The Random Oracle Model and the Ideal Cipher Model are Equivalent*

Jean-Sébastien Coron[1], Jacques Patarin[2], and Yannick Seurin[2,3]

[1] University of Luxembourg
[2] University of Versailles
[3] Orange Labs

Abstract. The Random Oracle Model and the Ideal Cipher Model are two well known idealised models of computation for proving the security of cryptosystems. At Crypto 2005, Coron *et al.* showed that security in the random oracle model implies security in the ideal cipher model; namely they showed that a random oracle can be replaced by a block cipher-based construction, and the resulting scheme remains secure in the ideal cipher model. The other direction was left as an open problem, *i.e.* constructing an ideal cipher from a random oracle. In this paper we solve this open problem and show that the Feistel construction with 6 rounds is enough to obtain an ideal cipher; we also show that 5 rounds are insufficient by providing a simple attack. This contrasts with the classical Luby-Rackoff result that 4 rounds are necessary and sufficient to obtain a (strong) pseudo-random permutation from a pseudo-random function.

1 Introduction

Modern cryptography is about defining security notions and then constructing schemes that provably achieve these notions. In cryptography, security proofs are often relative: a scheme is proven secure, assuming that some computational problem is hard to solve. For a given functionality, the goal is therefore to obtain an efficient scheme that is secure under a well known computational assumption (for example, factoring is hard). However for certain functionalities, or to get a more efficient scheme, it is sometimes necessary to work in some idealised model of computation.

The well known *Random Oracle Model* (ROM), formalised by Bellare and Rogaway [1], is one such model. In the random oracle model, one assumes that some hash function is replaced by a publicly accessible random function (the random oracle). This means that the adversary cannot compute the result of the hash function by himself: he must query the random oracle. The random oracle model has been used to prove the security of numerous cryptosystems, and it has lead to simple and efficient designs that are widely used in practice (such as PSS [2] and OAEP [3]). Obviously, a proof in the random oracle model is not fully satisfactory, because such proof does not imply that the scheme will remain secure when the random oracle is replaced by a concrete hash function (such as SHA-1). Numerous papers have shown artificial schemes that are provably secure in the ROM, but completely insecure when the RO is instantiated with any function family (see [7]). Despite these separation results, the ROM still appears to be a useful tool for proving the security of cryptosystems. For some functionalities, the ROM construction is actually the only known construction (for example, for non-sequential aggregate signatures [6]).

The *Ideal Cipher Model* (ICM) is another idealised model of computation, similar to the ROM. Instead of having a publicly accessible random function, one has a publicly accessible random block cipher (or ideal cipher). This is a block cipher with a κ-bit key and a n-bit input/output, that is chosen uniformly at random among all block ciphers of this form; this is equivalent to having a family of 2^κ independent random permutations. All parties including the

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adversary can make both encryption and decryption queries to the ideal block cipher, for any given key. As for the random oracle model, many schemes have been proven secure in the ICM [5, 10, 13, 15]. As for the ROM, it is possible to construct artificial schemes that are secure in the ICM but insecure for any concrete block cipher (see [4]). Still, a proof in the ideal cipher model seems useful because it shows that a scheme is secure against generic attacks, that do not exploit specific weaknesses of the underlying block cipher.

A natural question is whether the random oracle model and the ideal cipher model are equivalent models, or whether one model is strictly stronger than the other. Given a scheme secure with random oracles, is it possible to replace the random oracles with a block cipher-based construction, and obtain a scheme that is still secure in the ideal cipher model ? Conversely, if a scheme is secure in the ideal cipher model, is it possible to replace the ideal cipher with a construction based on functions, and get a scheme that is still secure when these functions are seen as random oracles ?

At Crypto 2005, Coron *et al.* [9] showed that it is indeed possible to replace a random oracle (taking arbitrary long inputs) by a block cipher-based construction. The proof is based on an extension of the classical notion of indistinguishability, called *indifferentiability*, introduced by Maurer *et al.* in [17]. Using this notion of indifferentiability, the authors of [9] gave the definition of an "indifferentiable construction" of one ideal primitive (F) (for example, a random oracle) from another ideal primitive (G) (for example an ideal block cipher). When a construction satisfies this notion, any scheme that is secure in the former ideal model (F) remains secure in the latter model (G), when instantiated using this construction. The authors of [9] proposed a slight variant of the Merkle-Damgard construction to instantiate a random oracle (see Fig. 1). Given any scheme provably secure in the random oracle model, this construction can replace the random oracle, and the resulting scheme remains secure in the ideal cipher model; other constructions have been analysed in [8].

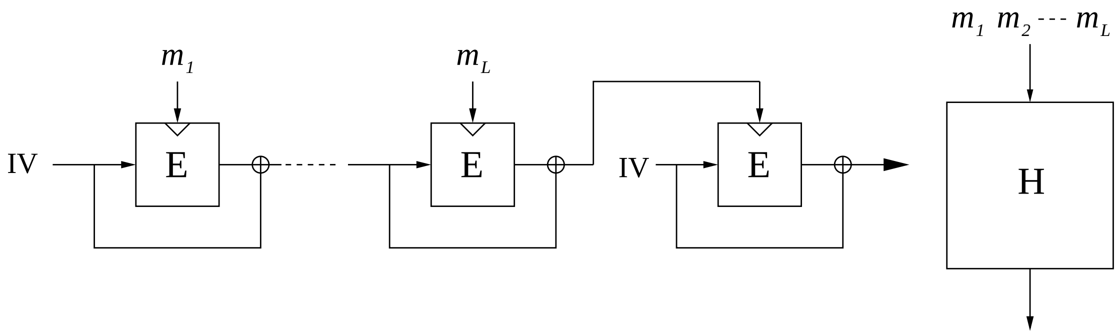


Fig. 1. A Merkle-Damgard like construction [9] based on ideal cipher E (left) to replace random oracle H (right). Messages blocks m_i's are used as successive keys for ideal-cipher E. IV is a pre-determined constant.

The other direction (constructing an ideal cipher from a random oracle) was left as an open problem in [9]. In this paper we solve this open problem and show that the Luby-Rackoff construction with 6 rounds is sufficient to instantiate an ideal cipher (see Fig. 2 for an illustration). Actually, it is easy to see that it is enough to construct a random *permutation* instead of an ideal cipher; namely, a family of 2^κ independent random permutations (*i.e.*, an ideal block cipher) can be constructed by simply prepending a k-bit key to the inner random oracle functions F_i's. Therefore in this paper, we concentrate on the construction of a random permutation. We also show that 5 rounds Luby-Rackoff is insecure by providing a simple attack; this shows that 6 rounds is actually optimal.

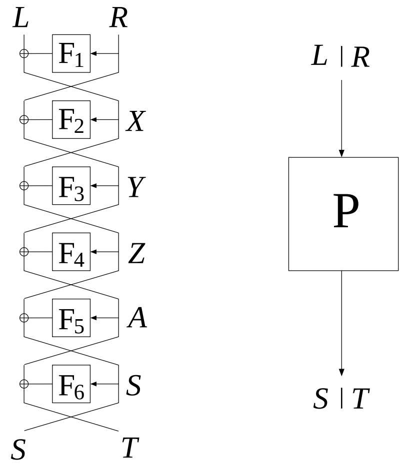


Fig. 2. The Luby-Rackoff construction with 6 rounds (left), to replace a random permutation P (right).

Our result shows that the random oracle model and the ideal cipher model are actually equivalent assumptions. It seems that up to now, many cryptographers have been reluctant to use the Ideal Cipher Model and have endeavoured to work in the Random Oracle Model, arguing that the ICM is richer and carries much more structure than the ROM. Our result shows that it is in fact not the case and that designers may use the ICM when they need it without making a stronger assumption than when working in the random oracle model. However, our security reduction is quite loose, which implies that in practice large security parameters should be used in order to replace an ideal cipher with a 6-round Luby-Rackoff.

We stress that the "indifferentiable construction" notion is very different from the classical indistinguishability notion. The well known Luby-Rackoff result that 4 rounds are enough to obtain a strong pseudo-random permutation from pseudo-random functions [16], is proven under the classical indistinguishability notion. Under this notion, the adversary has only access to the input/output of the Luby-Rackoff (LR) construction, and tries to distinguish it from a random permutation; in particular it does not have access to the input/output of the inner pseudo-random functions. On the contrary, in our setting, the distinguisher can make oracle calls to the inner round functions F_i's (see Fig. 2); the indifferentiability notion enables to accommodate these additional oracle calls in a coherent definition.

1.1 Related Work

One of the first paper to consider having access to the inner round functions of a Luby-Rackoff is [19]; the authors showed that Luby-Rackoff with 4 rounds remains secure if adversary has oracle access to the middle two round functions, but becomes insecure if adversary is allowed access to any other round functions.

In [14] a random permutation oracle was instantiated for a specific scheme using a 4-rounds Luby-Rackoff. More precisely, the authors showed that the random permutation oracle P in the Even-Mansour [13] block-cipher $E_{k_1,k_2}(m) = k_2 \oplus P(m \oplus k_1)$ can be replaced by a 4-rounds Luby-Rackoff, and block-cipher E remains secure in the random oracle model; for this specific scheme, the authors obtained a (much) better security bound than our general bound in this paper.

In [11], Dodis and Puniya introduced a different model for indifferentiability, called indifferentiability in the *honest-but-curious* model. In this model, the distinguisher is not allowed to make direct calls to the inner hash functions; instead he can only query the global Luby-Rackoff construction and get all the intermediate results. The authors showed that in this model, a Luby-Rackoff construction with a super-logarithmic number of rounds can replace an ideal cipher. The authors also showed that indifferentiability in the honest-but-curious model implies indifferentiability in the general model, for LR constructions with up to a logarithmic number of rounds. But because of this gap between logarithmic and super-logarithmic, the authors could not conclude about general indifferentiability of Luby-Rackoff constructions. Subsequent work by Dodis and Puniya [12] studied other properties (such as unpredictability and verifiablity) of the Luby-Rackoff construction when the intermediate values are known to the attacker.

We have an observation about indifferentiability in the honest-but-curious model: general indifferentiability does not necessarily imply indifferentiability in the honest-but-curious model. More precisely, we show in Appendix B that LR constructions with up to logarithmic number of rounds are *not* indifferentiable from a random permutation in the honest-but-curious model, whereas our main result in this paper is that 6-rounds LR is indifferentiable from a random permutation in the general model.

2 Definitions

In this section, we recall the notion of indifferentiability of random systems, introduced by Maurer *et al.* in [17]. This is an extension of the classical notion of indistinguishability, where one or more oracles are publicly available, such as random oracles or ideal ciphers.

We first motivate why such an extension is actually required. The classical notion of indistinguishability enables to argue that if some system S_1 is indistinguishable from some other system S_2 (for any polynomially bounded attacker), then any application that uses S_1 can use S_2 instead, without any loss of security; namely, any non-negligible loss of security would precisely be a way of distinguishing between the two systems. Since we are interested in replacing a random permutation (or an ideal cipher) by a Luby-Rackoff construction, we would like to say that the Luby-Rackoff construction is "indistinguishable" from a random permutation. However, when the distinguisher can make oracle calls to the inner round functions, one cannot say that the two systems are "indistinguishable" because they don't even have the same interface (see Fig. 2); namely for the LR construction the distinguisher can make oracle calls to the inner functions F_i's, whereas for the random permutation he can only query the input and receive the output and vice versa. This contrasts with the setting of the classical Luby-Rackoff result, where the adversary has only access to the input/output of the LR construction, and tries to distinguish it from a random permutation. Therefore, an extension of the classical notion of indistinguishability is required, in order to show that some ideal primitive (like a random permutation) can be constructed from another ideal primitive (like a random oracle).

Following [17], we define an *ideal primitive* as an algorithmic entity which receives inputs from one of the parties and delivers its output immediately to the querying party. The ideal primitives that we consider in this paper are random oracles and random permutations (or ideal ciphers). A *random oracle* [1] is an ideal primitive which provides a random output for each new query. Identical input queries are given the same answer. A *random permutation* is an ideal primitive that contains a random permutation $P : \{0,1\}^n \to \{0,1\}^n$. The ideal primitive provides oracle access to P and P^{-1}. An *ideal cipher* is an ideal primitive that models a random block cipher $E : \{0,1\}^\kappa \times \{0,1\}^n \to \{0,1\}^n$. Each key $k \in \{0,1\}^\kappa$ defines a random permutation $E_k = E(k, \cdot)$ on $\{0,1\}^n$. The ideal primitive provides oracle access to E and E^{-1}; that is, on query $(0, k, m)$, the primitive answers $c = E_k(m)$, and on query $(1, k, c)$, the primitive answers m such that $c = E_k(m)$. These oracles are available for any n and any κ.

The notion of indifferentiability [17] is used to show that an ideal primitive $\mathcal{P}$ (for example, a random permutation) can be replaced by a construction C that is based on some other ideal primitive $\mathcal{F}$ (for example, C is a LR construction based on a random oracle F):

Definition 1 ([17]). *A Turing machine C with oracle access to an ideal primitive $\mathcal{F}$ is said to be $(t_D, t_S, q, \varepsilon)$ indifferentiable from an ideal primitive $\mathcal{P}$ if there exists a simulator S with oracle access to $\mathcal{P}$ and running in time at most t_S, such that for any distinguisher D running in time at most t_D and making at most q queries, it holds that:*

$$\left|\Pr\left[D^{C^{\mathcal{F}},\mathcal{F}} = 1\right] - \Pr\left[D^{\mathcal{P},S^{\mathcal{P}}} = 1\right]\right| < \varepsilon$$

$C^{\mathcal{F}}$ is simply said to be indifferentiable from $\mathcal{F}$ if ε is a negligible function of the security parameter n, for polynomially bounded q, t_D and t_S.

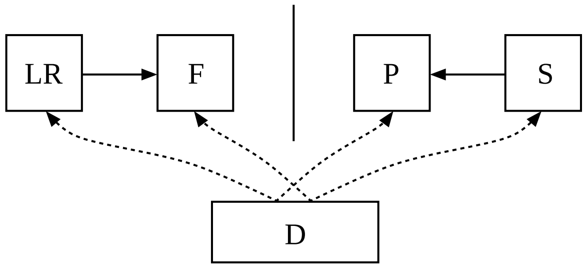


Fig. 3. The indifferentiability notion.

The previous definition is illustrated in Figure 3, where $\mathcal{P}$ is a random permutation, C is a Luby-Rackoff construction LR, and F is a random oracle. In this paper, for a 6-round Luby-Rackoff, we denote these random oracles $F_1, \ldots, F_6$ (see Fig. 2). Equivalently, one can consider a single random oracle F and encode in the first 3 bits input which round function $F_1, \ldots, F_6$ is actually called. The distinguisher has either access to the system formed by the construction LR and random oracle F, or to the system formed by the random permutation P and a simulator $\mathcal{S}$. In the first system (left), the construction LR computes its output by making calls to F (this corresponds to the round functions F_i's of the Luby-Rackoff); the distinguisher can also make calls to F directly. In the second system (right), the distinguisher can either query the random permutation P, or the simulator that can make queries to P. We see that the role of the simulator is to simulate the random oracles F_i's so that no distinguisher can tell whether it is interacting with LR and F, or with P and S. In other words, 1) the output of S should be indistinguishable from that of random oracles F_i's and 2) the output of S should look "consistent" with what the distinguisher can obtain from P. We stress that the simulator does not see the distinguisher's queries to P; however, it can call P directly when needed for the simulation. Note that the two systems have the same interface, so now it makes sense to require that the two systems should be indistinguishable.

To summarise, in the first system the random oracles F_i are chosen at random, and a permutation $C = LR$ is constructed from them with a 6 rounds Luby-Rackoff. In the second system the random permutation P is chosen at random and the inner round functions F_i's are simulated by a simulator with oracle access to P. Those two systems should be indistinguishable, that is the distinguisher should not be able to tell whether the inner round functions were chosen at random and then the Luby-Rackoff permutation constructed from it, or the random permutation was chosen at random and the inner round functions then "tailored" to match the permutation.

It is shown in [17] that the indifferentiability notion is the "right" notion for substituting one ideal primitive by a construction based on another ideal primitive. That is, if $C^{\mathcal{F}}$ is indifferentiable from ideal primitive $\mathcal{P}$, then $C^{\mathcal{F}}$ can replace $\mathcal{P}$ in any cryptosystem, and the resulting cryptosystem is at least as secure in the $\mathcal{F}$ model as in the $\mathcal{P}$ model; see [17] or [9] for a proof. Our main result in this paper is that the 6 rounds Luby-Rackoff construction is indifferentiable from a random permutation; this implies that such construction can replace a random permutation (or an ideal cipher) in any cryptosystem, and the resulting scheme remains secure in the random oracle model if the original scheme was secure in the random permutation (or ideal cipher) model.

3 Attack of Luby-Rackoff with 5 Rounds

In this section we show that 5 rounds are not enough to obtain the indifferentiability property. We do this by exhibiting for the 5 rounds Luby-Rackoff (see Fig. 4) a property that cannot be obtained with a random permutation.

Let Y and Y' be arbitrary values, corresponding to inputs of F_3 (see Fig. 4); let Z be another arbitrary value, corresponding to input of F_4. Let $Z' = F_3(Y) \oplus F_3(Y') \oplus Z$, and let:

$$X = F_3(Y) \oplus Z = F_3(Y') \oplus Z' \tag{1}$$
$$X' = F_3(Y') \oplus Z = F_3(Y) \oplus Z' \tag{2}$$

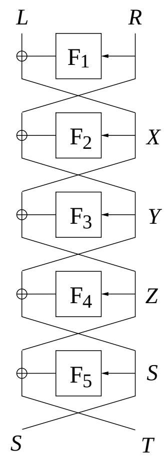


Fig. 4. 5-rounds Luby-Rackoff.

From X, X', Y and Y' we now define four couples (X_i, Y_i) as follows:

$$(X_0, Y_0) = (X, Y), \quad (X_1, Y_1) = (X', Y)$$
$$(X_2, Y_2) = (X', Y'), \quad (X_3, Y_3) = (X, Y')$$

and we let $L_i \| R_i$ be the four corresponding plaintexts; we have:

$$R_0 = Y_0 \oplus F_2(X_0) = Y \oplus F_2(X)$$
$$R_1 = Y_1 \oplus F_2(X_1) = Y \oplus F_2(X')$$
$$R_2 = Y_2 \oplus F_2(X_2) = Y' \oplus F_2(X')$$
$$R_3 = Y_3 \oplus F_2(X_3) = Y' \oplus F_2(X)$$

Let Z_0, Z_1, Z_2, Z_3 be the corresponding values as input of F_4; we have from (1) and (2):

$$Z_0 = X_0 \oplus F_3(Y_0) = X \oplus F_3(Y) = Z, \quad Z_1 = X_1 \oplus F_3(Y_1) = X' \oplus F_3(Y) = Z'$$
$$Z_2 = X_2 \oplus F_3(Y_2) = X' \oplus F_3(Y') = Z, \quad Z_3 = X_3 \oplus F_3(Y_3) = X \oplus F_3(Y') = Z'$$

Finally, let $S_i \| T_i$ be the four corresponding ciphertexts; we have:

$$S_0 = Y_0 \oplus F_4(Z_0) = Y \oplus F_4(Z), \quad S_1 = Y_1 \oplus F_4(Z_1) = Y \oplus F_4(Z')$$
$$S_2 = Y_2 \oplus F_4(Z_2) = Y' \oplus F_4(Z), \quad S_3 = Y_3 \oplus F_4(Z_3) = Y' \oplus F_4(Z')$$

We obtain the relations:

$$R_0 \oplus R_1 \oplus R_2 \oplus R_3 = 0, \quad S_0 \oplus S_1 \oplus S_2 \oplus S_3 = 0$$

Thus, we have obtained four pairs (plaintext, ciphertext) such that the xor of the right part of the four plaintexts equals 0 and the xor of the left part of the four ciphertexts also equals 0. For a random permutation, it is easy to see that such property can only be obtained with negligible probability, when the number of queries is polynomially bounded. Thus we have shown:

Theorem 1. *The Luby-Rackoff construction with* 5 *rounds is not indifferentiable from a random permutation.*

This contrasts with the classical Luby-Rackoff result, where 4 rounds are enough to obtain a pseudo-random permutation from pseudo-random functions.

4 Indifferentiability of 6 rounds Luby-Rackoff

We now prove our main result: the Luby-Rackoff construction with 6 rounds is indifferentiable from a random permutation.

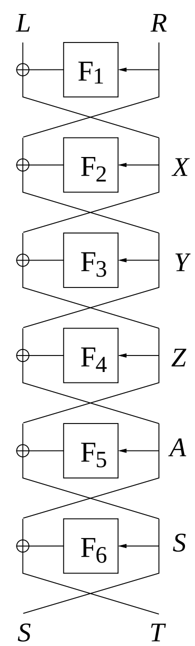
Theorem 2. *The LR construction with* 6 *rounds is* $(t_D, t_S, q, \varepsilon)$*-indifferentiable from a random permutation, with* $t_S = \mathcal{O}(q^4)$ *and* $\varepsilon = 2^{19} \cdot q^8/2^n$*, where* n *is the output size of the round functions.*

Note that here the distinguisher has unbounded running time; it is only bounded to ask q queries. As illustrated in Figure 3, we must construct a simulator $\mathcal{S}$ such that the two systems formed by (LR, F) and $(P, \mathcal{S})$ are indistinguishable. The simulator is constructed in Section 4.1, while the indistinguishability property is proved in Section 4.2.

4.1 The Simulator

We construct a simulator $\mathcal{S}$ that simulates the random oracles $F_1, \ldots, F_6$. For each function F_i the simulator maintains an history of already answered queries. We write $x \in F_i$ when x belongs to the history of F_i, and we denote by $F_i(x)$ the corresponding output. When we need to obtain $F_i(x)$ and x does not belong to the history of F_i, we write $F_i(x) \leftarrow y$ to determine that the answer to F_i query x will be y; we then add $(x, F_i(x))$ to the history of F_i. We denote by n the output size of the functions F_i's. We denote by LR and LR^{-1} the 6-round Luby-Rackoff construction as obtained from the functions F_i's.

We first provide an intuition of the simulator's algorithm. The simulator must make sure that his answers to the distinguisher's F_i queries are coherent with the answers to P queries that can be obtained independently by the distinguisher. In other words, when the distinguisher makes F_i queries to the simulator (possibly in some arbitrary order), the output generated by the corresponding Luby-Rackoff must be the same as the output from P obtained independently by the distinguisher. We stress that those P queries made by the distinguisher cannot be seen by the simulator; the simulator is only allowed to make his own P queries (as illustrated in Fig. 3). In addition, the simulator's answer to F_i queries must be statistically close to random functions.



The simulator's strategy is the following: when a "chain of 3 queries" has been made by the distinguisher, the simulator is going to define the values of all the other F_i's corresponding to this chain, by making a P or a P^{-1} query, so that the output of LR and the output of P are the same for the corresponding message. Roughly speaking, we say that we have a chain of 3 queries (x, y, z) when x, y, z are in the history of F_k, F_{k+1} and F_{k+2} respectively and $x = F_{k+1}(y) \oplus z$.

For example, if a query X to F_2 is received, and we have $X = F_3(Y) \oplus Z$ where Y, Z belong to the history of F_3 and F_4 respectively, then the triple (X, Y, Z) forms a 3-chain of queries. In this case, the simulator defines $F_2(X) \xleftarrow{\$} \{0,1\}^n$ and computes the corresponding $R = Y \oplus F_2(X)$. It also lets $F_1(R) \xleftarrow{\$} \{0,1\}^n$ and computes $L = X \oplus F_1(R)$. Then it makes a P-query to get $S \| T = P(L \| R)$. It also computes $A = Y \oplus F_4(Z)$. The values of $F_5(A)$ and $F_6(S)$ are then "adapted" so that the 6-round LR and the random permutation provide the same output, *i.e.* the simulator defines $F_5(A) \leftarrow Z \oplus S$ and $F_6(S) \leftarrow A \oplus T$, so that $\mathsf{LR}(L \| R) = P(L \| R) = S \| T$. In summary, given a F_2 query, the simulator looked at the history of (F_3, F_4)

and adapted the answers of (F_5, F_6). Additionally, the simulator must also look at the history of (F_6, F_1) and possibly adapt at (F_3, F_4).

More generally, given a query to F_k, the simulator proceeds according to Table 1 below; we denote by $+$ for looking downward in the LR construct and by $-$ for looking upward. The simulator must first simulate an additional call to F_i (column "Call"). Then the simulator can compute either $L\|R$ or $S\|T$ (as determined in column "Compute"). Given $L\|R$ (resp. $S\|T$) the simulator makes a P-query (resp. a P^{-1}-query) to obtain $S\|T = P(L\|R)$ (resp. $L\|R = P^{-1}(S\|T)$). Finally Table 1 indicates the index j for which the output of (F_j, F_{j+1}) is adapted (column "Adapt").

Query	Dir	History	Call	Compute	Adapt
F_1	-	(F_5, F_6)	F_4	$S\|T$	(F_2, F_3)
F_2	+	(F_3, F_4)	F_1	$L\|R$	(F_5, F_6)
F_2	-	(F_6, F_1)	F_5	$L\|R$	(F_3, F_4)
F_3	+	(F_4, F_5)	F_6	$S\|T$	(F_1, F_2)
F_4	-	(F_2, F_3)	F_1	$L\|R$	(F_5, F_6)
F_5	+	(F_6, F_1)	F_2	$S\|T$	(F_3, F_4)
F_5	-	(F_3, F_4)	F_6	$S\|T$	(F_1, F_2)
F_6	+	(F_1, F_2)	F_3	$L\|R$	(F_4, F_5)

Table 1. Simulator's behaviour.

Given a query x to F_k, with $2 \le k \le 3$, the simulator (when looking downward) must actually consider all 3-chains formed by (x, y, z) where $y \in F_{k+1}$ and $z \in F_{k+2}$. Therefore, for $k \le 2 \le 3$, one defines the following set:

$$\mathsf{Chain}(+1, x, k) = \{(y, z) \in (F_{k+1}, F_{k+2}) \mid x = F_{k+1}(y) \oplus z\}$$

where $+1$ corresponds to looking downward in the Luby-Rackoff construction. This corresponds to Lines $(F_2, +)$ and $(F_3, +)$ in Table 1.

Similarly, given a query t to F_k, with $4 \le k \le 5$, when looking upward the simulator must consider all 3-chains formed by (y, z, t) where $y \in F_{k-2}$ and $z \in F_{k-1}$; one defines the following set for $4 \le k \le 5$:

$$\mathsf{Chain}(-1, t, k) = \{(y, z) \in (F_{k-2}, F_{k-1}) \mid t = F_{k-1}(z) \oplus y\}$$

This corresponds to Lines $(F_4, -)$ and $(F_5, -)$ in Table 1.

Additionally one must consider the 3-chains obtained from a F_6 query S and looking in (F_1, F_2) history, with Line $(F_6, +)$:

$$\mathsf{Chain}(+1, S, 6) = \Big\{(R, X) \in (F_1, F_2) \mid \exists T,\ P\left(F_1\left(R\right) \oplus X\|R\right) = S\|T\Big\} \tag{3}$$

and symmetrically the 3-chains obtained from a F_1 query R and looking in (F_5, F_6) history, with Line $(F_1, -)$:

$$\mathsf{Chain}(-1, R, 1) = \Big\{(A, S) \in (F_5, F_6) \mid \exists L,\ P^{-1}(S\|TF_6(S) \oplus A) = L\|R\Big\} \tag{4}$$

One must also consider the 3-chains associated with (F_1, F_6) history, obtained either from a F_2 query X or a F_5 query A, with Lines $(F_2, -)$ and $(F_5, +)$. Given a F_2 query X, we consider all $R \in F_1$, and for each corresponding $L = X \oplus F_1(R)$, we determine whether $S \in F_6$, where

$S\|T = P(L\|R)$. Additionally, we also consider "virtual" 3-chains, where $S \notin F_6$, but S is such that $P(L'\|R') = S\|T'$ for some other $(R', X') \in (F_1, F_2)$, with $L' = X' \oplus F_1(R')$. Formally :

$$\begin{aligned}\mathsf{Chain}(-1, X, 2) = \Big\{ & (R, S) \in (F_1, \{0,1\}^n) \mid \exists T,\ P(X \oplus F_1(R)\|R) = S\|T \\ & \text{and } (S \in F_6,\ \text{or } \exists T',\ (R', X') \neq (R, X) \in (F_1, F_2), \\ & P(X' \oplus F_1(R')\|R') = S\|T') \Big\}\end{aligned} \tag{5}$$

$$\begin{aligned}\mathsf{Chain}(+1, A, 5) = \Big\{ & (R, S) \in (\{0,1\}^n, F_6) \mid \exists L,\ P^{-1}(S\|A \oplus F_6(S)) = L\|R \\ & \text{and } (R \in F_1,\ \text{or } \exists L', (A', S') \neq (A, S) \in (F_5, F_6), \\ & P^{-1}(A' \oplus F_6(S')\|S') = L'\|R) \Big\}\end{aligned} \tag{6}$$

When the simulator receives a query x for F_k, it then proceeds as follows:

$\mathsf{Query}(x, k)$:

1. If x is in the history of F_k then go to step 4.
2. $F_k(x) \stackrel{\$}{\leftarrow} \{0,1\}^n$
3. Call $\mathsf{ChainQuery}(x, k)$
4. Return $F_k(x)$.

The $\mathsf{ChainQuery}$ algorithm is used to handle all possible 3-chains created by the operation $F_k(x) \stackrel{\$}{\leftarrow} \{0,1\}^n$ at step 2:

$\mathsf{ChainQuery}(x, k)$:

1. If $k \in \{2, 3, 5, 6\}$, then for all $(y, z) \in \mathsf{Chain}(+1, x, k)$:
   (a) Call $\mathsf{CompleteChain}(+1, x, y, z, k)$.
2. If $k \in \{1, 2, 4, 5\}$, then for all $(y, z) \in \mathsf{Chain}(-1, x, k)$:
   (a) Call $\mathsf{CompleteChain}(-1, x, y, z, k)$.

The $\mathsf{CompleteChain}(b, x, y, z, k)$ works as follows: it computes the message $L\|R$ or $S\|T$ that corresponds to the 3-chain (x, y, z) given as input, without querying (F_j, F_{j+1}), where j is the index given in Table 1 (column "Adapt"). If $L\|R$ is first computed, then the simulator makes a P query to obtain $S\|T = P(L\|R)$; similarly, if $S\|T$ is first computed, then the simulator makes a P^{-1} query to obtain $L\|R = P^{-1}(S\|T)$. Eventually the output of functions (F_j, F_{j+1}) is adapted so that $\mathsf{LR}(L\|R) = S\|T$.

$\mathsf{CompleteChain}(b, x, y, z, k)$:

1. If $(b, k) = (-1, 2)$ and $z \notin F_6$, then call $\mathsf{Query}(z, 6)$, without considering in $\mathsf{ChainQuery}(z, 6)$ the 3-chain that leads to 3-chain (x, y, z).
2. If $(b, k) = (+1, 5)$ and $y \notin F_1$, then call $\mathsf{Query}(y, 1)$, without considering in $\mathsf{ChainQuery}(y, 1)$ the 3-chain that leads to 3-chain (x, y, z)
3. Given (b, k) and from Table 1:
   (a) Determine the index i of the additional call to F_i (column "Call").
   (b) Determine whether $L\|R$ or $S\|T$ must be computed first.
   (c) Determine the index j for adaptation at (F_j, F_{j+1}) (column "Adapt").
4. Call $\mathsf{Query}(x_i, i)$, where x_i is the input of F_i that corresponds to the 3-chain (x, y, z), without considering in $\mathsf{ChainQuery}(x_i, i)$ the 3-chain that leads to 3-chain (x, y, z).
5. Compute the message $L\|R$ or $S\|T$ corresponding to the 3-chain (x, y, z)
6. If $L\|R$ has been computed, make a P query to get $S\|T = P(L\|R)$; otherwise, make a P^{-1} query to get $L\|R = P^{-1}(S\|T)$.
7. Now all input values $(x_1, \ldots, x_6)$ to F_i corresponding to the 3-chain (x, y, z) are known. Additionally let $x_0 \leftarrow L$ and $x_7 \leftarrow T$.

8. If x_j is in the history of F_j or x_{j+1} is in the history of F_{j+1}, abort.
9. Define $F_j(x_j) \leftarrow x_{j-1} \oplus x_{j+1}$
10. Define $F_{j+1}(x_{j+1}) \leftarrow x_j \oplus x_{j+2}$
11. Call $\mathsf{ChainQuery}(x_j, j)$ and $\mathsf{ChainQuery}(x_{j+1}, j+1)$, without considering in $\mathsf{ChainQuery}(x_j, j)$ and $\mathsf{ChainQuery}(x_{j+1}, j)$ the 3-chain that leads to 3-chain (x, y, z).

Additionally the simulator maintains an upper bound B_{max} on the size of the history of each of the F_i's; if this bound is reached, then the simulator aborts. This terminates the description of the simulator.

We note that all the lines in Table 1 are necessary to ensure that the simulation of the F_i's is coherent with what the distinguisher can obtain independently from P. For example, if we suppress the line $(F_2, +)$ in the table, the distinguisher can make a query for Z to F_4, then Y to F_3 and $X = F_3(Y) \oplus Z$ to F_2, then $A = F_4(Z) \oplus Y$ to F_5 and since it is not possible anymore to adapt the output of (F_1, F_2), the simulator fails to provide a coherent simulation. We also note that a simulator would necessarily fail for a 5-rounds LR : using the same approach as in Section 3, the distinguisher can ask $F_2(Y)$ and $F_2(Y')$, then let X, X' such that $Z = X \oplus F_2(Y) = X' \oplus F_2(Y')$ which gives $Z' = X' \oplus F_2(Y) = X \oplus F_2(Y')$; when queried for $F_4(Z)$, the simulator can adapt the answer of $F_4(Z)$ corresponding to chain (X, Y, Z), and adapt the answer of $F_2(X')$ for chain (X', Y', Z); however, when queried for $F_4(Z')$, the simulator cannot adapt the answer of $F_4(Z')$ for both chains (X, Y', Z') and (X', Y, Z'). We also note that one could have taken 12 lines in Table 1 instead of 8, by taking both directions $+$ and $-$ for each of the F_i's, and without considering "virtual" 3-chains in equations (5) and (6); however in this case it seems harder to bound the simulator's running time.

Our simulator makes recursive calls to the Query and $\mathsf{ChainQuery}$ algorithms. The simulator aborts when the history size of one of the F_i's is greater than B_{max}. Therefore we must prove that despite these recursive calls, this bound B_{max} is never reached, except with negligible probability, for B_{max} polynomial in the system parameter. The main argument is that the number of 3-chains in the sets $\mathsf{Chain}(b, x, k)$ that involve the P permutation (equations (3), (4), (5) and (6)), must be upper bounded by the number of P/P^{-1}-queries made by the distinguisher, which is upper-bounded by q. This gives an upper-bound on the number of recursive queries to F_3, F_4, which in turn implies an upper-bound on the history of the other F_i's. Additionally, one must show that the simulator never aborts at Step 8 in the $\mathsf{CompleteChain}$ algorithm, except with negligible probability. This is summarised in the following lemma:

Lemma 1. *Let q be the maximum number of queries made by the distinguisher and let $B_{max} = 5q^2$. The simulator $\mathcal{S}$ runs in time $\mathcal{O}(q^4)$, and aborts with probability at most $2^{14} \cdot q^8/2^n$, while making at most $105 \cdot q^4$ queries to P or P^{-1}.*

Proof. See Appendix A

4.2 Indifferentiability

We now proceed to prove the indifferentiability result. As illustrated in Figure 3, we must show that given the previous simulator $\mathcal{S}$, the two systems formed by (LR, F) and $(P, \mathcal{S})$ are indistinguishable.

We consider a distinguisher $\mathcal{D}$ making at most q queries and outputting a bit γ. We define a sequence $\mathtt{Game}_0$, $\mathtt{Game}_1$, ... of modified distinguisher games. In the first game $\mathtt{Game}_0$, the distinguisher interacts with the system formed by the random permutation P and the previously defined simulator $\mathcal{S}$. In the subsequent games the system is modified so that in the last game the distinguisher interacts with (LR, F). We denote by S_i the event in game i that the distinguisher outputs $\gamma = 1$.

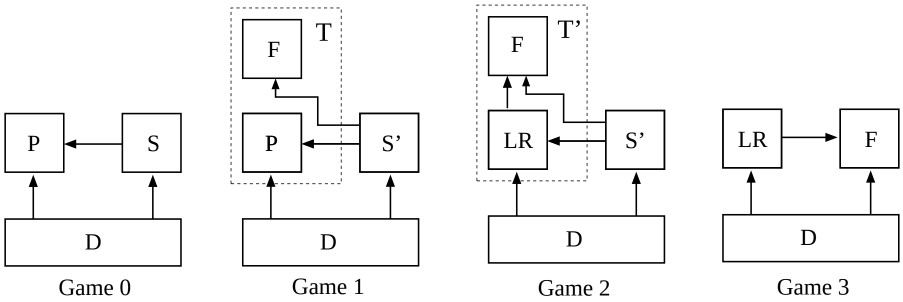


Fig. 5. Sequence of games for proving indifferentiability.

$\mathtt{Game}_0$: the distinguisher interacts with the simulator $\mathcal{S}$ and the random permutation P.

$\mathtt{Game}_1$: we make a minor change in the way F_i queries are answered by the simulator, to prepare a more important step in the next game. In $\mathtt{Game}_0$ we have that a F_i query for x can be answered in two different ways : either $F_i(x) \stackrel{\$}{\leftarrow} \{0,1\}$, or the value $F_i(x)$ is "adapted" by the simulator. In $\mathtt{Game}_1$, instead of letting $F_i(x) \stackrel{\$}{\leftarrow} \{0,1\}$, the new simulator $\mathcal{S}'$ makes a query to random oracle F_i which returns $F_i(x)$. Since we have simply replaced one set of random variables by a different, but identically distributed, set of random variables, we have:

$$\Pr[S_0] = \Pr[S_1]$$

$\mathtt{Game}_2$: we modify the way P and P^{-1} queries are answered. Instead of returning $P(L\|R)$ with random permutation P, the system returns $\mathsf{LR}(L\|R)$ by calling random oracles F_i's (and similarly for P^{-1} queries).

We must show that the distinguisher's view has statistically close distribution in $\mathtt{Game}_1$ and $\mathtt{Game}_2$. For this, we consider the subsystem $\mathcal{T}$ with random permutation P/P^{-1} and random oracle F_i's in $\mathtt{Game}_1$, and the subsystem $\mathcal{T}'$ with Luby-Rackoff LR and random oracle F_i's in $\mathtt{Game}_2$ (see Fig. 5). We show that the output of systems $\mathcal{T}$ and $\mathcal{T}'$ is statistically close; this in turn shows that the distinguisher's view has statistically close distribution in $\mathtt{Game}_1$ and $\mathtt{Game}_2$ [4].

In the following, we assume that the distinguisher eventually makes a sequence of F_i-queries corresponding to all previous P/P^{-1} queries made by the distinguisher; this is without loss of generality, because from any distinguisher $\mathcal{D}$ we can build a distinguisher $\mathcal{D}'$ with the same success probability that satisfies this property.

The outputs to F_i queries provided by subsystem $\mathcal{T}$ in $\mathtt{Game}_1$ and by subsystem $\mathcal{T}'$ in $\mathtt{Game}_2$ are the same, since in both cases these queries are answered by random oracles F_i. Therefore, we must show that the output to P/P^{-1} queries provided by $\mathcal{T}$ and $\mathcal{T}'$ have statistically close distribution, when the outputs to F_i queries provided by $\mathcal{T}$ or $\mathcal{T}'$ are fixed.

We can distinguish two types of P/P^{-1} queries to $\mathcal{T}$ or $\mathcal{T}'$:

- Type I : P/P^{-1} queries made by the distinguisher, or by the simulator during execution of the $\mathsf{CompleteChain}$ algorithm. From Lemma 1 there are at most $B_{max} + q \leq 6q^2$ such queries.
- Type II : P/P^{-1} queries made by the simulator when computing the sets $\mathsf{Chain}(+1, S, 6)$, $\mathsf{Chain}(-1, R, 1)$, $\mathsf{Chain}(+1, A, 5)$ and $\mathsf{Chain}(-1, X, 2)$, which are not of Type I. From Lemma 1 there are at most $Q_P = 105 \cdot q^4$ such queries.

[4] We do not claim that subsystems $\mathcal{T}$ and $\mathcal{T}'$ are indistinguishable for any possible sequence of queries (this is clearly false); we only show that $\mathcal{T}$ and $\mathcal{T}'$ have statistically close output for the particular sequence of queries made by the simulator and the distinguisher.

At the end of the distinguisher's queries, the CompleteChain algorithm has been executed for all 3-chains corresponding to P/P^{-1} queries of Type I. We consider one such P query $L\|R$ (the argument for P^{-1} query is similar) of Type I. In $\mathtt{Game}_2$ the answer $S\|T$ can be written as follows :

$$(S,T) = (L \oplus r_1 \oplus r_3 \oplus r_5, R \oplus r_2 \oplus r_4 \oplus r_6) \tag{7}$$

where $r_1 = F_1(R)$, $r_2 = F_2(X)$, $r_3 = F_3(Y)$, $r_4 = F_4(Z)$, $r_5 = F_5(A)$ and $r_6 = F_6(S)$, and (X, Y, Z, A) are defined in the usual way.

Let j be the index used at steps 9 and 10 of the corresponding CompleteChain execution, and let x_j, x_{j+1} be the corresponding inputs. If the simulator does not abort during CompleteChain, this implies that the values $r_j = F_j(x_j)$ and $r_{j+1} = F_{j+1}(x_{j+1})$ have not appeared before in the simulator's execution. This implies that $r_j = F_j(x_j)$ and $r_{j+1} = F_{j+1}(x_{j+1})$ have not appeared in a previous P/P^{-1}-query (since otherwise it would have been defined in the corresponding CompleteChain execution), and moreover $F_j(x_j)$ and $F_{j+1}(x_{j+1})$ have not been queried before to subsystem $\mathcal{T}'$. Since the values $r_j = F_j(x_j)$ and $r_{j+1} = F_{j+1}(x_{j+1})$ are defined by the simulator at steps 9 and 10 of CompleteChain, these values will not be queried later to $\mathcal{T}'$. Therefore we have that $r_j = F_j(x_j)$ and $r_{j+1} = F_{j+1}(x_{j+1})$ are not included in the subsystem $\mathcal{T}'$ output; $\mathcal{T}'$ output can only include randoms in $(r_1, \ldots, r_{j-1}, r_{j+2}, \ldots, r_6)$. Therefore, we obtain that for fixed randoms $(r_1, \ldots, r_{j-1}, r_{j+2}, \ldots, r_6)$, from equation (7) the distribution of $S\|T = \mathsf{LR}(L\|R)$ in $\mathtt{Game}_2$ is uniform in $\{0,1\}^{2n}$ and independent from the output of previous P/P^{-1} queries.

In $\mathtt{Game}_1$, the output to query $L\|R$ is $S\|T = P(L\|R)$; since there are at most $q + B_{max} \le 6 \cdot q^2$ Type I queries to P/P^{-1}, the statistical distance between $P(L\|R)$ and $\mathsf{LR}(L\|R)$ is at most $6 \cdot q^2/2^{2n}$. This holds for a single P/P^{-1} query of Type I. Since there are at most $6 \cdot q^2$ such queries, we obtain the following statistical distance δ between the outputs to Type I P/P^{-1} queries by system $\mathcal{T}$ and system $\mathcal{T}'$, conditioned on the event that the simulator does not abort :

$$\delta \le 6 \cdot q^2 \cdot \frac{6 \cdot q^2}{2^{2n}} \le \frac{36 \cdot q^4}{2^{2n}} \tag{8}$$

We now consider P/P^{-1} queries of Type II; from Lemma 1 there are at most $Q_P = 105 \cdot q^4$ such queries. We first consider the sets $\mathsf{Chain}(+1, S, 6)$ and $\mathsf{Chain}(-1, X, 2)$:

$$\begin{aligned}
\mathsf{Chain}(+1,S,6) &= \Big\{\ (R,X) \in (F_1,F_2) \mid \exists T,\ P\left(F_1\left(R\right) \oplus X\|R\right) = S\|T \Big\} \\
\mathsf{Chain}(-1,X,2) &= \Big\{\ (R,S) \in (F_1,\{0,1\}^n) \mid \exists T,\ P(X \oplus F_1(R)\|R) = S\|T \text{ and} \\
&\qquad (S \in F_6,\ \text{or } \exists T',\ (R',X') \ne (R,X) \in (F_1,F_2), \\
&\qquad\ P(X' \oplus F_1(R')\|R') = S\|T')\ \Big\}
\end{aligned}$$

and we consider a corresponding query $L\|R$ to P, where $L = F_1(R) \oplus X$. By definition this query is not of Type I, so no CompleteChain execution has occurred corresponding to this query. For $\mathsf{Chain}(+1, S, 6)$, given $(R,X) \in (F_1, F_2)$ we let $Y = F_2(X) \oplus R$; for $\mathsf{Chain}(-1, X, 2)$, with $R \in F_1$ we also write $Y = F_2(X) \oplus R$. If Y is not in the history of F_3, we let $F_3(Y) \xleftarrow{\$} \{0,1\}^n$; in this case, $Z = X \oplus F_3(Y)$ has the uniform distribution in $\{0,1\}^n$; this implies that Z belongs to the history of F_4 with probability at most $|F_4|/2^n \le 2q/2^n$. If Y belongs to the history of F_3, then we have that Z cannot be in the history of F_4, otherwise 3-chain (X, Y, Z) would already have appeared in CompleteChain algorithm, from Line $(F_2, +)$ and $(F_4, -)$ in Table 1. Therefore, we have that for all P queries of type II, no corresponding value of Z belongs to the history of F_4, except with probability at most $Q_P \cdot 2q/2^n$.

We now consider the sequence (L_i, R_i) of distinct P-queries of Type II corresponding to previous sets $\mathsf{Chain}(+1, S, 6)$ and $\mathsf{Chain}(-1, X, 2)$. We must show that in $\mathtt{Game}_2$ the output (S_i, T_i) provided by $\mathcal{T}'$ has a distribution that is statistically close to uniform, when the outputs to F_i queries provided by $\mathcal{T}'$ are fixed. We consider the corresponding sequence of (Y_i, Z_i); as explained previously, no Z_i belongs to the simulator's history of F_4, except with probability at most $Q_P \cdot 2q/2^n$. We claim that $F_4(Z_i) \oplus Y_i \neq F_4(Z_j) \oplus Y_j$ for all $1 \leq i < j \leq Q_P$, except with probability at most $(Q_P)^2/2^n$. Namely, if $Z_i = Z_j$ for some $i < j$, then $F_4(Z_i) \oplus Y_i = F_4(Z_j) \oplus Y_j$ implies $Y_i = Y_j$, which gives $(L_i, R_i) = (L_j, R_j)$, a contradiction since we have assumed the (L_i, R_i) queries to be distinct. Moreover, for all $i < j$ such that $Z_i \neq Z_j$, we have that $F_4(Z_i) \oplus Y_i = F_4(Z_j) \oplus Y_j$ happens with probability at most 2^{-n}; since there are at most $(Q_P)^2$ such i, j, we have that $F_4(Z_i) \oplus Y_i = F_4(Z_j) \oplus Z_j$ for some $i < j$ happens with probability at most $(Q_P)^2/2^n$.

This implies that the elements $A_i = Y_i \oplus F_4(Z_i)$ are all distinct, except with probability at most $(Q_P)^2/2^n$. Therefore elements $S_i = Z_i \oplus F_5(A_i)$ are uniformly and independently distributed in $\{0,1\}^n$; this implies that elements S_i are all distinct, except with probability at most $(Q_P)^2/2^n$, which implies that elements $T_i = A_i \oplus F_6(S_i)$ are uniformly and independently distributed in $\{0,1\}^n$. For each (S_i, T_i), the statistical distance with $P(L_i \| R_i)$ in $\texttt{Game}_1$ is therefore at most $Q_P/2^{2n}$. The previous arguments are conditioned on the event that no A_i or S_i belongs to the simulator's history for F_5 and F_6, which for each A_i or S_i happens with probability at most $B_{max}/2^n$. The reasoning for the sets $\mathsf{Chain}(-1, R, 1)$, $\mathsf{Chain}(+1, A, 5)$ is symmetric so we omit it. Therefore, the statistical distance δ_2 for the output of Type II P/P^{-1} queries between $\texttt{Game}_1$ and $\texttt{Game}_2$ is at most :

$$\delta_2 \leq 2 \cdot \left(\frac{Q_P \cdot 2q}{2^n} + 2 \cdot \frac{(Q_P)^2}{2^n} + \frac{(Q_P)^2}{2^{2n}} + \frac{Q_P \cdot B_{max}}{2^n} \right) \leq \frac{2^{16} \cdot q^8}{2^n} \tag{9}$$

Let denote by Abort the event that the simulator aborts in $\texttt{Game}_1$; we obtain from Lemma 1 and inequalities (8) and (9) :

$$|\Pr[S_2] - \Pr[S_1]| \leq \Pr[\mathsf{Abort}] + \delta + \delta_2 \leq \frac{2^{14} \cdot q^8}{2^n} + \frac{36 \cdot q^4}{2^{2n}} + \frac{2^{16} \cdot q^8}{2^n} \leq \frac{2^{17} \cdot q^8}{2^n}$$

$\texttt{Game}_3$: the distinguisher interacts with random system (LR, F). We have that system (LR, F) provides the same outputs as the system in $\texttt{Game}_2$ except if the simulator fails in $\texttt{Game}_2$. Namely, when the output values of (F_j, F_{j+1}) are adapted (steps 9 and 10 of $\mathsf{CompleteChain}$ algorithm), the values $F_j(x_j)$ and $F_{j+1}(x_{j+1})$ are the same as the one obtained directly from random oracles F_j and F_{j+1}, because in $\texttt{Game}_2$ the P/P^{-1} queries are answered using $\mathsf{LR}/\mathsf{LR}^{-1}$. Let denote by Abort_2 the event that simulator aborts in Game_2; we have :

$$\Pr[\mathsf{Abort}_2] \leq \Pr[\mathsf{Abort}] + |\Pr[S_2] - \Pr[S_1]| \leq \frac{2^{18} \cdot q^8}{2^n}$$

which gives :

$$|\Pr[S_3] - \Pr[S_2]| \leq \Pr[\mathsf{Abort}_2] \leq \frac{2^{18} \cdot q^8}{2^n}$$

From the previous inequalities, we obtain the following upper-bound on the distinguisher's advantage:

$$|\Pr[S_3] - \Pr[S_0]| \leq \frac{2^{19} \cdot q^8}{2^n}$$

which terminates the proof.

5 Conclusion

We have shown that the 6 rounds Feistel construction is indifferentiable from a random permutation, a problem that was left open in [9]. This shows that the random oracle model and the ideal cipher model are equivalent models. A natural question is whether our security bound in $q^8/2^n$ is optimal or not. One can try to obtain a better bound for 6 rounds, or to increase the number of rounds.

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A Proof of Lemma 1

We first give an upper bound on the size of the four sets $\mathsf{Chain}(+1, S, 6)$, $\mathsf{Chain}\ (-1, R, 1)$, $\mathsf{Chain}(+1, A, 5)$ and $\mathsf{Chain}(-1, X, 2)$. Given a S query to F_6, consider the set:

$$\mathsf{Chain}(+1, S, 6) = \Big\{(R, X) \in (F_1, F_2) \mid \exists T,\ P\left(F_1\left(R\right) \oplus X \| R\right) = S \| T\Big\} \tag{10}$$

If the distinguisher has not made a P query for $L\|R$, then the probability that $P(L\|R) = S\|T$ for some T is at most 2^{-n}; let denote by bad_6 this event. If the simulator has not aborted up to the i-th query made by the distinguisher, then from the simulator's definition, the history size of F_1 and F_2 is upper bounded by B_{max}. Therefore, for a single S query to F_6, the probability that bad_6 occurs is at most $(B_{max})^2/2^n$. Since there has been at most B_{max} such queries to F_6, this gives:

$$\Pr[\mathsf{bad}_6] \leq \frac{(B_{max})^3}{2^n}$$

If event bad_6 does not occur, the number of 3-chains in $\mathsf{Chain}(+1, S, 6)$ is then upper bounded by the number of P/P^{-1}-queries made by the distinguisher, which is at most q. This actually holds for the total number of 3-chains in $\mathsf{Chain}(+1, S, 6)$ and for all S queries to F_6, which must be upper bounded by q, unless event bad_6 occurs. Symmetrically, the same property holds for the set $\mathsf{Chain}(-1, R, 1)$, unless the corresponding event bad_1 occurs, with $\Pr[\mathsf{bad}_1] \leq (B_{max})^3/2^n$.

We now consider the set $\mathsf{Chain}(-1, X, 2)$:

$$\begin{aligned}\mathsf{Chain}(-1, X, 2) = \Big\{ & (R, S) \in (F_1, \{0,1\}^n) \mid \exists T,\ P(X \oplus F_1(R)\|R) = S\|T \text{ and} \\ & (S \in F_6,\ \text{ or } \exists T',\ (R', X') \neq (R, X) \in (F_1, F_2), \\ & P(X' \oplus F_1(R')\|R') = S\|T') \Big\}\end{aligned}$$

If the distinguisher has not made a query for $P(L\|R)$ where $L = X \oplus F_1(R)$, then the probability that $P(L\|R) = S\|T$ with $S \in F_6$ is at most $|F_6|/2^n$, where $|F_6|$ is the history size of F_6. Similarly, if the distinguisher has not made a query for $P(L\|R)$, then the probability that $P(L\|R) = S\|T$ where S is such that $P(X' \oplus F_1(R')\|R') = S\|T'$ for some T' and $(R', X') \in (F_1, F_2)$ is at most $|F_1| \cdot |F_2|/2^n$. Let denote by bad_2 the event that one of those two events occurs. For a single X query, event bad_2 happens with probability at most $|F_1| \cdot (|F_6|/2^n + |F_1| \cdot |F_2|/2^n)$; since there are at most B_{max} such queries, this gives :

$$\Pr[\mathsf{bad}_2] \leq B_{max} \cdot |F_1| \cdot \left(\frac{|F_6|}{2^n} + \frac{|F_1| \cdot |F_2|}{2^n} \right) \leq \frac{2 \cdot (B_{max})^4}{2^n}$$

Symmetrically, for set $\mathsf{Chain}(+1, A, 5)$, we have that the probability of corresponding event bad_5 is such that $\Pr[\mathsf{bad}_5] \leq 2 \cdot (B_{max})^4/2^n$.

Therefore, we obtain that the total number of 3-chains considered in the sets $\mathsf{Chain}(+1, S, 6)$, $\mathsf{Chain}(-1, R, 1)$, $\mathsf{Chain}(+1, A, 5)$ and $\mathsf{Chain}(-1, X, 2)$, for all S, R, A and X queries, is upper bounded by the number of P/P^{-1} queries made by the distinguisher (which is upper bounded by q), except if event $\mathsf{bad} = \mathsf{bad}_1 \vee \mathsf{bad}_2 \vee \mathsf{bad}_5 \vee \mathsf{bad}_6$ occurs, which happens with probability at most:

$$\Pr[\mathsf{bad}] \leq \frac{5(B_{max})^4}{2^n}$$

Those 3-chains (from Lines $(F_1, -)$, $(F_2, -)$, $(F_5, +)$ and $(F_6, +)$ in Table 1) are the only ones which can generate recursive calls to F_3 and F_4, since the other 3-chains (from Lines $(F_2, +)$, $(F_3, +)$, $(F_4, -)$ and $(F_5, -)$) always include elements in F_3 and F_4 histories. Since the distinguisher makes at most q queries to F_3 and F_4, the total size of F_3 and F_4 histories in the simulator is then upper-bounded by $q + q = 2 \cdot q$. The resulting number of 3-chains (corresponding to Lines $(F_2, +)$, $(F_3, +)$, $(F_4, -)$ and $(F_5, -)$) is then upper bounded by $(2q)^2 = 4q^2$; this is because those 3-chains always include elements from F_3 and F_4 history. This implies that the simulator makes at most $4q^2$ recursive queries to F_1, F_2, F_5 and F_6. Therefore, taking

$$B_{max} = 5 \cdot q^2$$

we obtain that the simulator does not reach the bound B_{max}, except if event bad has occurred, which happens with probability at most:

$$\Pr[\mathsf{bad}] \leq \frac{5(B_{max})^4}{2^n} \leq \frac{2^{12} \cdot q^8}{2^n} \tag{11}$$

We now proceed to bound the number Q_P of P/P^{-1} queries made by the simulator. The simulator makes P/P^{-1}-queries when computing the four sets $\mathsf{Chain}(+1, S, 6)$, $\mathsf{Chain}(-1, R, 1)$, $\mathsf{Chain}(+1, A, 5)$ and $\mathsf{Chain}(-1, X, 2)$ and also when completing a 3-chain with CompleteChain algorithm. From the previous bound B_{max} on the history of the F_i's, we obtain that the number of P/P^{-1}-queries made by the simulator is at most:

$$Q_P \leq 4 \cdot (B_{max})^2 + B_{max} \leq 105 \cdot q^4 \tag{12}$$

From this we have that the simulator runs in $\mathcal{O}(q^4)$ time, which completes the first part of the proof.

Additionally, we must show that the simulator never aborts at Step 8 in the CompleteChain algorithm, except with negligible probability. In the simulator, we have that a call to Query generates a call to ChainQuery, which can generate calls to CompleteChain, which recursively generate calls to Query (at Step 4) and also direct calls to ChainQuery (at Step 11), which in turn can generate recursive calls to CompleteChain.

In the following, we consider a sequence of recursive calls to CompleteChain, that do not contain intermediate calls to Query. For exemple, in a call to CompleteChain for Line $(F_2, +)$ and 3-chain (X, Y, Z), the value of $F_2(A)$ for $A = Y \oplus F_4(Z)$ is "adapted" by the simulator at Step 9, and a recursive call to $\mathsf{ChainQuery}(A, 5)$ occurs; if a 3-chain (A, Z', Y') with $(Y', Z') \in (F_3, F_4)$ exists, this will generate a recursive call to $\mathsf{CompleteChain}(-1, A, Y', Z', 5)$, which corresponds to Line $(F_5, -)$; we denote this as transition $(F_2, +) \rightarrow (F_5, -)$. In the following, we show that the only transitions that can appear with non-negligible probability are :

$$\begin{array}{ccc}
(F_5, +) & & (F_2, -) \\
\uparrow & & \uparrow \\
(F_2, +) & \longleftrightarrow & (F_5, -) \\
(F_1, -) & \longrightarrow & (F_3, +) \\
(F_6, +) & \longrightarrow & (F_4, -)
\end{array}$$

Formally, we denote by bad' the event that one of these events occurs :

1. Line $(F_1, -)$ or $(F_6, +)$ is called from a previous execution of CompleteChain.
2. Line $(F_5, +)$ is called from another CompleteChain execution than with $(F_2, +)$, or $(F_2, -)$ is called from another CompleteChain than $(F_5, -)$.
3. Line $(F_3, +)$ is called from another CompleteChain execution than $(F_1, -)$, or Line $(F_4, -)$ is called from another CompleteChain than $(F_6, +)$.
4. Line $(F_5, -)$ is called from another CompleteChain than $(F_2, +)$, or Line $(F_2, +)$ is called from another CompleteChain execution than $(F_5, -)$.
5. In Line $(F_5, -)$ with $\mathsf{CompleteChain}(-1, A, Y, Z, 5)$ the value $S = F_5(A) \oplus Z$ belongs to the history of F_6, or in Line $(F_2, +)$ with $\mathsf{CompleteChain}(+1, X, Y, Z, 2)$ the value $R = F_2(X) \oplus Y$ belongs to the history of F_1.

In the following, we consider any sequence of CompleteChain executions that does not contain an intermediate call to Query, and we show by recursion that bad' does not occur, except with negligible probability. The following Lemma shows that bad' does not occur for the first execution of CompleteChain in the sequence, except with negligible probability.

Lemma 2. *If* $\mathsf{Query}(A,5)$ *occurs, then for all* $(Y,Z) \in \mathsf{Chain}(-1,A,5)$*, the value* $S = F_5(A) \oplus Z$ *does not belong to the history of* F_6*, except with probability at most* $5q^2/2^n$*. Symmetrically, if* $\mathsf{Query}(X,2)$ *occurs, then for all* $(Y,Z) \in \mathsf{Chain}(+1,X,2)$*, the value* $R = F_2(X) \oplus Y$ *does not belong to the history of* F_1*, except with probability at most* $5q^2/2^n$*.*

Proof. If $\mathsf{Query}(A,5)$ occurs, then $F_5(A) \xleftarrow{\$} \{0,1\}^n$ occurs; then each $S = F_5(A) \oplus Z$ where $(Y,Z) \in \mathsf{Chain}(-1,A,5)$ has the uniform distribution in $\{0,1\}^n$; therefore, S does not belong to the history of F_6, except with probability at most $|F_6|/2^n \leq B_{max}/2^n \leq 5q^2/2^n$. By symmetry, the same property holds for $\mathsf{Query}(X,2)$.

□

Now we assume that event $\mathsf{bad'}$ has not occured for a given execution of $\mathsf{CompleteChain}$, and we show that $\mathsf{bad'}$ will not occur for the next recursive execution of $\mathsf{CompleteChain}$, if any. We start with $\mathsf{CompleteChain}$ for Line $(F_1,-)$ and corresponding 3-chain (R,A,S); for Line $(F_1,-)$ the "adapt" operation occurs for F_2 and F_3. Since $\mathsf{bad'}$ has not occured, $\mathsf{CompleteChain}$ for Line $(F_1,-)$ must originate from a $\mathsf{Query}(R,1)$ call. Therefore, $F_1(R) \xleftarrow{\$} \{0,1\}^n$ has occured, which implies that $X = F_1(R) \oplus L$ has the random distribution in $\{0,1\}^n$. This implies that no new 3-chain is generated from X, and Lines $(F_2,+)$ and $(F_2,-)$ are not called, except with negligible probability. Namely, the probability that there exists $(Y',Z') \in (F_3,F_4)$ such that $X = Y' \oplus F_2(Z')$ is at most $|F_3| \cdot |F_4|/2^n$, so Line $(F_2,+)$ gets recursively called with probability at most $|F_3| \cdot |F_4|/2^n$. Similarly, the probability that there exists $(R',S') \in (F_1,F_6)$ such that $P(F_1(R') \oplus X \| R') = S' \| T'$ for some T' is at most $|F_1| \cdot |F_6|/2^n$, and the probability that there exists $(R',R'',X'') \in (F_1,F_1,F_2)$ with $(R',X) \neq (R'',X'')$ and $P(F_1(R') \oplus X \| R') = S' \| T'$ and $P(F_1(R'') \oplus X'' \| R'') = S' \| T''$ for some T',T'', is at most $|F_1|^2 \cdot |F_2|/2^n$. Therefore, event $\mathsf{bad'}$ does not occur for a recursive execution of $\mathsf{CompleteChain}$, if any, except with probability at most $(|F_3| \cdot |F_4| + |F_1| \cdot |F_6| + |F_1|^2 \cdot |F_2|)/2^n \leq 2^7 \cdot q^6/2^n$.

We now consider $\mathsf{CompleteChain}$ for Line $(F_2,+)$ and 3-chain (X,Y,Z). For this line, the "adapt" operation occurs for F_5 and F_6. Since $\mathsf{bad'}$ has not occured, we have that $R = F_2(X) \oplus Y$ does not belong to the history of F_1; therefore $F_1(R) \xleftarrow{\$} \{0,1\}^n$ occurs and $L = F_1(R) \oplus X$ has the uniform distribution in $\{0,1\}^n$, which implies that S in $S \| T = P(L \| R)$ has the uniform distribution in $\{0,1\}^n$; therefore, Line $(F_6,+)$ cannot get recursively called, except with probability at most $|F_1| \cdot |F_2|/2^n$; moreover, we have that $F_5(A) = Z \oplus S$ has the uniform distribution in $\{0,1\}^n$; therefore, if $\mathsf{CompleteChain}$ gets recursively called for Line $(F_5,-)$ and 3-chain (A,Z',Y'), then $S' = F_5(A) \oplus Z'$ has the uniform distribution in $\{0,1\}^n$, which implies that S' does not belong to the history of F_6, except with probability at most $|F_6|/2^n$. This implies that event $\mathsf{bad'}$ does not occur for a recursive call to $\mathsf{CompleteChain}$, if any, except with probability at most $(|F_1| \cdot |F_2| + |F_6|)/2^n \leq 2^5 \cdot q^4/2^n$.

We now consider Line $(F_2,-)$ and corresponding 3-chain (X,R,S); for this line, the "adapt" operation occurs for F_3 and F_4. Since event $\mathsf{bad'}$ has not occured, we have that either the $\mathsf{CompleteChain}$ call comes from $\mathsf{Query}(X,2)$, or it comes from another $\mathsf{CompleteChain}$ with $(F_5,-)$. In the first case, $F_2(X) \xleftarrow{\$} \{0,1\}^n$ has occured, so $Y = F_2(X) \oplus R$ has the random distribution, and as previously, Line $(F_3,+)$ cannot get called, except with probability at most $|F_4| \cdot |F_5|/2^n$. In the second case, let denote by (A',Z',Y') the corresponding 3-chain for $\mathsf{CompleteChain}$ with $(F_5,-)$; since $\mathsf{bad'}$ has not occured, we have that $S' = F_5(A') \oplus Z'$ did not belong to the history of F_6; therefore $F_6(S') \xleftarrow{\$} \{0,1\}^n$ occured, which implies that R' in $L' \| R' = P^{-1}(S' \| F_6(S') \oplus A')$ is uniformly distributed in $\{0,1\}^n$, which implies that $F_2(X) = R' \oplus Y'$ is also uniformly distributed in $\{0,1\}^n$; as previously, this implies that $Y = F_2(X) \oplus R$ has the random distribution, and Line $(F_3,+)$ cannot get called, except with probability at most $|F_4| \cdot |F_5|/2^n$. Moreover, we have that $A = T \oplus F_6(S)$ with $S \| T = P(X \oplus F_1(R) \| R)$ was not in the history of F_5 when $\mathsf{CompleteChain}(-1, X, R, S, 2)$ was called (since otherwise Line $(F_5, +)$ would have been called before for 3-chain (A, S, R)); therefore $F_5(A) \xleftarrow{\$} \{0,1\}^n$ occured and $Z = S \oplus F_5(A)$ is uniformly distributed in $\{0,1\}^n$, which implies that Line $(F_4, -)$ does not get called, except with probability at most $|F_2| \cdot |F_3|/2^n$. Therefore, we obtain that event bad' does not occur for a recursive execution of CompleteChain, if any, except with probability at most $(|F_2| \cdot |F_3| + |F_4| \cdot |F_5|)/2^n \leq 2^5 \cdot q^3/2^n$.

We now consider CompleteChain for Line $(F_3, +)$ and 3-chain (Y, Z, A); for this line, the "adapt" operation occurs for F_1 and F_2; since event bad' has not occured for this line, we have that either $F_3(Y) \xleftarrow{\$} \{0,1\}^n$ occured, or CompleteChain for Line $(F_1, -)$ occured previously. In the first case, we have that $X = Z \oplus F_3(Y)$ has the random distribution in $\{0,1\}^n$, which implies that as previously no recursive call can be generated for $(F_2, +)$ or $(F_2, -)$, except with probability at most $2^7 \cdot q^6/2^n$. In the second case, let (R', S', A') be the corresponding 3-chain for $(F_1, -)$, and let $L' \| R' = P^{-1}(S' \| A' \oplus F_6(S'))$ and $Z' = F_5(A') \oplus S'$; we have $Y = A' \oplus F_4(Z')$. Since event bad' has not occured, we have that $F_1(R') \xleftarrow{\$} \{0,1\}^n$ occured, which implies that $X' = L' \oplus F_1(R')$ has the uniform distribution in $\{0,1\}^n$. Therefore, $F_3(Y) = X' \oplus Z'$ has the uniform distribution in $\{0,1\}^n$; this implies that $X = Z \oplus F_3(Y)$ has the uniform distribution, which as previously implies that no recursive call can be generated for $(F_2, +)$ or $(F_2, -)$, except with probability at most $2^7 \cdot q^6/2^n$. Moreover, we have that no recursive call to CompleteChain can occur for Line $(F_1, -)$; namely, denoting $S = Z \oplus F_5(A)$ and $L \| R = P^{-1}(S \| A \oplus F_6(S))$, if a recursive call to CompleteChain occurs for 3-chain (R, S'', A'') for some $(A'', S'') \in (F_5, F_6)$, this would give two 3-chains (R, S, A) and (R, S'', A'') and Line $(F_5, -)$ would have been called previously. Therefore, we obtain that event bad' does not occur for a recursive call to CompleteChain, if any, except with probability at most $2^7 \cdot q^6/2^n$.

The analysis for the remaining four lines of Table 1 is symmetric, so we omit it. Since each Line of Table 1 gets called at most $B_{max} = 5 \cdot q^2$ times, we obtain :

$$\Pr[\mathsf{bad}'] \leq 2 \cdot B_{max} \cdot \left(\frac{2^7 \cdot q^6 + 2^5 \cdot q^4 + 2^5 \cdot q^3 + 2^7 \cdot q^6}{2^n} \right) \leq \frac{2^{12} \cdot q^8}{2^n} \tag{13}$$

Now we proceed to show that CompleteChain never aborts at step 8, except with negligible probability. We assume that event bad' has not occured.

We first consider CompleteChain being called for Line $(F_1, -)$ in Table 1, and denote by (R, A, S) the corresponding 3-chain. We have that Line $(F_1, -)$ cannot have been called previously for the same R but with a different 3-chain (R, A', S'); otherwise this would give two 3-chains (R, A, S) and (R, A', S') for the same R; then Line $(F_5, +)$ would already have been called for 3-chain (R, A, S) or (R, A', S'), a contradiction. Since event bad' has not occured, we have that this call to CompleteChain comes from $\mathsf{Query}(R, 1)$ execution. Therefore, we have that operation $F_1(R) \xleftarrow{\$} \{0,1\}^n$ has occured; therefore, $X = F_1(R) \oplus L$ is in the history of F_2 with probability at most $|F_2|/2^n$. Moreover, if $Z = F_5(A) \oplus S$ is in the history of F_4, then $Y = F_4(Z) \oplus A$ cannot be in the history of F_3, otherwise from Lines $(F_3, +)$ and $(F_5, -)$ in Table 1 the 3-chain (Y, Z, A) would already have been completed, and R would already be in the history of F_1. Alternatively, if Z is not in the history of F_4, then the operation $F_4(Z) \xleftarrow{\$} \{0,1\}^n$ occurs, which implies that $Y = F_4(Z) \oplus A$ is in the history of F_3 with probability at most $|F_3|/2^n$. Therefore, we obtain that X is not in the history of F_2 and Y is not in the history of F_3, except with probability at most $(|F_2| + |F_3|)/2^n$; from the previous bound, this probability is at most $(B_{max} + 2q)/2^n \leq 6 \cdot q^2/2^n$. This implies that the simulator will not abort at Step 8 in CompleteChain for Line $(F_1, -)$, except with probability at most $6 \cdot q^2/2^n$.

We now proceed with a similar analysis for CompleteChain when called for the other lines in Table 1. We consider the line $(F_2, +)$, and let (X, Y, Z) the corresponding 3-chain. Since event bad' has not occured, we have that $R = Y \oplus F_2(X)$ does not belong to the history of F_1. Therefore $F_1(R) \xleftarrow{\$} \{0,1\}^n$ occurs, and therefore $L = F_1(R) \oplus X$ has the uniform distribution in $\{0,1\}^n$; this implies that S in $S\|T = P(L\|R)$ is uniformly distributed in $\{0,1\}^n$; therefore S is not in the history of F_6, except with probability at most $|F_6|/2^n$. Moreover, we can't have that $A = F_4(Z) \oplus Y$ is in the history of F_5, otherwise the 3-chain (Y, Z, A) would already have been completed (from Lines $(F_3, +)$ and $(F_5, -)$ in Table 1), and X would already be in the history of F_2. Therefore, we obtain that A is not in the history of F_5 and S is not in the history of F_6, except with probability at most $|F_6|/2^n \leq 5 \cdot q^2/2^n$.

We now consider the line $(F_2, -)$ with query X to F_2 and 3-chain (X, R, S). Since event bad' has not occured, from the previous analysis we have that $Y = F_2(X) \oplus R$ has the random distribution, and so Y belongs to the history of F_3 with probability at most $|F_3|/2^n$. From the previous analysis, we also have that $Z = S \oplus F_5(A)$ is uniformly distributed, which implies that Z belongs to the history of F_4 with probability at most $|F_4|/2^n$. Therefore, the simulator aborts at Step 8 with probability at most $(|F_3| + |F_4|)/2^n \leq 4q/2^n$.

We now consider the line $(F_3, +)$, with (Y, Z, A) the corresponding 3-chain. Since event bad' has not occured, we have that either $\mathsf{Query}(Y, 3)$ occured, or $\mathsf{CompleteChain}$ with Line $(F_1, -)$ occured. In the first case, we have that $F_3(Y) \xleftarrow{\$} \{0,1\}^n$ occured, which implies that $X = F_3(Y) \oplus Z$ has the uniform distribution, which implies that X does not belong to the history of F_2, except with probability at most $|F_2|/2^n$. In the second case, let (R', A', S') be the corresponding 3-chain for Line $(F_1, -)$. Let denote $Z' = S' \oplus F_5(A')$ and $L'\|R' = P^{-1}(S'\|A' \oplus F_6(S'))$. Since 3-chain (R', A', S') has generated a recursive call to $\mathsf{CompleteChain}$ with line $(F_3, +)$ and 3-chain (Y, Z, A), we have $Y = A \oplus F_4(Z) = A' \oplus F_4(Z')$. Moreover we must have $(Z, A) \neq (Z', A')$, which implies from the previous equality that $Z \neq Z'$. From the previous analysis, we have that $F_1(R') \xleftarrow{\$} \{0,1\}^n$ occured, and Line $(F_1, -)$ was called only once for R'. For Line $(F_1, -)$ the "adapt" operation occured for $F_2(X')$ and $F_3(Y)$, where $X' = L' \oplus F_1(R')$, with $F_3(Y) = X' \oplus Z'$. Therefore, $F_3(Y) = L' \oplus Z' \oplus F_1(R')$ has the uniform distribution in $\{0,1\}^n$; this implies that $X = F_3(Y) \oplus Z$ has the uniform distribution in $\{0,1\}^n$; moreover, with $Z \neq Z'$ we have that $X \neq X'$, which implies that X does not belong to the history of F_2, except with probability at most $|F_2|/2^n$. Moreover, R in $L\|R = P^{-1}(S\|A \oplus F_6(S))$ cannot be in the history of F_1, since otherwise Line $(F_5, +)$ or $(F_1, -)$ would already have been called for 3-chain (A, S, R), and Y would already be in the history of F_3. Therefore, the simulator does not abort at Step 8, except with probability at most $|F_2|/2^n \leq 5 \cdot q^2/2^n$.

The analysis for the four remaining lines of Table 1 is completely symmetric, so we omit it. Let denote by Abort_8 the event that the simulator aborts at Step 8 in the $\mathsf{CompleteChain}$ algorithm. Since each line gets called at most $B_{max} = 5q^2$ times, we obtain :

$$\Pr[\mathsf{Abort}_8] \leq \Pr[\mathsf{bad}'] + 2 \cdot B_{max} \cdot \frac{6 \cdot q^2 + 5 \cdot q^2 + 4 \cdot q + 5 \cdot q^2}{2^n} \leq \frac{2^{13} \cdot q^8}{2^n}$$

Let denote by Abort the event that the simulator aborts; this occurs if the simulator aborts at step 8 of $\mathsf{CompleteChain}$, or if the bound B_{max} is reached (event bad). Using inequality (11), this gives :

$$\Pr[\mathsf{Abort}] \leq \Pr[\mathsf{Abort}_8] + \Pr[\mathsf{Bad}] \leq \frac{2^{14} \cdot q^8}{2^n}$$

which terminates the proof of Lemma 1.

B A Note on Indifferentiability in the Honest-but-curious Model

In this section, we show that LR with up to logarithmic number of rounds is *not* indifferentiable from a random permutation in the honest-but-curious model [11]; combined with our main result in the general model, this provides a separation between the two models. Note that this observation does not contradict any result formally proven in [11]; it only shows that honest-but-curious indifferentiability is not necessarily weaker than general indifferentiability.

Roughly speaking, in the honest-but-curious indifferentiability model, the distinguisher can not query the F_i's directly. It can only make two types of queries: direct queries to the LR/LR^{-1} construction, and queries to the LR/LR^{-1} construction where in addition the intermediate results of the F_i's is provided. When interacting with the random permutation P and a simulator $\mathcal{S}$, the first type of query is sent directly to P, while the second type is sent to $\mathcal{S}$ who makes the corresponding query to P, and in addition provides a simulated transcript of intermediate F_i results. Note that the simulator $\mathcal{S}$ is not allowed to make additional queries to P apart from forwarding the queries from the distinguisher; see [11] for a precise definition.

The authors of [11] define the notion of a *transparent construction*. Roughly speaking, this is a construction C^F such that the value of random oracle $F(x)$ can be computed efficiently for any x, by making a polynomial number of queries to C^F and getting the F outputs used by C^F to answer each query. The authors show that Luby-Rackoff with up to logarithmic number of rounds is a transparent construction. Namely the authors construct an extracting algorithm E such that when given oracle access to LR and the intermediate values F_i used to compute LR, the value $F_i(x)$ can be computed for any x at any round i. We note that algorithm E does not make queries to LR^{-1}, only to LR.

Algorithm E implies that for a LR construction with up to logarithmic number of rounds, it is possible to find an input message $L\|R$ such that the value S in $S\|T = \mathsf{LR}(L\|R)$ has a predetermined value, by only making forward queries to LR; namely this is how algorithm E can obtain $F_\ell(S)$, where ℓ is the last round. But this task is clearly impossible with a random permutation P : it is infeasible to find $L\|R$ such that S in $S\|T = P(L\|R)$ has a pre-determined value while only making forward queries to P. This implies that a simulator in the honest-but-curious model will necessarily fail (recall that such simulator only forwards queries from the distinguisher to P and cannot make his own queries to P/P^{-1}). Therefore, LR with up to logarithmic number of rounds is *not* indifferentiable from a random permutation in the honest-but-curious model. Since our main result is that LR with 6 rounds is indifferentiable from a random permutation in the general model, this provides a separation between the two models.