variant, collisions can be found much faster than by brute force search. The work shows that the approach used by Chabaud and Joux [1] in their analysis of SHA-0 is extensible to that particular variant of SHA-256. The message expansion as a building block on its own was studied in [7,11].

Finally, we discuss previous work that focuses on step-reduced but otherwise unmodified SHA-256. The first study was done by Mendel *et al.* [8]. The results obtained are a practical 18-step collision and a differential characteristic for 19-step SHA-224 collision. Also an example of a pseudo-near-collision for 22-step SHA-256 is given. Similar techniques have been studied by Matusiewicz [7] and recently also by Sanadhya and Sarkar [14]. By using a different technique, Nikolić and Biryukov [9] obtained collisions for up to 21 steps and non-random behaviour in the form of semi free-start near-collisions for up to 25 steps. Very recently, Sanadhya and Sarkar [13] showed a collision example for 22 steps of SHA-256, likely based on a similar technique.

## 1.2 Our Contribution

We extend the work of Nikolić and Biryukov [9] to collisions for 23- and 24-step SHA-256 with respective time complexities of  $2^{18}$  and  $2^{28.5}$  reduced SHA-256 compression function evaluations. We also give several

**Table 1.** Comparison of our results with the known results in the literature for each type. Effort is expressed in (equivalent) calls to the respective reduced compression functions.

function	steps	type	effort	source	example
SHA-256	18	collision	$2^0$	[8]	yes
SHA-256	20	collision	$2^{1.58}$	[9]	no
SHA-256	21	collision	$2^{15}$	[9]	yes
SHA-256	22	collision	$2^9$	[13]	yes
SHA-256	<b>23</b>	collision	$2^{18}$	this work	yes
SHA-256	<b>24</b>	collision	$2^{28.5}$	this work	yes
SHA-256	23	semi-free start collision	$2^{17}$	[9]	yes
SHA-256	<b>24</b>	semi-free start collision	$2^{17}$	this work	no
SHA-224	<b>25</b>	free start collision	$2^{17}$	this work	no
SHA-256	22	free start near-collision	$2^0$	[8]	yes
SHA-256	25	semi-free start near-collision	$2^{34}$	[9]	yes
SHA-256	<b>31</b>	free start near-collision	$2^{32}$ , Table 6	this work	no

weaker collision style attacks on a larger number of rounds. Our results are summarised in Table 1.

The structure of this paper is as follows. We give a short description of SHA-256 in Sect. 2. Section 3 gives an alternative description of the semi-free start collision attack by Nikolić and Biryukov [9], which will make the subsequent description of the new attacks easier to understand. We then discuss our collision attacks on 23- and 24-step SHA-256 in Sect. 4. Further extensions can be found in the appendix. Finally, Sect. 5 concludes.

## 2 Description of SHA-256

This section gives a short description of the SHA-256 hash function, using the notations from Table 2. For a detailed specification, we refer to [10].

The compression function of SHA-256 consists of a message expansion, which transforms a 512-bit message block into 64 expanded message words  $W_i$  of 32 bits each, and a state update transformation. The latter updates eight 32-bit state variables  $A, \dots, H$  in 64 identical steps, each using one expanded message word. The message expansion can be defined recursively as follows.

$$W_i = \begin{cases} M_i & 0 \leq i < 16 \\ \sigma_1(W_{i-2}) + W_{i-7} + \sigma_0(W_{i-15}) + W_{i-16} & 16 \leq i < 64 \end{cases} \quad (1)$$

**Table 2.** The notations used in this paper.

$X \ggg s$	$X$ rotated over $s$ bits to the right
$X \gg s$	$X$ shifted over $s$ bits to the right
$\overline{X}$	One's complement of $X$
$X \oplus Y$	Bitwise exclusive OR of $X$ and $Y$
$X + Y$	Addition of $X$ and $Y$ modulo $2^{32}$
$X - Y$	Subtraction of $X$ and $Y$ modulo $2^{32}$
$A_i, \dots, H_i$	State variables at step $i$ , for the first message
$A'_i, \dots, H'_i$	Idem, for the second message
$W_i$	$i$ -th expanded message word of the first message
$W'_i$	$i$ -th expanded message word of the second message
$\delta X$	Additive difference in $X$ , <i>i.e.</i> , $X' - X$
$\delta\sigma_0(X)$	Additive difference in $\sigma_0(X)$ , <i>i.e.</i> , $\sigma_0(X') - \sigma_0(X)$

The functions  $\sigma_0(X)$  and  $\sigma_1(X)$  are given by

$$\begin{aligned}\sigma_0(X) &= (X \ggg 7) \oplus (X \ggg 18) \oplus (X \gg 3) , \\ \sigma_1(X) &= (X \ggg 17) \oplus (X \ggg 19) \oplus (X \gg 10) .\end{aligned}\quad (2)$$

The state update transformation updates two of the state variables in every step. It uses the bitwise Boolean functions  $f_{\text{ch}}$  and  $f_{\text{maj}}$  as well as the GF(2)-linear functions  $\Sigma_0$  and  $\Sigma_1$ .

$$\begin{aligned}f_{\text{ch}}(X, Y, Z) &= XY \oplus \overline{X}Z , \\ f_{\text{maj}}(X, Y, Z) &= XY \oplus YZ \oplus XZ , \\ \Sigma_0(X) &= (X \ggg 2) \oplus (X \ggg 13) \oplus (X \ggg 22) , \\ \Sigma_1(X) &= (X \ggg 6) \oplus (X \ggg 11) \oplus (X \ggg 25) .\end{aligned}\quad (3)$$

The following equations describe the state update transformation, where  $K_i$  is a step constant.

$$\begin{aligned}T_1 &= H_i + \Sigma_1(E_i) + f_{\text{ch}}(E_i, F_i, G_i) + K_i + W_i , \\ T_2 &= \Sigma_0(A_i) + f_{\text{maj}}(A_i, B_i, C_i) , \\ A_{i+1} &= T_1 + T_2 , \quad B_{i+1} = A_i , \quad C_{i+1} = B_i , \quad D_{i+1} = C_i , \\ E_{i+1} &= D_i + T_1 , \quad F_{i+1} = E_i , \quad G_{i+1} = F_i , \quad H_{i+1} = G_i .\end{aligned}\quad (4)$$

After 64 steps, the initial state variables are fed forward using word-wise addition modulo  $2^{32}$ .

### 3 Review of the Nikolić-Biryukov Semi-Free Start Collision Attack

In this section, we review the 23 step semi-free start collision attack by Nikolić and Biryukov [9], because new results we present in this paper are extensions of this attack. The notations we use are given in Table 2.

**Table 3.** A 9 step differential, using additive differences (left) and conditions on the value (right). Zero differences resp. unconstrained values are denoted by blanks.

step	$\delta A$	$\delta B$	$\delta C$	$\delta D$	$\delta E$	$\delta F$	$\delta G$	$\delta H$	$\delta W$	$A$	$B$	$C$	$D$	$E$	$F$	$G$	$H$
8										$\alpha$				$\gamma$			
9									1	$\alpha$	$\alpha$			$\gamma+1$	$\gamma$		
10	1				1				-1	-1	$\alpha$	$\alpha$		-1	$\gamma+1$	$\gamma$	
11		1			-1	1			$\delta_1$	$\alpha$	-1	$\alpha$	$\alpha$	$\epsilon$	-1	$\gamma+1$	$\gamma$
12			1			-1	1		$\delta_2$	$\alpha$	$\alpha$	-1	$\alpha$	$\beta$	$\epsilon$	-1	$\gamma+1$
13				1			-1	1		$\alpha$	$\alpha$	$\alpha$	-1	$\beta$	$\beta$	$\epsilon$	-1
14					1			-1			$\alpha$	$\alpha$	$\alpha$	-1	$\beta$	$\beta$	$\epsilon$
15						1						$\alpha$	$\alpha$	0	-1	$\beta$	$\beta$
16							1		1				$\alpha$	-2	0	-1	$\beta$
17								1	-1						-2	0	-1
18																-2	0

### 3.1 A Nine Step Differential

The attack uses a nine step differential which is presented in Table 3. All additive differences are fixed, as well as some of the actual values. Fixing these values ensures that the differential is followed, as will be explained later. The constants  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\epsilon$  are determined by the attack algorithm. The first difference is inserted via the message word  $W_9$ . In order to obtain a 23 step semi-free start collision, we require that there are no differences in expanded message words other than those indicated in Table 3. In other words, only  $W_9$ ,  $W_{10}$ ,  $W_{11}$ ,  $W_{12}$ ,  $W_{16}$  and  $W_{17}$  can have a difference.

### 3.2 The Attack

The attack algorithm consists of two phases. The first phase finds suitable values for the constants  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\epsilon$  as well as two expanded message words,  $W_{16}$  and  $W_{17}$ . The entire complexity of the attack can be attributed to this phase. A detailed description of this phase of the attack will be given in Sect. 3.4, as it is more instructive to describe the second phase first.

### 3.3 The Second Phase of the Attack

The second phase of the attack finds, when given suitable values for  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\epsilon$ ,  $W_{16}$  and  $W_{17}$ , a pair of messages and a set of initial values that

lead to a semi-free start collision for 23 steps of SHA-256. It works by carefully fixing the internal state at step 11 and then computing forward and backward. At each step, the expanded message word  $W_i$  is computed such that the differential from Table 3 is followed. In three steps, extra conditions appear, involving only the constants determined by the first phase of the attack. The first phase guarantees that the constants are such that these conditions are satisfied.

The second phase of the attack has a negligible complexity and is guaranteed to succeed. Since there is still a lot of freedom left, many 23 step semi-free start collisions can be found with only a negligible extra effort, by repeating this second phase several times.

1. Start at step 11 by fixing the state variables in this step,  $A_{11}, \dots, H_{11}$  as indicated in Table 3. The constants  $\alpha, \beta, \gamma$  and  $\epsilon$  are given by the first phase of the attack.
2. Calculate  $W_{11}$  such that  $A_{12} = \alpha$  and  $W'_{11}$  such that  $A'_{12} = \alpha$ . This can be done by simple rearranging (4). Compute the value of  $E_{12}$ , which we will refer to as  $\beta$ . Note that the value of  $\beta$  only depends on  $\alpha$ , since

$$A_{12} = \alpha = \Sigma_0(\alpha) + f_{\text{maj}}(\alpha, -1, \alpha) + \beta - \alpha \Rightarrow \beta = \alpha - \Sigma_0(\alpha) \quad . \quad (5)$$

3. In a similar way, calculate  $W_{12}$  such that  $E_{13} = \beta$  and  $W'_{12}$  such that  $E'_{13} = \beta$ . This also guarantees that  $A_{13} = A'_{13}$  because the majority function absorbs the difference in  $C_{12}$ .
4. Calculate  $W_{13}$  such that  $E_{14} = -1$  and set  $W'_{13} = W_{13}$ . Now, see Table 3, the additive difference  $\delta E_{14}$  should be equal to 1. Computing this difference yields the condition

$$\delta E_{14} = f_{\text{ch}}(\beta, \beta, \epsilon - 1) - f_{\text{ch}}(\beta, \beta, \epsilon) + 2 = 1 \quad . \quad (6)$$

This condition only involves the constants  $\beta$  and  $\epsilon$ , and can thus already be ensured by a proper choice of these constants in phase one of the attack. Note that this condition also ensures that  $\delta A_{14} = 0$ .

5. Calculate  $W_{14}$  such that  $E_{15} = 0$  and set  $W'_{14} = W_{14}$ . Since the values of  $E_{14}$  and  $E'_{14}$  were chosen to be fixed points of the function  $\Sigma_1$  in the previous step,  $\delta \Sigma_1(E_{14}) = \delta E_{14} = 1$  cancels with  $\delta H_{14} = -1$ . Also,  $f_{\text{ch}}$  absorbs the difference in  $E_{14}$ , no new differences are introduced into the state.
6. Calculate  $W_{15}$  such that  $E_{16} = -2$  and set  $W'_{15} = W_{15}$ . The difference in  $F_{15}$  is absorbed by  $f_{\text{ch}}$ .

7. The value for  $W_{16}$  is computed in phase one of the attack. The difference  $\delta W_{16} = 1$  is cancelled by the output of  $f_{\text{ch}}$ . Indeed, since the binary representation of  $E_{16} = -2$  is  $111 \dots 10_b$ , the  $f_{\text{ch}}$  function passes only the difference in the least significant bit.
8. Also the value for  $W_{17}$  is computed in phase one of the attack. The difference  $\delta W_{17} = 1$  cancels with  $\delta H_{17} = 1$ , thereby eliminating the final difference in the state variables. Thus, a collision is reached after step 17.
9. Now, go back to step 11 and proceed in the backward direction. Make an arbitrary choice for  $W_{10}$ . The differential from Table 3 is followed because of the careful choice of the state variables in step 11.
10. Make an arbitrary choice for  $W_9$ , and proceed one step backward. The difference  $\delta W_9 = 1$  cancels with  $\delta A_{10}$  and with  $\delta E_{10}$  such that there is a zero difference in the state variables  $A_9$  through  $H_9$ . Now randomly choose  $W_8$  down to  $W_2$  and calculate backward. Because no new differences appear in these expanded message words, there is also a zero difference in the state variables  $A_2$  through  $H_2$ .
11. It is not possible to freely choose  $W_0$  or  $W_1$  as 16 expanded message words have already been chosen, *i.e.*,  $W_2$  until  $W_{17}$ . Hence, these are computed using the message expansion in the backward direction. Although some of the message words used to compute  $W_0$  and  $W_1$  have differences, these differences always cancel out.
12. Continuing forward from step 18 again, note that the collision is preserved as long as no new differences are introduced via the expanded message words. From the message expansion, it follows that

$$\begin{aligned}
\delta W_{18} &= \delta \sigma_1(W_{16}) + \delta W_{11} \\
&= \sigma_1(W_{16} + 1) - \sigma_1(W_{16}) - \Sigma_1(\epsilon - 1) + \Sigma_1(\epsilon) \\
&\quad - f_{\text{ch}}(\epsilon - 1, 0, \gamma + 1) + f_{\text{ch}}(\epsilon, -1, \gamma + 1) = 0 . \quad (7)
\end{aligned}$$

Phase one of the attack will have to ensure that  $W_{16}$ ,  $\gamma$  and  $\epsilon$  are chosen such that this condition is satisfied.

13. In step 19, it follows from the message expansion that the following condition needs to be satisfied:

$$\begin{aligned}
\delta W_{19} &= \delta \sigma_1(W_{17}) + \delta W_{12} \\
&= \sigma_1(W_{17} - 1) - \sigma_1(W_{17}) \\
&\quad - f_{\text{ch}}(\beta, \epsilon - 1, 0) + f_{\text{ch}}(\beta, \epsilon, -1) = 0 . \quad (8)
\end{aligned}$$

Again, this condition only depends on  $W_{17}$  and the constants  $\beta$  and  $\epsilon$ , so phase one of the attack can ensure that this condition is satisfied.

14. In steps 20–22, the message expansion guarantees that no new differences are introduced. In step 23, however, a difference of 1 is impossible to avoid, hence the attack stops after 23 steps.

### 3.4 The First Phase of the Attack

The goal of the first phase of the attack is to determine suitable values for the constants  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\epsilon$ , as well as two expanded message words,  $W_{16}$  and  $W_{17}$ . Suitable values imply that the three conditions that are required to be satisfied in the second phase of the attack are indeed satisfied. This is achieved as follows.

1. Make a random choice for  $\gamma$  and  $\epsilon$ , and search for a value of  $W_{16}$  such that condition (7) is satisfied. There exists a simple, generic method to solve equations of this form, which is described in Appendix B. We note however that for this particular case, a faster method exists. An exhaustive search over every possible value of  $W_{16}$  resulted in the observation that only 6 181 additive differences can ever be achieved. These can be stored in a lookup table, together with one or more solutions for each difference. Hence, solving an equation of the form  $\sigma_1(x+1) - \sigma_1(x) = \delta$  can be done with a simple table lookup. If no solution exists, simply retry with different choices for  $\gamma$  and/or  $\epsilon$ . If the right hand side difference is selected uniformly at random, the probability that the equation has a solution is about  $2^{-19.5}$ , so we expect to have to repeat this step about  $2^{19.5}$  times.
2. Satisfying the other two conditions can be done independently of the first. Make a random choice for  $\alpha$ , and compute  $\beta$  using (5), *i.e.*,  $\beta = \alpha - \Sigma_0(\alpha)$ . Now check condition (6), which states that

$$f_{\text{ch}}(\beta, \beta, \epsilon - 1) - f_{\text{ch}}(\beta, \beta, \epsilon) = -1 \quad . \quad (9)$$

As described in [9], this equation is satisfied if the bits of  $\beta$  are zero in the positions where the bits of  $\epsilon - 1$  and  $\epsilon$  differ. This occurs with a probability of approximately 1/3, so this condition is fairly easy to satisfy.

3. The last condition, (8), is of the same form as the first condition, provided it is rewritten as

$$\sigma_1(W'_{17} + 1) - \sigma_1(W'_{17}) = f_{\text{ch}}(\beta, \epsilon, -1) - f_{\text{ch}}(\beta, \epsilon - 1, 0) \quad . \quad (10)$$

Also, it becomes independent of  $\epsilon$  if we assume that the previous condition is satisfied, because that implies that the bits of  $\beta$  are zero where  $\epsilon$  and  $\epsilon - 1$  differ. Hence, the right hand side is equal to  $\bar{\beta}$ .



Note that, because not all conditions depend on all of the constants that are determined in this phase of the attack, the first condition can be treated independently of the last two. Thus, the first and last step of this phase of the attack are executed about  $2^{19.5}$  times and the second step about  $2^{21}$  times. Note also that one of these steps requires much less work than an evaluation of the compression function of (reduced) SHA-256 — a bit less than one step. Hence, the overall time complexity of the attack, when expressed in SHA-256 compression function evaluations, is below  $2^{17}$ . If we consider the first phase to be a precomputation, the time complexity is only one reduced SHA-256 evaluation.

## 4 Our Collision Attacks on SHA-256

In this section we describe a novel, practical collision attack on SHA-256, reduced to 23 steps. It has a time complexity of about  $2^{18}$  evaluations of the reduced SHA-256 compression function. We also extend this to 24 steps of SHA-256, with an expected time complexity of  $2^{28.5}$  compression function evaluations.

### 4.1 23 Step Collision

Our collision attack for SHA-256 reduced to 23 steps consists of two parts. First, we construct a semi-free start collision for 23 steps, based on the attack from Sect. 3. Then we transform this semi-free start collision into a real collision.

*Finding “Good” Constants.* Finding a 23 step semi-free start collision is done using the same attack as described in Sect. 3, with a slight change to the first phase. In Sect. 3.4 it was described how to find constants  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\epsilon$  such that there exist values for  $W_{16}$  and  $W_{17}$  ensuring that the conditions (7) and (8) are satisfied. There are still some degrees of freedom left in this process. Indeed, it is possible to determine the constants  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\epsilon$  such that there are *many* values for  $W_{16}$  and  $W_{17}$  satisfying (7) and (8).

We performed a partially exhaustive search for such good constants. Condition (7) depends only on  $\epsilon$  and  $\gamma$ . An exhaustive search can be performed with approximately  $2^{37}$  evaluations of (7), because for each value of  $\epsilon$ , only some of the bits in  $\gamma$  can have an influence. We found several values for  $\epsilon$  and  $\gamma$  for which more than  $2^{29}$  choices for  $W_{16}$  ensure that (7) is satisfied, for example

$$\gamma = 0000017c_x \quad \text{and} \quad \epsilon = 7f5f7200_x . \quad (11)$$

Conditions (6) and (8) depend on  $\beta$  (which in turn depends on  $\alpha$ ) and  $\epsilon$ . Recall from Sect. 3.4 that (8) becomes independent of  $\epsilon$  if we assume (6) is satisfied. Hence again, an exhaustive search is feasible. With  $\epsilon$  as in (11), the following values for  $\alpha$  and  $\beta$  are one of many optimal choices:

$$\alpha = 00b321e3_x \quad \text{and} \quad \beta = fcffe000_x . \quad (12)$$

There are  $2^{16}$  possible  $W_{17}$ 's which satisfy (8) with these constants. Thus, these values for  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\epsilon$  gives us an additional freedom of  $2^{45}$  in the choice of  $W_{16}$  and  $W_{17}$ . This phase can be considered a precomputation, or alternatively, one can reduce the effort spent in this phase by only searching a smaller part of the available search space, which likely leads to less optimal results. It may however be a worthwhile trade-off in practice.

*Transforming into a Collision.* Note that only 7 expanded message words are actually fixed to a certain value when constructing a semi-free start collision. Indeed, only  $W_{11}$  until  $W_{17}$  are really fixed, if we ignore the freedom we still have left in  $W_{16}$  and  $W_{17}$  for now. The others are chosen arbitrarily or computed from the message expansion when necessary. Using this freedom, it is possible to construct many semi-free start collisions with only a negligible extra effort. But it is also possible to use this freedom in a controlled manner to transform the semi-free start collision into a real collision.

To this end, we first introduce an alternative description of SHA-256. In older variants of the same design strategy like MD5 or SHA-1, only a single state variable is updated in every step. This naturally leads to a description where only the first state variable is considered. Something similar can be done with the SHA-2 hash functions, even though in the standard description, two state variables are updated in every step.

From the state update equations (4), we derive a series of equations which express the inputs of the  $i$ -th state update transformation,  $A_i, \dots, H_i$ , as a function of only  $A_i$  through  $A_{i-7}$ .

$$\begin{aligned} A_i &= A_i , \\ B_i &= A_{i-1} , \\ C_i &= A_{i-2} , \\ D_i &= A_{i-3} , \\ E_i &= A_{i-4} + A_i - \Sigma_0(A_{i-1}) - f_{\text{maj}}(A_{i-1}, A_{i-2}, A_{i-3}) , \\ F_i &= A_{i-5} + A_{i-1} - \Sigma_0(A_{i-2}) - f_{\text{maj}}(A_{i-2}, A_{i-3}, A_{i-4}) , \\ G_i &= A_{i-6} + A_{i-2} - \Sigma_0(A_{i-3}) - f_{\text{maj}}(A_{i-3}, A_{i-4}, A_{i-5}) , \\ H_i &= A_{i-7} + A_{i-3} - \Sigma_0(A_{i-4}) - f_{\text{maj}}(A_{i-4}, A_{i-5}, A_{i-6}) . \end{aligned} \quad (13)$$

Substituting these into (4) yields an alternative description requiring only a single state variable. This description can be written as

$$A_{i+1} = F(A_i, A_{i-1}, A_{i-2}, A_{i-3}, A_{i-4}, A_{i-5}, A_{i-6}) + A_{i-7} + W_i . \quad (14)$$

The function  $F(\cdot)$  encapsulates (4) and (13) except for the addition of the expanded message word  $W_i$  and the state variable  $A_{i-7}$ .

From (13) it is clear that one can easily transform the internal state in the standard description,  $\langle A_i, \dots, H_i \rangle$ , to the internal state in the alternative description,  $\langle A_i, \dots, A_{i-7} \rangle$  and vice versa. Analogous to what is done for MD5 and SHA-1, the initial values can be defined as  $A_{-7}, \dots, A_0$ . Since control over one expanded message word  $W_i$  gives full control over one state variable  $A_{i+1}$ , control over eight consecutive expanded message words gives full control over the entire internal state.

This alternative description of SHA-256 can be used to transform a 23 step semi-free start collision for SHA-256 into a real collision.

1. Set  $\langle A_0, \dots, A_{-7} \rangle$  to the SHA-256 initial values, in the alternative description. Make arbitrary choices for  $W_0, W_1$  and  $W_2$ . Recompute the first three steps.
2. The eight message words  $W_3$  until  $W_{10}$  are now modified such that  $A_4$  until  $A_{11}$  remain unchanged. In the original description of SHA-256, this implies that the internal state at step 11, *i.e.*,  $\langle A_{11}, \dots, H_{11} \rangle$  does not change, and thus we connect to the rest of the semi-free start collision. More specifically, for every  $i$ ,  $3 \leq i \leq 10$ , the new value of the  $i$ -th message word is computed as

$$W_i = A_{i+1} - F(A_i, A_{i-1}, A_{i-2}, A_{i-3}, A_{i-4}, A_{i-5}, A_{i-6}) - A_{i-7} . \quad (15)$$

In the message words  $W_9$  and  $W_{10}$  there is an additive difference of 1 and  $-1$ , respectively. This does not pose a problem since the construction of the semi-free start condition guarantees that these will have the intended effect, regardless of the values of  $W_9$  and  $W_{10}$ , see Sect. 3.3.

3. Now we need to verify again if conditions (7) and (8) are still satisfied, since they depend on  $W_{16}$  and  $W_{17}$ , which may have changed. If the conditions are not satisfied, simply restart and make different choices for  $W_0, W_1$  and/or  $W_2$ .

Recall however that we have spent extra effort in the first phase of the attack to choose the constants  $\alpha, \beta, \gamma$  and  $\epsilon$  such that there are many values for  $W_{16}$  and  $W_{17}$  that satisfy the conditions. For the constants given in (11) and (12), there are  $2^{45}$  allowed values for these two

**Table 4.** Example Colliding Message Pair for 23-Step Reduced SHA-256.

$M$	29f1ebfb <sub>x</sub>	4468041a <sub>x</sub>	1e6565b6 <sub>x</sub>	4cc17e75 <sub>x</sub>
	4ea4f993 <sub>x</sub>	33a77104 <sub>x</sub>	864a828d <sub>x</sub>	1dcec3d2 <sub>x</sub>
	d33d7b02 <sub>x</sub>	bcd4a2d7 <sub>x</sub>	3b10201d <sub>x</sub>	39953548 <sub>x</sub>
	8e127f2b <sub>x</sub>	0304fc01 <sub>x</sub>	e7118577 <sub>x</sub>	43b12ca7 <sub>x</sub>
$M'$	29f1ebfb <sub>x</sub>	4468041a <sub>x</sub>	1e6565b6 <sub>x</sub>	4cc17e75 <sub>x</sub>
	4ea4f993 <sub>x</sub>	33a77104 <sub>x</sub>	864a828d <sub>x</sub>	1dcec3d2 <sub>x</sub>
	d33d7b02 <sub>x</sub>	bcd4a2d8 <sub>x</sub>	3b10201c <sub>x</sub>	3995d548 <sub>x</sub>
	91129f2a <sub>x</sub>	0304fc01 <sub>x</sub>	e7118577 <sub>x</sub>	43b12ca7 <sub>x</sub>
$H$	c77405ea <sub>x</sub>	8bfe2016 <sub>x</sub>	ff0531b6 <sub>x</sub>	a89b81f6 <sub>x</sub>
	e98cf052 <sub>x</sub>	491a6c62 <sub>x</sub>	fd009a40 <sub>x</sub>	3969dc83 <sub>x</sub>

expanded message words. This translates into a probability of  $2^{-19}$  that the conditions (7) and (8) are indeed still satisfied. We hence expect to have to repeat this procedure about  $2^{19}$  times. Every trial requires an effort equivalent to about 10 steps of SHA-256.

4. After a successful modification of the first message words, the expanded message words  $W_{18}$  until  $W_{22}$  need to be recomputed, and also the corresponding steps need to be redone. The construction of the semi-free start collision still guarantees that no differences will be introduced.

If we consider the first phase to be a precomputation, the overall attack complexity is about  $2^{18}$  evaluations of the compression function of SHA-256 reduced to 23 steps. An example collision pair for 23-step reduced SHA-256 is given in Table 4.

## 4.2 24 Step Collision

The same approach can be extended to 24 steps of SHA-256. The entire attack is simply shifted down by one step. Because of this, one more message word,  $W_0$ , needs to be computed from the message expansion in the reverse direction when constructing a 24-step semi-free start collision. From the message expansion in the reverse direction, it follows that the additive difference in this word is

$$\delta W_0 = \delta W_{16} - \delta \sigma_1(W_{14}) - \delta W_9 - \delta \sigma_0(W_1) . \quad (16)$$

None of these expanded message words has a difference, so also  $\delta W_0 = 0$ .

When turning the semi-free start collision into a collision, however, the value of the expanded message word  $W_{16}$  (which was  $W_{15}$ , a non-expanded message word, in the 23-step attack) should not change. In a

straightforward extension of the 23-step collision attack to 24 steps, this extra condition would only be satisfied with a probability of  $2^{-32}$ . Using the available freedom in a better way, this can be improved tremendously.

1. Set  $\langle A_0, \dots, A_{-7} \rangle$  to the SHA-256 initial values. Make an arbitrary choice for  $W_0$  and recompute the first step. Now, it follows from (4) that  $(A_2 - W_1)$  is a constant

$$c_1 = A_2 - W_1 . \quad (17)$$

2. The new value of  $W_9$  is determined from (15), *i.e.*, it depends on  $A_2$  through  $A_{10}$ . The state variables  $A_5$  through  $A_{10}$  have already been fixed in the semi-free start collision. If we additionally fix  $A_4$  and  $A_3$  to arbitrary values, it is possible to compute the sum of  $W_9$  and  $A_2$ ,

$$c_2 = W_9 + A_2 = A_{10} - F(A_9, \dots, A_3) . \quad (18)$$

3. From (1) and (17)-(18), it follows that

$$\begin{aligned} W_{16} &= \sigma_1(W_{14}) + W_9 + \sigma_0(W_1) + W_0 , \\ &= \sigma_1(W_{14}) + c_2 - c_1 - W_1 + \sigma_0(W_1) + W_0 , \\ &= c_3 + \sigma_0(W_1) - W_1 . \end{aligned} \quad (19)$$

It is easy to find a suitable value for  $W_1$  that ensures that  $W_{16}$  has the proper value, if it exists. It suffices to guess the 15 least significant bits of  $W_1$  to compute all 32 bits of  $W_1$ , satisfying the above condition with probability  $2^{-14}$ . A conservative estimate is that each trial requires an effort equivalent to one step update of SHA-256.

4. Now there is no more freedom left, as all the internal state variables have been fixed. The corresponding message words can be found from (15) and the message expansion. Just like in the 23-step collision attack, however, there are still some conditions left. As explained in Sect. 4.1, these are satisfied with a probability of  $2^{-19}$ .

Hence, the overall expected time complexity is equivalent to about  $2^{19}$ .  $(2^{14} + 10)$  SHA-256 step computations, or about  $2^{28.5}$  evaluations of the SHA-256 compression function reduced to 24 steps. An example collision pair for 24-step reduced SHA-256 is given in Table 5. An extension of this attack method beyond 24 steps fails, because then a difference in the first message word,  $W_0$ , becomes unavoidable. In [13], another differential than the one shown in Table 3 is used to find 22-step collisions for SHA-256. We tried to use this differential in our extended attacks, but even for 23 steps, using this differential fails.

**Table 5.** Example Colliding Message Pair for 24-Step Reduced SHA-256.

$M$	0187e08e <sub>x</sub>	865cedaf <sub>x</sub>	5b69e21a <sub>x</sub>	e0f7485e <sub>x</sub>
	50b98993 <sub>x</sub>	217e4650 <sub>x</sub>	51e3cf65 <sub>x</sub>	c2997c68 <sub>x</sub>
	2c267e16 <sub>x</sub>	82ffa4e9 <sub>x</sub>	37b5af09 <sub>x</sub>	5b28721d <sub>x</sub>
	1be35597 <sub>x</sub>	7ff22aa1 <sub>x</sub>	e807a758 <sub>x</sub>	c1519aaa <sub>x</sub>
$M'$	0187e08e <sub>x</sub>	865cedaf <sub>x</sub>	5b69e21a <sub>x</sub>	e0f7485e <sub>x</sub>
	50b98993 <sub>x</sub>	217e4650 <sub>x</sub>	51e3cf65 <sub>x</sub>	c2997c68 <sub>x</sub>
	2c267e16 <sub>x</sub>	82ffa4e9 <sub>x</sub>	37b5af0a <sub>x</sub>	5b28721c <sub>x</sub>
	1be3f597 <sub>x</sub>	82f24aa0 <sub>x</sub>	e807a758 <sub>x</sub>	c1519aaa <sub>x</sub>
$H$	1584074c <sub>x</sub>	8b810a94 <sub>x</sub>	01ea31b1 <sub>x</sub>	81bffd02 <sub>x</sub>
	d29c817d <sub>x</sub>	e4e04b51 <sub>x</sub>	b9f5ac4f <sub>x</sub>	6b34d1f8 <sub>x</sub>

### 4.3 Extensions

Further extensions using weaker attack models are discussed in Appendix A. In Sect. A.1, a semi-free start collision attack on 24 step SHA-256 is described. Section A.2 considers SHA-224 and shows free start collisions for one more step, *i.e.*, 25 steps of SHA-224. Finally, in Sect. A.3, free-start near collision attacks for up to 31 steps of SHA-256 are explored.

## 5 Conclusion

Our results push the limit to cryptanalysis for step reduced but otherwise unmodified SHA-256; we found practical collisions for up to 24 steps. For almost half of the steps (31 out of 64) non-random properties are detectable in practice. Even though we did not perform a detailed analysis, we expect very similar results for SHA-512.

### Acknowledgements

The authors would like to thank Vincent Rijmen for his advice. We acknowledge the use of the VIC computer cluster of K.U.Leuven, which was used to obtain most of the experimental results presented in this paper. Finally, thanks to Ralph Wernsdorf for bringing [13] to our attention.

### References

1. Florent Chabaud and Antoine Joux. Differential Collisions in SHA-0. In Hugo Krawczyk, editor, *CRYPTO*, volume 1462 of *LNCS*, pages 56–71. Springer, 1998.
2. Henri Gilbert and Helena Handschuh. Security Analysis of SHA-256 and Sisters. In Mitsuru Matsui and Robert J. Zuccherato, editors, *Selected Areas in Cryptography*, volume 3006 of *LNCS*, pages 175–193. Springer, 2003.

3. Philip Hawkes, Michael Paddon, and Gregory G. Rose. On corrective patterns for the SHA-2 family. Cryptology ePrint Archive, Report 2004/207, August 2004. <http://eprint.iacr.org/>.
4. Marko Holbl, Christian Rechberger, and Tatjana Welzer. Searching for messages conforming to arbitrary sets of conditions in SHA-256. In *Proceedings of WEWORC 2007*, LNCS. Springer, 2008. To appear.
5. Helger Lipmaa and Shiho Moriai. Efficient algorithms for computing differential properties of addition. In Mitsuru Matsui, editor, *FSE*, volume 2355 of *Lecture Notes in Computer Science*, pages 336–350. Springer, 2001.
6. Helger Lipmaa, Johan Wallén, and Philippe Dumas. On the additive differential probability of exclusive-or. In Bimal K. Roy and Willi Meier, editors, *FSE*, volume 3017 of *Lecture Notes in Computer Science*, pages 317–331. Springer, 2004.
7. Krystian Matusiewicz, Josef Pieprzyk, Norbert Pramstaller, Christian Rechberger, and Vincent Rijmen. Analysis of simplified variants of SHA-256. In *Proceedings of WEWoRC 2005*, LNI P-74, pages 123–134, 2005.
8. Florian Mendel, Norbert Pramstaller, Christian Rechberger, and Vincent Rijmen. Analysis of Step-Reduced SHA-256. In Matthew J. B. Robshaw, editor, *FSE*, volume 4047 of *LNCS*, pages 126–143. Springer, 2006.
9. Ivica Nikolić and Alex Biryukov. Collisions for Step-Reduced SHA-256. In Kaisa Nyberg, editor, *Fast Software Encryption, 15th International Workshop, FSE 2008, Lausanne, Switzerland, March 26-28, 2007*, LNCS. Springer, 2008. To appear.
10. National Institute of Standards and Technology (NIST). FIPS-180-2: Secure Hash Standard, August 2002. Available online at <http://www.itl.nist.gov/fipspubs/>.
11. Norbert Pramstaller, Christian Rechberger, and Vincent Rijmen. Preliminary Analysis of the SHA-256 Message Expansion. NIST - First Cryptographic Hash Workshop, October 31-November 1, 2005.
12. Somitra Kumar Sanadhya and Palash Sarkar. New Local Collisions for the SHA-2 Hash Family. In Kil-Hyun Nam and Gwangsoo Rhee, editors, *ICISC*, volume 4817 of *LNCS*, pages 193–205. Springer, 2007.
13. Somitra Kumar Sanadhya and Palash Sarkar. 22-step collisions for SHA-2. arXiv e-print archive, arXiv:0803.1220v1, March 2008. <http://de.arxiv.org/abs/0803.1220>.
14. Somitra Kumar Sanadhya and Palash Sarkar. Attacking Reduced Round SHA-256. In *Proceedings of ACNS 2008*, LNCS. Springer, 2008. To appear.
15. Hirotaka Yoshida and Alex Biryukov. Analysis of a SHA-256 Variant. In Bart Preneel and Stafford E. Tavares, editors, *Selected Areas in Cryptography*, volume 3897 of *LNCS*, pages 245–260. Springer, 2005.

## A Further Extensions

This appendix explores further extensions towards weaker collision-style attacks on a larger number of steps of SHA-256 and SHA-224.

### A.1 Semi-Free Start Collisions for 24 Steps of SHA-256

This section presents a straightforward extension of the 23 step semi-free start collision attack of Nikolić and Biryukov [9], which was described

in Sect. 3. We keep the entire attack algorithm unchanged, but shift everything down by a single step. Because of this, one more message word,  $W_0$ , needs to be computed from the message expansion in the reverse direction. From the message expansion in the reverse direction, it follows that the additive difference in this word is

$$\delta W_0 = \delta W_{16} - \delta \sigma_1(W_{14}) - \delta W_9 - \delta \sigma_0(W_1) \quad . \quad (20)$$

None of these expanded message words has a difference, so also  $\delta W_0 = 0$ . This yields 24 step semi-free start collisions of SHA-256 with the same complexity ( $2^{17}$ ).

## A.2 Free Start Collisions for 25 Steps of SHA-224

SHA-224 differs from SHA-256 in two ways. First, it has different initial values, and second, the output is truncated to the leftmost 224 bits. We can thus extend the 24-step semi-free start collision of SHA-256 from Sect. 4.2 to a 25-step free start collision of SHA-224 by simply shifting the same attack down one more step. Now a difference will inevitably appear in  $W_0$ , which propagates to the initial value  $H_0$ . The other initial values,  $A_0$  through  $G_0$  still have a zero difference. Because the word  $H$  is truncated away in SHA-224, this results in free start collisions for 25 steps of SHA-224.

## A.3 Free-Start Near Collisions of SHA-256

Extending the attack to more steps is possible, provided that some differences are allowed both in the initial value and in the hash result, *i.e.*, when considering free-start near collisions. The starting point is again the 23-step semi-free start collision attack from Sect. 3. It is extended by adding a number of extra backward and forward steps. For a given number of steps, the attacker can choose how to split the required extra steps into backward and forward steps.

As explained in Sect. 4.2, no difference is introduced in the first backward step. Note also that the diffusion of differences is slower in the backward direction than in the forward direction. A difference introduced in an expanded message word  $W_i$  affects both  $A_{i+1}$  and  $E_{i+1}$  in the forward direction, as opposed to only  $H_i$  when going in the backward direction. Thus, in the forward direction, all state words can be affected by a single difference in an expanded message word after only four rounds. In the backward direction, this takes eight rounds.



**Table 6.** Experimental results of the free-start near collision attack on SHA-256. For each number of steps, only the combination of forward/backward steps that gave the best results is shown. For comparison, the expected numbers of solutions for a generic birthday attack with an equal effort are also given.

steps	bwd.	fwd.	$k_{\min}$	2-logarithm of the number of solutions with $k$								
				$\leq 8$	$\leq 16$	$\leq 24$	$\leq 32$	$\leq 40$	$\leq 48$	$\leq 56$	$\leq 64$	
25	1	1	2	31.95	32.00	32.00	32.00	32.00	32.00	32.00	32.00	32.00
26	2	1	8	24.17	31.55	31.99	32.00	32.00	32.00	32.00	32.00	32.00
27	3	1	11	$-\infty$	15.41	26.20	30.65	31.89	32.00	32.00	32.00	32.00
28	4	1	18	$-\infty$	$-\infty$	8.77	20.41	27.24	30.63	31.80	31.99	32.00
29	5	1	32	$-\infty$	$-\infty$	$-\infty$	1.58	14.31	22.86	28.19	30.93	32.00
30	6	1	43	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	10.73	19.58	25.68	32.00
31	6	2	53	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	6.34	15.50	32.00
Birthday Attack			57	-143.41	-108.84	-80.49	-56.36	-35.51	-17.37	-1.57	12.14	32.00

We have done several experiments, each equivalent to an effort of  $2^{32}$  reduced SHA-256 compression function evaluations, which are summarised in Table 6. The first three columns give the total number of steps, and the number of extra backward and extra forward steps, respectively. The fourth column gives  $k_{\min}$ , the smallest Hamming distance found. The last eight columns contain the 2-logarithm of the number of solutions with a Hamming distance  $k$  of at most 8, 16,  $\dots$ , 64 bits. For comparison, also the expected values for a generic birthday attack with an equal effort of  $2^{32}$  is given.

For a generic (free-start) near collision attack on an ideal  $n$ -bit hash function, using the birthday paradox with an effort of  $2^w$  compression function evaluations, the lowest expected Hamming distance is the lowest  $k$  for which

$$2^{2w} \cdot \left( \sum_{i=0}^k 2^{-n} \binom{n}{i} \right) \geq 1 . \quad (21)$$

For instance, with  $w = 32$  and for SHA-256 (*i.e.*,  $n = 256$ ), this gives  $k = 57$  bits. Our attack performs significantly better for up to 30 steps of SHA-256. For 31 steps, we still found 208 free-start near collisions with a Hamming distance of at most 57 bits, whereas a birthday attack is only expected to find one with the same effort.

## B Solving $\mathcal{L}(x + \delta) = \mathcal{L}(x) + \delta'$

This section describes a generic method to solve equations of the form  $\mathcal{L}(x + \delta) = \mathcal{L}(x) + \delta'$  where  $\delta$  and  $\delta'$  are given  $n$ -bit additive differences, and  $\mathcal{L}$  is an  $n$ -bit to  $n$ -bit GF(2)-linear transformation. This is a similar problem to the ones studied by Lipmaa and Moriai [5] and Lipmaa *et al.* [6].

Consider the modular addition  $x + \delta$  and let  $\Delta = (x + \delta) \oplus x$ . This addition is described by the following equations, where  $x_i$  is the  $i$ -th bit of  $x$  and the  $c_i$ 's are the carry bits:

$$\begin{aligned} (x + \delta)_i &= x_i \oplus \delta_i \oplus c_i & c_i &= \delta_i \oplus \Delta_i \\ c_{i+1} &= f_{\text{maj}}(x_i, \delta_i, c_i) & c_{i+1} &= f_{\text{maj}}(x_i, \delta_i, \delta_i \oplus \Delta_i) \quad . \quad (22) \\ c_0 &= 0 & c_0 &= 0 \end{aligned}$$

Hence, once we fix both the additive difference  $\delta$  and the XOR difference  $\Delta$ , all the  $c_i$ 's are fixed. Some of the  $x_i$ 's are also fixed: when  $\Delta_i = 1$  and  $i + 1 < n$ , it must hold that  $x_i = c_{i+1} = \delta_{i+1} \oplus \Delta_{i+1}$ . This means that the allowed values for  $x$  lie in an affine space. Note that not all additive differences are consistent with all XOR differences, *i.e.*, the following conditions must be satisfied

$$\begin{cases} c_0 = \delta_0 \oplus \Delta_0 = 0 \\ \delta_i = \delta_{i+1} \oplus \Delta_{i+1} \quad \text{when } \Delta_i = 0 \text{ and } i + 1 < n \end{cases} \quad . \quad (23)$$

Solving an equation of the form  $\mathcal{L}(x + \delta) = \mathcal{L}(x) + \delta'$  can be done as follows. Let  $\Delta' = (\mathcal{L}(x) + \delta') \oplus \mathcal{L}(x)$ , *i.e.*, the XOR-difference associated with the modular addition  $\mathcal{L}(x) + \delta'$ . Since  $\mathcal{L}(x + \delta) = \mathcal{L}(x) + \delta'$  and  $\mathcal{L}$  is GF(2)-linear, it follows that  $\Delta' = \mathcal{L}(\Delta)$ . We can thus simply enumerate all the XOR-differences  $\Delta$  consistent with  $\delta$ , compute  $\Delta' = \mathcal{L}(\Delta)$  and check if this is consistent with  $\delta'$ . If it is, both additions restrict  $x$  to a (different) affine space. The intersection of these spaces, which can be computed by solving a system of linear equations over GF(2), gives the solutions  $x$  for the chosen  $\Delta$ . Note that the intersection may be empty. If no solutions are found for any value of  $\Delta$ , the equation  $\mathcal{L}(x + \delta) = \mathcal{L}(x) + \delta'$  has no solutions.

The time complexity of this method is proportional to the minimum of the number of XOR differences consistent with  $\delta$  or  $\delta'$ . This follows from the fact that one can easily modify the method to choose  $\Delta'$  instead of  $\Delta$ .