

# Attacking Step Reduced SHA-2 Family in a Unified Framework

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**Abstract.** In this work, we make a detailed analysis of local collisions and their applicability to obtain collisions for step reduced SHA-2 hash family. Our analysis explains previously reported collisions for up to 22-step SHA-2 hash functions with probability one.

We provide new and improved attacks against 23 and 24-step SHA-256 using a local collision given by Sanadhya and Sarkar (SS) at ACISP '08. The computational efforts for the 23-step and 24-step attacks are respectively  $2^{12.5}$  and  $2^{28.5}$  calls to the corresponding step reduced SHA-256. Using a look-up table having  $2^{32}$  entries the computational effort for finding 24-step collisions can be reduced to  $2^{14.5}$  calls. We exhibit colliding message pairs for both the 23 and 24-step attacks. The previous work on 23 and 24-step SHA-256 attacks is due to Indestege et al. and utilizes the local collision presented by Nikolić and Biryukov (NB) at FSE '08. The reported computational efforts are  $2^{18}$  and  $2^{28.5}$  respectively.

Building on our analysis, we describe the first collision attacks against 25, 26 and 27-step SHA-256 which succeed with efforts  $2^{48}$ ,  $2^{80}$  and  $2^{112}$  calls to the respective step-reduced SHA-256 and using a look-up table having  $2^{32}$  entries. Both the NB and the SS local collision can be used to obtain these attacks. While these results improve and extend all existing reduced round SHA-2 attacks, they do not threaten the security of the full SHA-2 family.

**Keywords:** SHA-2 family, reduced round collisions, cryptanalysis.

## 1 Introduction

Cryptanalysis of SHA-2 family has recently gained momentum due to the important work of Nikolić and Biryukov [5]. Prior work on finding collisions for step reduced SHA-256 was done in [3,4] and [8]. These earlier works used local collisions valid for the XOR linearized version of SHA-256 from [1] and [7]. On the other hand, the work [5] used a local collision which is valid for the actual SHA-256.

The authors in [5] developed techniques to handle nonlinear functions and the message expansion of SHA-2 to obtain collisions for up to 21-step SHA-256. The 21-step attack of [5] succeeded with probability  $2^{-19}$ . Using similar techniques, but utilizing a different local collision, [10] showed an attack against 20-step SHA-2 which succeeds with probability one and an attack against 21-step SHA-256 which succeeds with probability  $2^{-15}$ . Further work [9,6] developed collision attacks against 21 and 22 step SHA-2 family which succeed with probability one. Very recently, Indestege et al. [2] have developed attacks against 23 and 24 step SHA-256. They utilize the local collision from [5] in these attacks.

**OUR CONTRIBUTIONS.** We consider all 9-step local collisions using additive differentials in a general and unified framework. Using a combinatorial analysis, the local collisions from [5] and [10] are obtained as special cases. We show how the general analysis can be used to explain recently obtained collisions for up to 22-step SHA-2 with probability one.

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Collisions for 23 and 24-step SHA-256 are described. Our analysis shows that both the NB and the SS local collisions can be used for this purpose. The 23-step collision using the SS local collision requires  $2^{12.5}$  calls to 23-step SHA-256. This improves upon the previously reported effort of  $2^{18}$  calls obtained in [2] using the NB local collision.

The computational effort for the 24-step collisions is  $2^{28.5}$  calls to 24-step SHA-256. While this equals the previously reported computational effort [2], we present complete details of our method including a guess-then-verify algorithm to solve a nonlinear equation arising in the analysis. This equation can also be solved using a table look-up. This brings down the computational cost of obtaining 24-step SHA-256 collisions to  $2^{14.5}$  while requiring a look-up table having  $2^{32}$  entries, where each entry consists of 8 bytes.

Examples of 23 and 24-step collisions are presented. For the case of 23-step collisions, two examples are presented. The first one uses a local collision from Steps 8 to 15, while the second one uses a local collision from Steps 9 to 16. The second 23-step collision can be seen as a simplified application of the technique for obtaining 24-step collisions.

The work in [2] describes 23 and 24-step collisions as a two-part procedure; first obtain a pseudo-collision and then convert it into a collision. In contrast, our analysis is direct and shows that such a two-part description is unnecessary.

A consequence of our detailed analysis allows us to extend the 24-step collision to describe 25, 26 and 27-step attacks. Our description of these attacks use the SS local collision and a similar analysis can also be used with the NB local collision. The computational efforts are  $2^{48}$ ,  $2^{80}$  and  $2^{112}$  calls to the respective step-reduced SHA-256. Additionally, the above mentioned look-up table will be required. These are the first reported attacks against 25, 26 and 27-step SHA-256. In contrast, we note that [2] mentions that it is *not* possible to extend their technique to attack 25-step SHA-256. A summary of results on collision attacks against reduced SHA-2 family is given in Table 1.

**Table 1.** Summary of results against reduced SHA-2 family. Effort is expressed as either the probability of success or as the number of calls to the respective reduced round hash function.

Work	Hash Function	Steps	Effort		Local Collision utilized	Attack Type
			Prob.	Calls		
[3,4]	SHA-256	18		*	GH [1]	Linear
[8]	SHA-256	18		**	SS <sub>5</sub> [7]	"
[5]	SHA-256	20	$\frac{1}{3}$		NB [5]	Non-linear
		21	$2^{-19}$		"	"
[10]	SHA-256/SHA-512	18,20	1	1	SS [10]	"
	SHA-256	21	$2^{-15}$		"	"
[9]	SHA-256/SHA-512	21	1	1	"	"
[6]	SHA-256/SHA-512	22	1	1	"	"
[2]	SHA-256	23		$2^{18}$	NB [5]	"
		24		$2^{28.5}$	"	"
This work	SHA-256	23		$2^{12.5}$	SS [10]/NB [5]	"
		24		$2^{28.5}$	"	"
		24		$2^{14.5}$ †	"	"
		25		$2^{48}$ †	"	"
		26		$2^{80}$ †	"	"
		27		$2^{112}$ †	"	"

\* It is mentioned in [3,4] that the effort is  $2^0$  but no details are provided.

\*\* Effort is given as running a C-program for about 30–40 minutes on a standard PC.

† A table containing  $2^{32}$  entries, each entry of size 8 bytes, is required.

## 2 Preliminaries

In this paper we use the following notation:

- Message words:  $W_i \in \{0, 1\}^n$ ,  $W'_i \in \{0, 1\}^n$ ;  $n$  is 32 for SHA-256 and 64 for SHA-512.
- Colliding message pair:  $\{W_0, W_1, W_2, \dots, W_{15}\}$  and  $\{W'_0, W'_1, W'_2, \dots, W'_{15}\}$ .
- Expanded message pair:  $\{W_0, W_1, W_2, \dots, W_{N-1}\}$  and  $\{W'_0, W'_1, W'_2, \dots, W'_{N-1}\}$ .  
The number of steps  $N$  is 64 for SHA-256 and 80 for SHA-512.
- The internal registers for the two messages at step  $i$ :  $\text{REG}_i = \{a_i, \dots, h_i\}$  and  $\text{REG}'_i = \{a'_i, \dots, h'_i\}$ .
- $\text{ROTR}^k(x)$ : Right rotation of an  $n$ -bit string  $x$  by  $k$  bits.
- $\text{SHR}^k(x)$ : Right shift of an  $n$ -bit string  $x$  by  $k$  bits.
- $\oplus$ : bitwise XOR;  $+$ ,  $-$ : addition and subtraction modulo  $2^n$ .
- $\delta X = X' - X$  where  $X$  is an  $n$ -bit quantity.

### 2.1 SHA-2 Hash Family

Eight registers are used in the evaluation of SHA-2. In Step  $i$ , the 8 registers are updated from  $(a_{i-1}, b_{i-1}, c_{i-1}, d_{i-1}, e_{i-1}, f_{i-1}, g_{i-1}, h_{i-1})$  to  $(a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i)$ . For more details, see Section B.

By the form of the round update function, we have the following relation.

#### Cross Dependence Equation (CDE).

$$e_i = a_i + a_{i-4} - \Sigma_0(a_{i-1}) - f_{MAJ}(a_{i-1}, a_{i-2}, a_{i-3}). \quad (1)$$

Later, we make extensive use of this relation.

## 3 A General Non-Linear Differential Path

We use a differential technique to find a 9-round local collision. The idea is to use modular differentials which was first used for SHA-2 by Nikolić and Biryukov [5]. Given a word  $w$ , we define

$$x = -\delta \Sigma_0^i(w) - \delta f_{MAJ}(w, 0, 0); \quad y = -\delta f_{MAJ}^{i+1}(0, w, 0); \quad z = -\delta f_{MAJ}^{i+2}(0, 0, w). \quad (2)$$

The general differential path and corresponding message differences are shown in Table 2.

The important thing to note about the differential path shown in Table 2 is that it puts no restrictions on the actual message words  $W_i, \dots, W_{i+8}$  and  $W'_i, \dots, W'_{i+8}$ . Starting at any value for the registers  $a$  to  $h$ , and using any given non-zero  $w$ , and any  $W_i, \dots, W_{i+8}$ , we simply run the compression function step-by-step and define the words  $x, y, z$ , the respective  $\delta W_i$ s and consequently the respective  $W'_i$ s. All the steps are deterministic and hence with probability one, we obtain  $W'_i$ s which collide with  $W_i$ s.

The expression for  $x$  has the term  $\delta \Sigma_0^i(w)$ . This map is invariant on 0 and  $-1$ . Consequently, by setting either  $(w = 1$  and  $a_i = -1)$  or  $(w = -1$  and  $a_i = 0)$ , we can achieve  $\delta \Sigma_0^i(w) = w$ . The condition  $(w = -1$  and  $a_i = 0)$  is the dual of the condition  $(w = 1$  and  $a_i = -1)$  in the following sense. We have defined  $\delta X = X' - X$  and so  $\delta W_i = w$  means  $W'_i = W + w$ ; if we had defined  $\delta X$  to be  $X - X'$ , then  $W'_i$  would have been  $W - w$ . Consequently, without loss of generality one can ignore the case  $-w$ . Majority function can be simplified by ensuring that two of its inputs are equal. This rule can be used to simplify the expressions of  $x, y$  and  $z$  to values such as 0 or  $-1$ .

**The Nikolić-Biryukov (NB) local collision [5].** A special case of Table 2 is obtained by putting  $(w, x, y, z) = (1, -1, 0, 0)$ . This was the first reported local collision for SHA-2 using modular differentials [5]. We will call this the NB local collision.

**Table 2.** General 9-step nonlinear local collision for SHA-256.

Differential Path										Message Word Differences
Step $i$	$\delta W_i$	$\delta a_i$	$\delta b_i$	$\delta c_i$	$\delta d_i$	$\delta e_i$	$\delta f_i$	$\delta g_i$	$\delta h_i$	
$i-1$	0	0	0	0	0	0	0	0	0	$\delta W_i = w;$
$i$	$w$	$w$	0	0	0	$w$	0	0	0	$\delta W_{i+1} = x - \delta \Sigma_1^i(w) - \delta f_{IF}^i(w, 0, 0);$
$i+1$	$\delta W_{i+1}$	0	$w$	0	0	$x$	$w$	0	0	$\delta W_{i+2} = y - \delta \Sigma_1^{i+1}(x) - \delta f_{IF}^{i+1}(x, w, 0);$
$i+2$	$\delta W_{i+2}$	0	0	$w$	0	$y$	$x$	$w$	0	$\delta W_{i+3} = z - \delta \Sigma_1^{i+2}(y) - \delta f_{IF}^{i+2}(y, x, w);$
$i+3$	$\delta W_{i+3}$	0	0	0	$w$	$z$	$y$	$x$	$w$	$\delta W_{i+4} = -w - \delta \Sigma_1^{i+3}(z) - \delta f_{IF}^{i+3}(z, y, x);$
$i+4$	$\delta W_{i+4}$	0	0	0	0	$w$	$z$	$y$	$x$	$\delta W_{i+5} = -x - \delta \Sigma_1^{i+4}(w) - \delta f_{IF}^{i+4}(w, z, y);$
$i+5$	$\delta W_{i+5}$	0	0	0	0	0	$w$	$z$	$y$	$\delta W_{i+6} = -y - \delta f_{IF}^{i+5}(0, w, z);$
$i+6$	$\delta W_{i+6}$	0	0	0	0	0	0	$w$	$z$	$\delta W_{i+7} = -z - \delta f_{IF}^{i+6}(0, 0, w);$
$i+7$	$\delta W_{i+7}$	0	0	0	0	0	0	0	$w$	$\delta W_{i+8} = -w.$
$i+8$	$\delta W_{i+8}$	0	0	0	0	0	0	0	0	

**The Sanadhya-Sarkar (SS) Differential Path [10].** The case from Table 2 obtained by putting  $z = 0$  was studied in [10]. The special case of  $(w, x, y, z) = (1, -1, -1, 0)$  turned out to be important in [10] and it is also important in the current work. We will call this the SS local collision.

After simplifying the values of  $x, y$  and  $z$ , it is possible to simplify the expressions for some of the  $\delta W_i$ s. We are interested in obtaining conditions to ensure that some of the  $\delta W_i$ s are zero. For example, the condition  $z = 0$  and  $e_{i+6} = -1$  ensures  $\delta W_{i+7} = 0$ ; while the condition ( $e_{i+5} = 0$  and  $y = -z$ ) or ( $e_{i+5} = -1$  and  $y = -w$ ) ensures  $\delta W_{i+6} = 0$ . The conditions for achieving  $\delta W_{i+4} = \delta W_{i+5} = 0$  are a little more complicated. The complete details for the simplification of  $\delta \Sigma_0$ ; Majority and the  $\delta W_i$ s are given in Section C.

### 3.1 Obtaining up to 22-Round Collisions

For obtaining collisions with more than 16 rounds, we need to consider the message expansion. The initial free words are  $W_0, \dots, W_{15}$  and from  $W_{16}$  onwards, the words are computed using the message expansion recursion given by (14). For clarity some initial words are shown in Table 3.

**Table 3.** Message expansion from  $W_{16}$  to  $W_{26}$ .

$W_{16} = \sigma_1(W_{14}) + W_9 + \sigma_0(W_1) + W_0$
$W_{17} = \sigma_1(W_{15}) + W_{10} + \sigma_0(W_2) + W_1$
$W_{18} = \sigma_1(W_{16}) + W_{11} + \sigma_0(W_3) + W_2$
$W_{19} = \sigma_1(W_{17}) + W_{12} + \sigma_0(W_4) + W_3$
$W_{20} = \sigma_1(W_{18}) + W_{13} + \sigma_0(W_5) + W_4$
$W_{21} = \sigma_1(W_{19}) + W_{14} + \sigma_0(W_6) + W_5$
$W_{22} = \sigma_1(W_{20}) + W_{15} + \sigma_0(W_7) + W_6$
$W_{23} = \sigma_1(W_{21}) + W_{16} + \sigma_0(W_8) + W_7$
$W_{24} = \sigma_1(W_{22}) + W_{17} + \sigma_0(W_9) + W_8$
$W_{25} = \sigma_1(W_{23}) + W_{18} + \sigma_0(W_{10}) + W_9$
$W_{26} = \sigma_1(W_{24}) + W_{19} + \sigma_0(W_{11}) + W_{10}$

The basic technique is to place a single local collision from steps  $i$  to  $i + 8$  for a suitably chosen  $i$  and then ensure that message expansion does not interfere with this collision. All  $\delta W_j$  with  $0 \leq j \leq 15$

and  $j \notin \{i, i+1, \dots, i+8\}$  are set to 0. Doing this does not place any constraint on either  $W_0, \dots, W_{15}$  or on  $W'_0, \dots, W'_{15}$ .

Additionally, some  $\delta W_k$  with  $k \in \{i, i+1, \dots, i+8\}$  are also set to 0. This requires setting some of the  $a_i$ s and  $e_j$ s to specific values. This is achieved by setting the message word for the corresponding round to a specific value. Note, however, that using a single message word we cannot set both  $a$  and  $e$  registers to desired values. Also, by the CDE (Equation 1), fixing  $a_{i-4}$  to  $a_i$  sets  $e_i$  to a fixed value.

**18-Round Collisions [10].** Deterministic 18-round collisions are easy to obtain by setting  $i = 3$  (i.e., the local collision spans from  $i = 3$  to  $i + 8 = 11$ ) and ensuring  $\delta W_{i+6}$  and  $\delta W_{i+7}$  are both zeros. For example, if  $y = z = 0$ , then the setting  $e_{i+5} = 0$  and  $e_{i+6} = -1$  ensures  $\delta W_{i+6} = \delta W_{i+7} = 0$  for any choice of  $w$ .

**20-Round Collisions [5,10].** Deterministic 20-round collisions can be obtained by setting  $i = 5$  (i.e., the local collision spans from  $i = 5$  to  $i + 8 = 13$ ) and ensuring  $\delta W_{i+4} = \delta W_{i+5} = \delta W_{i+6} = \delta W_{i+7} = 0$ . Details of how this can be done is described in Section D.1.

**21-Round Collisions [9].** Nikolić-Biryukov show how to find local collisions for SHA-256 with probability  $2^{-19}$  using  $(w, x, y, z) = (1, -1, 0, 0)$ . This was improved to probability  $2^{-15}$  using  $(w, x, y, z) = (1, -1, -1, 0)$  in [10]. But, none of these works could provide 21-round SHA-512 collisions.

In a later work [9], deterministic 21-round collisions were shown for both SHA-256 and SHA-512 using  $(w, x, y, z) = (1, -1, -1, 0)$ . In [9], it was also shown that the NB local collision, i.e.,  $(w, x, y, z) = (1, -1, 0, 0)$  is unlikely to produce 21-step SHA-512 collisions. Some details of the deterministic 21-round collisions are given in Section D.2.

**22-Round Collisions [6].** The technique of [9] was extended to obtain deterministic 22-round collisions in [6]. Some details of how this can be done is given in Section D.3. Again, it is unlikely that the NB local collision can be used to produce 22-step SHA-512 collisions [6].

## 4 A General Idea for Obtaining 23 to 27-Round Collisions

Obtaining deterministic collisions up to 22 rounds did not require the (single) local collision to extend beyond step 15. For obtaining collisions for more number of rounds, we will need to start the local collision at Step 8 (or farther) and hence the local collision will end at Step 16 (or farther). This will require us to analyze the message expansion more carefully.

For obtaining collisions up to 22 rounds, we also needed to consider message expansion. But, following Nikolić-Biryukov, we ensured that there were no differences in message words from Step 16 onwards. However, now that we consider the local collision to end at Step 16 (or farther), this will necessarily mean that one or more  $\delta W_i$  (for  $i \geq 16$ ) will be non-zero. This will require a modification of the Nikolić-Biryukov strategy. Instead of requiring  $\delta W_i = 0$  for  $i \geq 16$ , we will require  $\delta W_i = 0$  for a few  $i$ 's after the local collision ends. So, supposing that the local collision ends at Step 16 and we want a 23-round collision, then  $\delta W_{16}$  is necessarily  $-w$  and we will require  $\delta W_{17} = \dots = \delta W_{22} = 0$ .

In the rest of the paper, we will only consider SHA-256 and not SHA-512. The techniques depend to some extent on the nature of the expansion function  $\sigma_1$  used in SHA-256. More importantly, we require to solve some nonlinear equations involving 32-bit quantities. The differential properties of  $\sigma_1$  are summarized in Section B.1. For SHA-512, this will involve 64-bit quantities and hence solving such equations will be much more difficult.

#### 4.1 A Class of Local Collisions

A local collision of the type shown in Table 2 is completely determined by the values of  $w, x, y$  and  $z$  and the values of  $\delta W_i$  to  $\delta W_{i+8}$ . We need to consider some special values for the  $\delta W$ s. Let

$$(\delta W_i, \dots, \delta W_{i+8}) = (w, -w, \delta_1, \delta_2, 0, 0, 0, u, -w). \quad (3)$$

The value of  $u$  is either 0 or  $w$  and the values of  $\delta_1$  and  $\delta_2$  will be explained later. Using the form of the  $\delta W$ s from Table 2, Equation 3 gives rise to the following 9 equations.

$$\begin{aligned} \text{(A)} \quad \delta W_i &= & &= w; \\ \text{(B)} \quad \delta W_{i+1} &= x - \delta \Sigma_1^i(w) - \delta f_{IF}^i(w, 0, 0) &= &= -w; \\ \text{(C)} \quad \delta W_{i+2} &= y - \delta \Sigma_1^{i+1}(x) - \delta f_{IF}^{i+1}(x, w, 0) &= &= \delta_1; \\ \text{(D)} \quad \delta W_{i+3} &= z - \delta \Sigma_1^{i+2}(y) - \delta f_{IF}^{i+2}(y, x, w) &= &= \delta_2; \\ \text{(E)} \quad \delta W_{i+4} &= -w - \delta \Sigma_1^{i+3}(z) - \delta f_{IF}^{i+3}(z, y, x) &= &= 0; \\ \text{(F)} \quad \delta W_{i+5} &= -x - \delta \Sigma_1^{i+4}(w) - \delta f_{IF}^{i+4}(w, z, y) &= &= 0; \\ \text{(G)} \quad \delta W_{i+6} &= -y - \delta f_{IF}^{i+4}(0, w, z) &= &= 0; \\ \text{(H)} \quad \delta W_{i+7} &= -z - \delta f_{IF}^{i+4}(0, 0, w) &= &= u; \\ \text{(I)} \quad \delta W_{i+8} &= & &= -w. \end{aligned}$$

The values of  $x, y$  and  $z$  from (2) are the following.

$$x = -\delta \Sigma_0^i(w) - \delta f_{MAJ}(w, 0, 0); \quad y = -\delta f_{MAJ}^{i+1}(0, w, 0); \quad z = -\delta f_{MAJ}^{i+2}(0, 0, w).$$

We now set conditions on the values for  $a$  and the  $e$  registers to obtain desired values for  $x, y$  and  $z$  and also to satisfy the values of  $\delta W$ s. The following are easy to verify.

1. If  $a_i = -1$  and  $a_{i-1} = a_{i-2} = \alpha$ , then  $x = -1$ .
2. If  $a_{i+1} = a_{i-1}$ , then  $y = 0$ ; if  $a_{i+1} = \overline{a_{i-1}}$ , then  $y = -1$ .
3. If  $a_{i+2} = a_{i+1}$ , then  $z = 0$ ; if  $a_{i+2} = \overline{a_{i+1}}$ , then  $z = -1$ .

**Note.** In the following, we will only consider  $z = 0$ . (Equations arising from the case  $z = -1$  are more complicated and require further analysis.) So, we will have  $a_{i+2} = a_{i+1}$ . Let this common value be  $\beta$ . Further, if  $\beta = \alpha$ , then  $y = 0$  and if  $\beta = \overline{\alpha}$ , then  $y = -1$ . These and other values of  $a$  and  $e$  registers are shown in Table 4.

**Table 4.** Values of  $a$  and  $e$  register for the  $\delta W$ s given by (3) to hold. We have  $w = 1, x = -1$  and  $z = 0$ . If  $\beta = \alpha$ , then  $y = 0$ , while if  $\beta = \overline{\alpha}$ , then  $y = -1$ . If  $y = 0$ , then  $\lambda = \alpha - \Sigma_0(\alpha)$ , while if  $y = -1$ , then  $\lambda = \alpha + \overline{\alpha} + 1 - \Sigma_0(\overline{\alpha})$ . The value of  $u$  is either 0 or  $w$ . By the CDE, we have  $\lambda = \beta + \alpha - \Sigma_0(\beta) - f_{MAJ}(\beta, -1, \alpha)$ . Thus, the independent quantities are  $\alpha, \gamma$  and  $\mu$ .

index	$i-2$	$i-1$	$i$	$i+1$	$i+2$	$i+3$	$i+4$	$i+5$	$i+6$
$a$	$\alpha$	$\alpha$	$-1$	$\beta$	$\beta$				
$e$	$\gamma$	$\gamma+1$	$-1$	$\mu$	$\lambda$	$\lambda+y$	$-1$	$y$	$-1-u$

The values shown in Table 4 have been chosen so that the conditions on  $\delta W_{i+1}$  and  $\delta W_{i+5}$  to  $\delta W_{i+7}$  hold with probability one. Consider, for example,  $\delta W_{i+1}$ . From (B), we have

$$\delta W_{i+1} = x - \delta \Sigma_1^i(w) - \delta f_{IF}^i(w, 0, 0)$$

$$\begin{aligned}
&= x - (\Sigma_1(e_i + w) - \Sigma_1(e_i)) - (f_{IF}(e_i + w, e_{i-1}, e_{i-2}) - f_{IF}(e_i, e_{i-1}, e_{i-2})) \\
&= -1 - (0 - (-1)) - (e_{i-2} - e_{i-1}) \\
&= -2 - \gamma + \gamma + 1 \\
&= -1.
\end{aligned}$$

Similarly, Equations (F), (G) and (H) can be verified. Equations (C), (D) and (E) on the other hand give rise to the following conditions on the values of  $\alpha$ ,  $\gamma$  and  $\mu$ .

$$\left. \begin{aligned}
\delta_1 &= y - \Sigma_1(\mu + x) + \Sigma_1(\mu) - f_{IF}(\mu + x, 0, \gamma + 1) + f_{IF}(\mu, -1, \gamma + 1) \\
\delta_2 &= -\Sigma_1(\lambda + y) + \Sigma_1(\lambda) - f_{IF}(\lambda + y, \mu + x, 0) + f_{IF}(\lambda, \mu, -1) \\
w &= -f_{IF}(\lambda + y, \lambda + y, \mu + x) + f_{IF}(\lambda + y, \lambda, \mu).
\end{aligned} \right\} \quad (4)$$

The special case of these equations with  $y = 0$  have been reported in [2] and a method for solving them has been discussed. For the case  $y = -1$ , the following strategy (which is somewhat similar to the case  $y = 0$  given in [2]) can be used to solve the equations.

- The third equation has a solution if and only if both  $\lambda$  and  $\mu$  are odd.
- Given that the third equation holds, the second equation simplifies to  $\delta_2 = -\Sigma_1(\lambda - 1) + \Sigma_1(\lambda) + (\lambda - 1)$ . For odd values of  $\delta$  occurring in the distribution of  $\sigma_1(W) - \sigma_1(W - 1)$ , it is possible to solve this equation for odd  $\lambda$ . The maximum value of  $\text{freq}_\delta$  for odd  $\delta$  is  $2^{16}$ . One example is  $\delta = \text{ff006001}$  as shown in Table 11. (As mentioned earlier, if  $\text{freq}_\delta > 2^{16}$ , then  $\delta$  is even and for such values of  $\delta$ , we could not find any odd  $\lambda$  satisfying the second equation.)
- Given such a  $\lambda$ , it is possible to invert the equation  $\lambda = \alpha + \bar{\alpha} + 1 - \Sigma_0(\bar{\alpha})$  to obtain a suitable value of  $\alpha$ .
- Now, we choose a  $\gamma$  and obtain an odd  $\mu$  such that the first equation holds.

Solving all the three equations for  $\alpha, \gamma$  and  $\mu$  can be done in a few seconds on a current PC. Examples are provided in Table 5.

**Table 5.** Values leading to collisions for different number of rounds. Here  $w = 1$ ,  $x = y = -1$  and  $z = 0$ . The value of  $i$  denotes the start point of the local collision, i.e., the local collision is placed from Step  $i$  to  $i + 8$ .

(# rnds, $i$ )	$\delta_1$	$\delta_2$	$w$	$\alpha$	$\lambda$	$\gamma$	$\mu$
(23, 8)	0	ff006001	0	32b308b2	051f9f7f	684e62b7	041fff81
(23, 9)							
(24, 10)							
(25, 11)	00006000	ff006001	1	32b308b2	051f9f7f	98e3923b	fbe05f81
(26, 12)							
(27, 13)							

## 5 23-Round Collisions

We show that by suitably placing a local collision of the type described in Section 4.1 and using proper values for  $\alpha, \gamma$  and  $\mu$ , it is possible to obtain several 23-round collisions for SHA-256.

### 5.1 Case $i = 8$

The local collision is started at  $i = 8$  and ends at  $i = 16$ . We have  $w = 1, x = -1, z = 0$  and we choose  $y = -1$  by setting  $\beta = \bar{\alpha}$ . Also, we set  $u = 0$  and  $\delta_1 = 0$ . We need to choose a suitable value for  $\delta_2$  which is the value of  $\delta W_{i+3} = \delta W_{11}$ . For this case, we let  $\delta = \delta_2$ .

Since the local collision ends at Step 16, it necessarily follows that  $\delta W_{16} = -1$ . Consequently, we need to consider  $\delta W_{18}$  to ensure that it is zero. Since the collisions starts at  $i = 8$ , all  $\delta W_j$  for  $0 \leq j \leq 7$  are zero. Consequently, we can write  $\delta W_{18} = \delta \sigma_1(W_{16}) + \delta W_{11}$ , where  $\delta \sigma_1(W_{16}) = \sigma_1(W_{16} - 1) - \sigma_1(W_{16})$ . So, for  $\delta W_{18}$  to be zero, we need  $\delta W_{11} = -\delta \sigma_1(W_{16})$ , so that  $\delta W_{11}$  should be one of the values which occur in the distribution of  $\sigma_1(W) - \sigma_1(W - 1)$  for some  $W$ . Table 5 shows an example of solution of (4) in this case.

Obtaining proper values for the constants only ensures that the local collision holds from Steps  $i$  to  $i + 8$  as expected. It does not, however, guarantee that the reduced round collision holds. In the present case, we need to have  $\delta W_{18}$  to be zero. This will happen only if  $W_{16}$  takes a value such that  $\sigma_1(W_{16} - 1) - \sigma_1(W_{16})$  is equal to  $-\delta$ . This can be ensured probabilistically in the following manner. The  $\delta$  that we have used (shown in Table 5) is such that  $\text{freq}_\delta = 2^{16}$  and so by trying approximately  $2^{16}$  possible random choices of  $W_0$  and  $W_1$  we expect a proper value for  $W_{16}$  and hence, a 23-round collision. This shows that it is possible to obtain a 23-round collision with probability  $2^{-16}$ . In [2], the corresponding probability is  $2^{-19}$ . So, our technique is an improvement.

Since  $i = 8$ , from Table 4, we see that  $a_6$  to  $a_{10}$  get defined and  $e_6$  to  $e_{14}$  get defined. Using CDE, the values of  $e_9$  down to  $e_6$  is set by fixing values of  $a_5$  down to  $a_2$ . In other words, the values of  $a_2$  to  $a_{10}$  are fixed. Now, consider

$$e_{14} = \Sigma_1(e_{13}) + f_{IF}(e_{13}, e_{12}, e_{11}) + a_{10} + e_{10} + K_{14} + W_{14}.$$

Note that in this equation all values other than  $W_{14}$  have already been fixed. So,  $W_{14}$  and hence  $\sigma_1(W_{14})$  is also fixed. Now, from the update function of the  $a$  register, we can write

$$W_9 = a_9 - \Sigma_0(a_8) - f_{MAJ}(a_8, a_7, a_6) - \Sigma_1(e_8) - f_{IF}(e_8, e_7, e_6) - e_5 - K_9.$$

In the right hand side, all quantities other than  $e_5$  have fixed values. Using CDE,

$$e_5 = a_5 + a_1 - \Sigma_0(a_4) - f_{MAJ}(a_4, a_3, a_2).$$

Again in the right hand side, all quantities other than  $a_1$  have fixed values. So, we can write  $W_9 = C - a_1$ , where  $C$  is a fixed value. (This relation has already been observed in [2].)

Now,

$$a_1 = \Sigma_0(a_0) + f_{MAJ}(a_0, b_0, c_0) + \Sigma_1(e_0) + f_{IF}(e_0, f_0, g_0) + h_0 + K_1 + W_1$$

where  $a_0$  and  $e_0$  depend on  $W_0$  whereas  $b_0, c_0, f_0, g_0$  and  $h_0$  depend only on IV and hence are constants. Thus, we can write  $a_1 = \Phi(W_0) + W_1$ , where

$$\Phi(W_0) = \Sigma_0(a_0) + f_{MAJ}(a_0, b_0, c_0) + \Sigma_1(e_0) + f_{IF}(e_0, f_0, g_0) + h_0 + K_1.$$

We write  $\Phi(W_0)$  to emphasize that this depends only on  $W_0$ . At this point, we can write

$$\begin{aligned} W_{16} &= \sigma_1(W_{14}) + W_9 + \sigma_0(W_1) + W_0 \\ &= \sigma_1(W_{14}) + C - \Phi(W_0) - W_1 + \sigma_0(W_1) + W_0 \\ &= D - \Phi(W_0) - W_1 + \sigma_0(W_1) + W_0. \end{aligned}$$



There are  $2^{16}$  values of  $W_{16}$  for which  $\sigma(W_{16} - 1) - \sigma(W_{16})$  equals  $\delta$ . So, we have to solve this equation for  $W_0$  and  $W_1$  such that  $W_{16}$  is one of these  $2^{16}$  possible values. The simplest way to do this is to try out random choices of  $W_0$  and  $W_1$  until  $W_{16}$  takes one of the desired values. On an average, success is obtained after  $2^{16}$  trials. Each trial corresponds to about a single step of SHA-256 computation. So, the total cost of finding suitable  $W_0$  and  $W_1$  is about  $2^{12.5}$  tries of 23-step SHA-256 computations.

For each such solution  $(W_0, W_1)$  and an arbitrary choice of  $W_{15}$  we obtain a 23-round collision for SHA-256. Note that after  $W_0$  and  $W_1$  has been obtained everything else is deterministic, i.e., no random tries are required. The task of obtaining a suitable  $W_0$  and  $W_1$  can be viewed as a pre-computation of the type required to find the values of  $\alpha, \gamma$  and  $\mu$ . Then, the actual task of finding collisions becomes deterministic. An example of a collision obtained using this method is given in Table 7.

## 5.2 Case $i = 9$

It is possible to place the local collision from Step 9 to Step 17 and then perform an analysis to show that it is possible to deterministically obtain 23-step collisions for both  $y = 0$  and  $y = -1$ . We do not provide these details, since essentially the same technique with an additional constraint is required for 24-round collision for which we provide complete details. An example of a collision obtained using this method is given in Table 8.

## 5.3 Relation to the 23-Round Collision from [2]

The NB local collision (i.e.,  $(w, x, y, z) = (1, -1, 0, 0)$ ) has been used in [2]. The local collision was placed from Step 9 to Step 17. It was remarked in [2] that they were not able to use the local collision with  $(w, x, y, z) = (1, -1, -1, 0)$  (i.e., the SS local collision) to obtain 23-round collisions.

In comparison, we have shown that the SS local collision gives rise to two kinds of 23-round collision. The first one is obtained by placing the local collision from Steps 8 to 16, and the second one is obtained by placing the local collision from Steps 9 to 17. Examples of both kinds are given in the Appendix.

The description of the attack in [2] is quite complicated. First they consider a 23-round pseudo-collision which is next converted into 23-round collision. This two-step procedure is unnecessary. Our detailed combinatorial analysis allows us to directly describe the attacks. In fact, our analytical framework can also be used to explain how one may obtain a 23-round collision from the NB local collision by placing a local collision from Steps 9 to 17. We omit the details since a lot of it would be repetitive.

## 6 24-Round Collisions

**Note.** In this and the next section, we will be working with the SS local collision, i.e., with  $(w, x, y, z) = (1, -1, -1, 0)$ . (The same analysis also goes through with small changes for the NB local collision, i.e., with  $(w, x, y, z) = (1, -1, 0, 0)$  which has been used in [2].)

The local collision described in Section 4.1 is placed from Step  $i = 10$  to Step  $i + 8 = 18$  with  $w = 1$ ,  $x = y = -1$ ,  $z = 0$  and  $u = 1$ . The values of  $\delta_1, \delta_2$  as well as suitable values of  $\alpha, \gamma$  and  $\mu$  need to be chosen.

Since, the collision ends at Step 18 and  $u = 1$ , we will have  $\delta W_{17} = 1$  and  $\delta W_{18} = -1$ . As a result, to ensure  $\delta W_{19} = \delta W_{20} = 0$ , we need to have  $\delta_1 = \delta W_{12} = -(\sigma_1(W_{17} + 1) - \sigma_1(W_{17}))$  and  $\delta_2 = \delta W_{13} = -(\sigma_1(W_{18} - 1) - \sigma_1(W_{18}))$ . Based on the differential behaviour of  $\sigma_1$  described in Section B.1, we should try to choose  $\delta_1$  and  $\delta_2$  such that  $\text{freq}_{-\delta_1}$  and  $\text{freq}_{\delta_2}$  are as high as possible. (Here  $-\delta_1$  denotes  $-\delta_1 \bmod 2^{32}$ .) But, at the same time, the chosen  $\delta_1$  and  $\delta_2$  must be such that (4) are satisfied.

As mentioned earlier, if we choose  $\delta_2$  such that  $\text{freq}_{\delta_2} > 2^{16}$ , then it is not possible to solve (4). So we choose  $\delta_2 = \text{ff006001}$  with  $\text{freq}_{\delta_2} = 2^{16}$ . Also, we choose  $\delta_1 = 00006000$  so that  $-\delta_1 = \text{ffffa000}$  and  $\text{freq}_{-\delta_1} = 2^{29} + 2^{26}$ . For these values of  $\delta_1$  and  $\delta_2$ , it is possible to solve (4) to obtain suitable  $\alpha, \gamma$  and  $\mu$ , which in turn determine  $\beta = \bar{\alpha}$  and  $\lambda$ . An example of these values is shown in Table 5. The same values also hold for obtaining 23-step collision by placing a local collision from Step 9 to 17.

Now we consider Table 4. This table tells us what the values of the different  $a$  and  $e$ -registers need to be. Since messages up to  $W_{15}$  are free, we can set values for  $a$  and  $e$  registers up to Step 15. But, we see that  $e_{16} = -1 - u = -2$ . This can be achieved by setting  $W_{16}$  to

$$W_{16} = e_{16} - \Sigma_1(e_{15}) - f_{IF}(e_{15}, e_{14}, e_{13}) - a_{12} - e_{12} - K_{16}. \quad (5)$$

Since we want  $e_{16} = -2$  and all other values on the right hand side are constants, we have that  $W_{16}$  is a constant value. On the other hand,  $W_{16}$  is defined by message recursion. So, we have to ensure that  $W_{16}$  takes the correct value. In addition, we need to ensure that  $W_{17}$  and  $W_{18}$  take values such that  $\sigma_1(W_{17} + 1) - \sigma_1(W_{17}) = -\delta_1$  and  $\sigma_1(W_{18} - 1) - \sigma_1(W_{18}) = -\delta_2$ .

Since  $i = 10$ , from Table 4, we see that  $a_8$  to  $a_{12}$  have to be set to fixed values and  $e_8$  to  $e_{16}$  have to be set to fixed values. Using CDE, the values of  $e_{11}$  down to  $e_8$  are determined by  $a_7$  to  $a_4$ . So, the values of  $a_0$  to  $a_3$  are free and correspondingly the choices of words  $W_0$  to  $W_3$  are free.

We have already seen that  $W_{16}$  is a fixed value. Note that

$$\left. \begin{aligned} W_{14} &= e_{14} - \Sigma_1(e_{13}) - f_{IF}(e_{13}, e_{12}, e_{11}) - a_{10} - e_{10} - K_{14} \\ W_{15} &= e_{15} - \Sigma_1(e_{14}) - f_{IF}(e_{14}, e_{13}, e_{12}) - a_{11} - e_{11} - K_{15}. \end{aligned} \right\} \quad (6)$$

Since for both equations, all the quantities on the right hand side are fixed values, so are  $W_{14}$  and  $W_{15}$ .

Using CDE twice, we can write

$$\left. \begin{aligned} W_9 &= -W_1 + C_4 + f_{MAJ}(a_4, a_3, a_2) - \Phi_0 \\ W_{10} &= -W_2 + C_5 + f_{MAJ}(a_5, a_4, a_3) - \Phi_1 \\ W_{11} &= -W_3 + C_6 + f_{MAJ}(a_6, a_5, a_4) - \Phi_2 \end{aligned} \right\} \quad (7)$$

where

$$\left. \begin{aligned} C_i &= e_{i+5} - \Sigma_1(e_{i+4}) - f_{IF}(e_{i+4}, e_{i+3}, e_{i+2}) - 2a_{i+1} - K_{i+5} + \Sigma_0(a_i) \\ \Phi_i &= \Sigma_0(a_i) + f_{MAJ}(a_i, b_i, c_i) + \Sigma_1(e_i) + f_{IF}(e_i, f_i, g_i) + h_i + K_{i+1}. \end{aligned} \right\} \quad (8)$$

Using the expressions for  $W_9, W_{10}$  and  $W_{11}$  we obtain the following expressions for  $W_{16}, W_{17}$  and  $W_{18}$ .

$$\left. \begin{aligned} W_{16} &= \sigma_1(W_{14}) + C_4 - W_1 + f_{MAJ}(a_4, a_3, a_2) - \Phi_0 + \sigma_0(W_1) + W_0 \\ W_{17} &= \sigma_1(W_{15}) + C_5 - W_2 + f_{MAJ}(a_5, a_4, a_3) - \Phi_1 + \sigma_0(W_2) + W_1 \\ W_{18} &= \sigma_1(W_{16}) + C_6 - W_3 + f_{MAJ}(a_6, a_5, a_4) - \Phi_2 + \sigma_0(W_3) + W_2. \end{aligned} \right\} \quad (9)$$

We need to ensure that  $W_{16}$  has the desired value given by (5) and that  $W_{17}$  and  $W_{18}$  take values which lead to desired values for  $\delta\sigma_1(W_{17})$  and  $\delta\sigma_1(W_{18})$  as explained above.

The only free quantities are  $W_0$  to  $W_3$  which determine  $a_0$  to  $a_3$ . The value of  $C_4$  depends on  $e_8, e_7$  and  $e_6$ , where  $e_8$  has a fixed value and  $e_7$  and  $e_6$  are in turn determined using CDE by  $a_3$  and  $a_2$ . Similarly,  $C_5$  is determined by  $e_9, e_8$  and  $e_7$ ; where  $e_9, e_8$  have fixed values and  $e_7$  is determined using  $a_3$ . The value of  $C_6$  on the other hand is fixed. Coming to the  $\Phi$  values,  $\Phi_0$  is determined only by  $W_0$ ;  $\Phi_1$  determined by  $W_0$  and  $W_1$ ; and  $\Phi_2$  determined by  $W_0, W_1$  and  $W_2$ . Let

$$D = W_{16} - (\sigma_1(W_{14}) + C_4 + f_{MAJ}(a_4, a_3, a_2) - \Phi_0 + W_0). \quad (10)$$

If we fix  $W_0$  and  $a_3, a_2$ , then the value of  $D$  gets fixed and we need to find  $W_1$  such that the following equation holds.

$$D = -W_1 + \sigma_0(W_1). \quad (11)$$

A guess-then-verify algorithm can be used to solve this equation. By guessing a total of 18 bits (15 least significant bits of  $W_1$  and three other possible carry bits), it is possible to reconstruct the entire  $W_1$  and then verify whether the reconstructed value is correct. Thus, by trying a total of  $2^{18}$  combinations, it is possible to determine whether (11) has a solution and if so to find all possible solutions. The algorithm is given in Section E. (We note that in [2], it has been remarked that by guessing the least 15 bits of  $W_1$  the entire  $W_1$  can be reconstructed and with probability  $2^{-14}$  it is going to be correct. No details are provided; in particular, the guess-then-verify algorithm that we provide in Section E is not present in [2].)

In our experiments we found that for almost every other value of  $D$ , Equation (11) has solutions, the number of solutions being one or two. So, for a random choice of  $D$ , we consider (11) to hold with probability  $\approx 1$ .

**Solving (11) using table look-up.** An alternative approach would be to use a pre-computed table. For each of the  $2^{32}$  possible  $W_1$ s, prepare a table of entries  $(W_1, -W_1 + \sigma_0(W_1))$  sorted on the second column. Then all solutions (if there are any) for (11) can be found by a simple look-up into the table using  $D$ . The table would have  $2^{32}$  entries and if a proper index structure is used, then the look-up can be done very fast. We have not implemented this method.

Given  $a_1, b_1, \dots, h_1$  and  $a_2$  the value of  $W_2$  gets uniquely defined; similarly, given  $a_2, b_2, \dots, h_2$  and  $a_3$ , the value of  $W_3$  gets uniquely defined. The equations are the following.

$$\left. \begin{aligned} W_2 &= a_2 - (\Sigma_0(a_1) + f_{MAJ}(a_1, b_1, c_1) + h_1 + \Sigma_1(e_1) + f_{IF}(e_1, f_1, g_1) + K_2) \\ W_3 &= a_3 - (\Sigma_0(a_2) + f_{MAJ}(a_2, b_2, c_2) + h_2 + \Sigma_1(e_2) + f_{IF}(e_2, f_2, g_2) + K_3) \end{aligned} \right\} \quad (12)$$

The strategy for determining suitable  $W_0, \dots, W_3$  is the following.

1. Make random choices for  $W_0$  and  $a_2, a_3$ .
2. Run SHA-256 with  $W_0$  and determine  $\Phi_0$ .
3. From  $a_3$  and  $a_2$  determine  $e_7$  and  $e_6$  using CDE.
4. Determine  $C_4$  using (8) and then  $D$  using (10).
5. Solve (11) for  $W_1$  using the guess-then-verify algorithm in Section E.
6. Run SHA-256 with  $W_1$  to define  $a_1, \dots, h_1$ .
7. Determine  $\Phi_1$  using (8) and then  $W_2$  using (12).
8. Run SHA-256 with  $W_2$  to define  $a_2, \dots, h_2$ .
9. Determine  $\Phi_2$  using (8) and then  $W_3$  using (12).
10. Compute  $W_{17}$  and  $W_{18}$  using (9).
11. If  $\sigma_1(W_{17} + 1) - \sigma_1(W_{17}) = -\delta_1$  and  $\sigma_1(W_{18} - 1) - \sigma_1(W_{18}) = \delta_2$ , then return  $W_0, W_1, W_2$  and  $W_3$ .

The values of  $W_0, W_1, W_2$  and  $W_3$  returned by this procedure ensure that the local collision ends properly at Step 18 and that  $\delta W_j = 0$  for  $j = 19, \dots, 23$ . This provides a 24-round collision. An example of a collision obtained using this method is given in Table 9.

**Estimate of computation effort.** Step 5 involves a computation of  $2^{18}$  operations, where each operation is much faster than a single step of SHA-256; by our assessment the time for each operation

is around  $2^{-4}$  times the cost of a single step of SHA-256. Thus, the time for Step 5 is about  $2^{14}$  single SHA-256 steps.

By the choice of  $\delta_1$ , the equality  $\sigma_1(W_{17} + 1) - \sigma_1(W_{17}) = -\delta_1$  holds roughly with probability  $2^{-3}$  while by the choice of  $\delta_2$  the equality  $\sigma_1(W_{18} - 1) - \sigma_1(W_{18}) = \delta_2$  holds roughly with probability  $2^{-16}$  and we obtain success in Step 11 with roughly  $2^{-19}$  probability. So, the entire procedure needs to be carried out around  $2^{19}$  times to obtain a collision.

The time for executing the entire procedure once is about  $(2^{14} + 3)$  single SHA-256 steps which is about  $2^{9.5}$  24-step SHA-256 computations. Hence, success is obtained after about  $2^{28.5}$  24-step SHA-256 computations. In our experiments, we found that the computation effort required to find  $W_0, \dots, W_3$  actually turns out to less than the estimated effort of  $2^{28.5}$  24-step SHA-256 computations. The value of  $2^{28.5}$  matches the figure given in [2]. But, [2] does not provide the detailed analysis of their cost.

If (11) is solved using a table look-up, then the cost estimate changes quite a lot. The cost of Step 5 reduces to about a single SHA-256 step so that the overall cost reduces to about  $2^{14.5}$  24-step SHA-256 computations. The trade-off is that we need to use a look-up table having  $2^{32}$  entries.

## 7 25 to 27-Step Collisions

The attack described in the previous section can be extended to 25, 26 and 27 steps. In Table 6, we provide the steps which will be spanned by the local collision and the constraints on the  $e$  registers from Step 16 onwards. Examples of the constants that can be used for these cases are shown in Table 5. Increase in the number of constraints decreases the probability of finding a collision. Still, an esti-

**Table 6.** Summary of details for 24 to 27-step collisions with  $(w, x, y, z) = (1, -1, -1, 0)$ .

# steps	Extent of LC	constraints beyond Step 16
24	10 to 18	$e_{16} = -2$
25	11 to 19	$e_{16} = -1, e_{17} = -2$
26	12 to 20	$e_{16} = -1, e_{17} = -1, e_{18} = -2$
27	13 to 21	$e_{16} = \lambda - 1, e_{17} = -1, e_{18} = -1, e_{19} = -2$

mate of the effort required shows that collisions can be found with less than  $2^{128}$  computations of the corresponding step-reduced SHA-256. Hence, these constitute valid attacks on up to 27-step SHA-256.

In [2] it is mentioned that their technique cannot be extended to 25 or more steps. This is not correct. Our description of the attacks uses the SS local collision, i.e.,  $(w, x, y, z) = (1, -1, -1, 0)$ . But, a similar analysis with the NB local collision (used in [2]), i.e.,  $(w, x, y, z) = (1, -1, 0, 0)$  also provides (different) 25, 26 and 27-step attacks.

### 7.1 Analysis of 25-Step Collisions

The local collision is placed from Step 11 to Step 19 with  $w = 1, x = y = -1$  and  $z = 0$ . So, we have to consider Table 4 with  $i = 11$ . This means that for the local collision to end properly, we must have  $e_{16} = -1$  and  $e_{17} = -2$ . Additionally, to ensure  $\delta W_j = 0$  for  $20 \leq j \leq 24$ , we must have  $\sigma_1(W_{18} + 1) - \sigma_1(W_{18}) = -\delta_1$  and  $\sigma_1(W_{19} - 1) - \sigma_1(W_{19}) = \delta_2$ .

The values of  $e$  register are fixed from Step 9 onwards. To ensure this, using CDE, the values of  $a$  registers are fixed from Step 5 onwards. This means that the values of  $a_0$  to  $a_4$  are free and consequently, the words  $W_0$  to  $W_4$  are also free.

Since the values of  $e_9$  to  $e_{17}$  and  $a_9$  to  $a_{13}$  are fixed, to ensure these values,  $W_{14}, W_{15}, W_{16}$  and  $W_{17}$  must also assume fixed values. Equations (6) and (5) provide the values of  $W_{14}, W_{15}$  and  $W_{16}$ . The value of  $W_{17}$  is  $W_{17} = e_{17} - \Sigma_1(e_{16}) - f_{IF}(e_{16}, e_{15}, e_{14}) - a_{13} - e_{13} - K_{17}$ . Due to message expansion, we can now write the following equations.

$$\begin{aligned} W_{16} - \sigma_1(W_{14}) &= C_4 + f_{MAJ}(a_4, a_3, a_2) - \Phi_0 + W_0 - W_1 + \sigma_0(W_1) \\ W_{17} - \sigma_1(W_{15}) &= C_5 + f_{MAJ}(a_5, a_4, a_3) - \Phi_1 + W_1 - W_2 + \sigma_0(W_2) \\ W_{18} - \sigma_1(W_{16}) &= C_6 + f_{MAJ}(a_6, a_5, a_4) - \Phi_2 + W_2 - W_3 + \sigma_0(W_3) \\ W_{19} - \sigma_1(W_{17}) &= C_7 + f_{MAJ}(a_7, a_6, a_5) - \Phi_3 + W_3 - W_4 + \sigma_0(W_4). \end{aligned}$$

The expressions for  $C_j$  and  $\Phi_k$  are given by (8). The important thing to note is that  $\Phi_k$  is completely determined by the choice of the words  $W_0, \dots, W_k$ . On the other hand,  $C_j$  is determined by  $e_{j+5}, e_{j+4}, e_{j+3}, e_{j+2}$  and  $a_{j+1}, a_j$ . Using CDE, this means that  $C_j$  is determined by  $a_{j+1}, \dots, a_{j-2}$ . Thus,  $C_4$  is determined by  $a_5, a_4, a_3, a_2$ ;  $C_5$  is determined by  $a_6, a_5, a_4, a_3$ ;  $C_6$  is determined by  $a_7, a_6, a_5, a_4$ ; and  $C_7$  is determined by  $a_8, a_7, a_6, a_5$ ;

The following strategy can now be followed to obtain 25-round collisions. Choose  $W_0$  randomly and run SHA-256 for one step so as to define  $a_0, \dots, h_0$  and consequently  $\Phi_0$ . Randomly choose  $a_2, a_3, a_4$ . Then  $C_4$  gets defined and we have to consider an equation of the form  $D = -W_1 + \sigma_0(W_1)$  where  $D = W_{16} - \sigma_1(W_{14}) - C_4 - f_{MAJ}(a_4, a_3, a_2) + \Phi_0 - W_0$ . Using table look-up, this equation can be solved for  $W_1$ . Then,  $a_1, \dots, h_1$  gets fixed and since  $a_2, a_3, a_4$  have already been chosen, the words  $W_2, W_3, W_4$  also get fixed. At this point, there is no more flexibility left in the system. We verify whether the condition on  $W_{17}$  holds and whether the obtained values of  $W_{18}$  and  $W_{19}$  satisfy the condition on  $\delta_1$  and  $\delta_2$ . If all of these hold, we have a 25-round collision.

The probability that the condition on  $W_{17}$  holds is  $2^{-32}$  while the probability that the values of  $W_{18}$  and  $W_{19}$  are jointly proper is  $2^{-19}$ . Consequently, the overall probability is about  $2^{-51}$ . If we repeat the above procedure about  $2^{51}$  times then we expect to obtain a 25-round collision. The time for executing the procedure once is about 5 steps of SHA-256 and hence the total time for executing the procedure  $2^{51}$  times will be about  $2^{48}$  computations of 25-round SHA-256.

## 7.2 Analysis of 26-Step SHA-256

In this case, the local collision is placed from Step 12 to 20, i.e., we have  $i = 12$  in Table 4 with  $w = 1$ ,  $x = y = -1$  and  $z = 0$ . In other words, the entire attack is slid down one step. The values of the words  $W_{14}$  to  $W_{18}$  are still fixed by the constraints on the  $a$  and  $e$  registers given by Table 4. Due to the sliding down of the attack by one step, we have one more constraint for the local collision to hold. If we follow the analysis for the 25-step collision, we see that this new constraint holds with probability  $2^{-32}$ . Again following the time estimate for the 25-step collision, we see that the time estimate for 26-step collision comes to around  $2^{80}$  computations of 26-round SHA-256.

## 7.3 Analysis of 27-Step SHA-256

If we consider 27-step collision and place a local collision from Step 13 to 21, then we have constraints on  $a_{11}$  and  $e_{11}$  onwards. The value of  $W_{14}$  depends on  $e_{10}$  which using CDE, in turn depends on  $a_6$ ; and  $a_6$  is now free and so the value of  $W_{14}$  is no longer fixed. The way around is to choose a value of  $a_6$  so that the value of  $W_{14}$  becomes fixed. Then the previous analysis goes through with an extra 32-bit constraint. This pushes up the cost estimate of finding 27-round collisions to about  $2^{112}$  computations of 27-step SHA-256. Further details of the attack is given in Section F.

## 7.4 More Number of Steps

The method of attack described so far cannot be meaningfully extended beyond 27 steps. For each step, we are including new 32-bit constraints which pushes up the effort by a factor of  $2^{32}$ . Since the effort for 27-step collision is already  $2^{112}$ , multiplying by a factor of  $2^{32}$  will cross the birthday bound for SHA-256.

Perhaps more fundamentally, we are using only a single local collision. Since the local collision is nonlinear in nature, it is difficult to combine two or more such collisions. Further progress in analysis of step-reduced SHA-256 collisions will require some method to combined more than one (linear or non-linear) local collision.

**Note:** The work [2] acknowledges the VIC computer cluster of K.U. Leuven for obtaining most of their experimental results. Not having access to such excellent computational resources, we used only a standard PC. This necessitated a detailed and careful analysis of the nonlinear equations and the computational effort to solve them.

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## A Colliding Message Pairs

Examples of colliding message pairs for 23-step and 24-step SHA-256 using the standard IV are shown in Tables 7, 8 and 9.

**Table 7.** Colliding message pair for 23-step SHA-256 with standard IV. These messages utilize a single local collision starting at Step  $i = 8$ .

$W_1$	0-7	122060e3	000f813f	d92d3fc6	ea4a475f	fb0c6581	dc4558c4	d86428b4	6e2ca576
	8-15	c8d597bf	6372d4c2	ddb721c	79d654c4	f0064002	a894b7b6	91b7628e	3224db20
$W_2$	0-7	122060e3	000f813f	d92d3fc6	ea4a475f	fb0c6581	dc4558c4	d86428b4	6e2ca576
	8-15	c8d597c0	6372d4c1	ddb721c	78d6b4c5	f0064002	a894b7b6	91b7628e	3224db20

**Table 8.** Colliding message pair for 23-step SHA-256 with standard IV. These messages utilize a single local collision starting at Step  $i = 9$ .

$W_1$	0-7	c201bef2	14cc32c9	3b80da44	d8212037	8987161d	a790cb4a	53b8d726	89e9a288
	8-15	3edd76e0	05f41ddc	9ebc0fc3	e099698a	2eaec58f	e7060b78	95d7030d	6bf777c0
$W_2$	0-7	c201bef2	14cc32c9	3b80da44	d8212037	8987161d	a790cb4a	53b8d726	89e9a288
	8-15	3edd76e0	05f41ddd	9ebc0fc2	e099c98a	2daf2590	e7060b78	95d7030d	6bf777c0

**Table 9.** Colliding message pair for 24-step SHA-256 with standard IV. These messages utilize a single local collision starting at Step  $i = 10$ .

$W_1$	0-7	657adf63	06c066d7	90f0b709	95a3e1d1	c3017f24	fad6c2bf	dff43685	6abff0da
	8-15	e6cfc63f	de8fb4c1	c20ca05b	f74815cc	c2e789d9	208e7105	cc08b6cf	70171840
$W_2$	0-7	657adf63	06c066d7	90f0b709	95a3e1d1	c3017f24	fad6c2bf	dff43685	6abff0da
	8-15	e6cfc63f	de8fb4c1	c20ca05c	f74815cb	c2e7e9d9	1f8ed106	cc08b6cf	70171840

## B Details of the SHA-2 Hash Family

Eight registers are used in the evaluation of SHA-2. The initial value in the registers is specified by an  $8 \times n$  bit IV,  $n=32$  for SHA-256 and  $n = 64$  for SHA-512. In Step  $i$ , the 8 registers are updated from  $(a_{i-1}, b_{i-1}, c_{i-1}, d_{i-1}, e_{i-1}, f_{i-1}, g_{i-1}, h_{i-1})$  to  $(a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i)$  according to the following Equations:

$$\left. \begin{aligned}
 a_i &= \Sigma_0(a_{i-1}) + f_{MAJ}(a_{i-1}, b_{i-1}, c_{i-1}) + \Sigma_1(e_{i-1}) \\
 &\quad + f_{IF}(e_{i-1}, f_{i-1}, g_{i-1}) + h_{i-1} + K_i + W_i \\
 b_i &= a_{i-1} \\
 c_i &= b_{i-1} \\
 d_i &= c_{i-1} \\
 e_i &= d_{i-1} + \Sigma_1(e_{i-1}) + f_{IF}(e_{i-1}, f_{i-1}, g_{i-1}) \\
 &\quad + h_{i-1} + K_i + W_i \\
 f_i &= e_{i-1} \\
 g_i &= f_{i-1} \\
 h_i &= g_{i-1}
 \end{aligned} \right\} \quad (13)$$

The functions  $f_{IF}$  and the  $f_{MAJ}$  are three variable boolean functions defined as:

$$\begin{aligned}
 f_{IF}(x, y, z) &= (x \wedge y) \oplus (\neg x \wedge z), \\
 f_{MAJ}(x, y, z) &= (x \wedge y) \oplus (y \wedge z) \oplus (z \wedge x).
 \end{aligned}$$

For SHA-256, the functions  $\Sigma_0$  and  $\Sigma_1$  are defined as:

$$\begin{aligned}
 \Sigma_0(x) &= ROTR^2(x) \oplus ROTR^{13}(x) \oplus ROTR^{22}(x), \\
 \Sigma_1(x) &= ROTR^6(x) \oplus ROTR^{11}(x) \oplus ROTR^{25}(x).
 \end{aligned}$$

For SHA-512, the corresponding functions are:

$$\begin{aligned}\Sigma_0(x) &= ROTR^{28}(x) \oplus ROTR^{34}(x) \oplus ROTR^{39}(x), \\ \Sigma_1(x) &= ROTR^{14}(x) \oplus ROTR^{18}(x) \oplus ROTR^{41}(x).\end{aligned}$$

For a  $t$ -bit words  $\alpha, \beta, \gamma$  and integer  $i$ , we use the short-hands given in Table 10.

**Table 10.** Some useful short-hands.

$\begin{aligned}\delta\Sigma_1^i(\alpha) &= \Sigma_1(e_i + \alpha) - \Sigma_1(e_i) \\ &= \Sigma_1(e'_i) - \Sigma_1(e_i). \\ \delta\Sigma_0^i(\alpha) &= \Sigma_0(a_i + \alpha) - \Sigma_0(a_i) \\ &= \Sigma_0(a'_i) - \Sigma_0(a_i).\end{aligned}$	$\begin{aligned}\delta f_{IF}^i(\alpha, \beta, \gamma) &= f_{IF}(e_i + \alpha, f_i + \beta, g_i + \gamma) - f_{IF}(e_i, f_i, g_i) \\ &= f_{IF}(e'_i, f'_i, g'_i) - f_{IF}(e_i, f_i, g_i). \\ \delta f_{MAJ}^i(\alpha, \beta, \gamma) &= f_{MAJ}(a_i + \alpha, b_i + \beta, c_i + \gamma) - f_{IF}(a_i, b_i, c_i) \\ &= f_{MAJ}(a'_i, b'_i, c'_i) - f_{IF}(a_i, b_i, c_i).\end{aligned}$
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Given the message words  $W_0, W_1, \dots, W_{15}$ , for  $i \geq 16$ ,  $W_i$  is computed as follows.

$$W_i = \sigma_1(W_{i-2}) + W_{i-7} + \sigma_0(W_{i-15}) + W_{i-16} \quad (14)$$

For SHA-256, the functions  $\sigma_0$  and  $\sigma_1$  are defined as:

$$\begin{aligned}\sigma_0(x) &= ROTR^7(x) \oplus ROTR^{18}(x) \oplus SHR^3(x), \\ \sigma_1(x) &= ROTR^{17}(x) \oplus ROTR^{19}(x) \oplus SHR^{10}(x).\end{aligned}$$

And for SHA-512, they are defined as:

$$\begin{aligned}\sigma_0(x) &= ROTR^1(x) \oplus ROTR^8(x) \oplus SHR^7(x), \\ \sigma_1(x) &= ROTR^{19}(x) \oplus ROTR^{61}(x) \oplus SHR^6(x).\end{aligned}$$

### B.1 Differential Properties of $\sigma_1$

The linear function  $\sigma_1$  of SHA-256 used in the message expansion has very poor differential properties with respect to modular addition. Consider the distribution of  $\delta = \sigma_1(W) - \sigma_1(W - 1)$  as  $W$  ranges over all  $2^{32}$  values. It has already been observed in [2] that  $\delta$  takes only 6181 values and there are several values of  $\delta$  which occur for more than  $2^{29}$  or more values of  $W$ .

Let  $\text{freq}_\delta$  be the number of  $W$  such that  $\delta = \sigma_1(W) - \sigma_1(W - 1)$ . It is quite easy to prepare a list of  $(\delta, \text{freq}_\delta)$  values. For each of the  $2^{32}$  values of  $W$ , compute  $\delta = \sigma_1(W) - \sigma_1(W - 1)$ . If this  $\delta$  has been obtained earlier, then increment the frequency for this  $\delta$ ; else insert  $(\delta, \text{freq}_\delta = 1)$  into the list. To do this efficiently, we need a suitable index structure for searching and inserting into the list. A height balanced tree (or AVL tree) is the optimal solution; but, for the current application, a simple (data structure) hash technique is good enough and is the technique we implemented.

Some values of  $(\delta, \text{freq}_\delta)$  are given in Table 11. Interestingly, we have observed that if  $\text{freq}_\delta$  is greater than  $2^{16}$ , then  $\delta$  is always even.

## C Simplifications

The differential path by itself is not useful for obtaining longer round collisions. To do this, we need to simplify the expressions and obtain conditions, as was done by Nikolić-Biryukov [5]. This is done using several rules which are actually sufficient conditions. The rules and their consequences are described below.



**Table 11.** Some examples of high frequency values of  $\delta = \sigma_1(W) - \sigma_1(W - 1)$ .

$\delta$	$\text{freq}_\delta$	$\delta$	$\text{freq}_\delta$
ffff6000	$2^{29} + 2^{26} + 2^{25}$	0000a000	$2^{29} + 2^{26} + 2^{25}$
ffffa000	$2^{29} + 2^{26}$	00006000	$2^{29} + 2^{26}$
ff006001	$2^{16}$	ff005fff	$2^{16}$

### C.1 Simplifying $\delta\Sigma_0$

There is only one occurrence of  $\Sigma_0$  in all the expressions and that is in the expression for  $x$ . In both SHA-256 and SHA-512,  $\Sigma_0$  is a linear operator which is invariant only on 0 and  $-1$ . Note that  $-1 = \text{fffffff}$  for SHA-256 and  $-1 = \text{ffffffffffffffffffff}$  for SHA-512.

Since  $\delta\Sigma_0^i(w) = \Sigma_0(a_i + w) - \Sigma_0(a_i)$  an easy way to satisfy this is to ensure that both  $a_i$  and  $a_i + w$  are either 0 or  $-1$ .

**Rule 1:** Ensure that  $\delta\Sigma_0^i(w) = w$  by either ( $w = 1$  and  $a_i = -1$ ) or ( $w = -1$  and  $a_i = 0$ ).

### C.2 Simplifying Majority

The output of  $f_{MAJ}(a, b, c)$  can be predicted with probability one if two of the inputs are equal. Based on this, we make the following rule.

**Rule 2:** Simplify each occurrence of  $f_{MAJ}$  by making two of the inputs equal.

This rule has several consequences. The function  $f_{MAJ}$  is used only in the definitions of  $x$ ,  $y$  and  $z$ . Consider, for example  $x$  which, after the application of Rule 1, is equal to

$$x = -w - f_{MAJ}(a_i + w, a_{i-1}, a_{i-2}) + f_{MAJ}(a_i, a_{i-1}, a_{i-2}).$$

There are three ways to apply Rule 2 to this occurrence of  $f_{MAJ}$ . These are:

1. Set  $a_{i-1} = a_{i-2}$  which implies  $x = -w$ ;
2. set  $a_{i-1} = a_i + w$ ,  $a_i = a_{i-2}$  which implies that  $x = -2w$ ;
3. set  $a_{i-2} = a_i + w$ ,  $a_i = a_{i-1}$  which also implies that  $x = -2w$ .

So applying Rule 2 to  $x$  implies that either  $x = -w$  (in which case  $a_{i-1} = a_{i-2}$ ) or  $x = -2w$  (in which case either ( $a_{i-1} = a_i + w$  and  $a_i = a_{i-2}$ ) or ( $a_{i-2} = a_i + w$  and  $a_i = a_{i-1}$ )).

Similar reasoning applies to the expressions for  $y$  and  $z$ . Now, if we simultaneously apply Rule 2 to all the three occurrences of  $f_{MAJ}$ , then there are eight possible values of  $(w, x, y, z)$  which are listed as Cases (I) to (VIII) in Table 12. The related sufficient conditions are given in Table 13, where we consider only the case  $w = 1, a_i = -1$ , since the other case  $w = -1, a_i = 0$  is the dual of the first one.

**Table 12.** Different cases for  $(w, x, y, z)$ .

(I)	(II)	(III)	(IV)
$(w, -w, 0, 0)$	$(w, -w, 0, -w)$	$(w, -w, -w, 0)$	$(w, -w, -w, -w)$
(V)	(VI)	(VII)	(VIII)
$(w, -2w, 0, 0)$	$(w, -2w, 0, -w)$	$(w, -2w, -w, 0)$	$(w, -2w, -w, -w)$

**Table 13.** Result of applying Rules 1 and 2. For this table, we have  $w = 1$  and  $a_i = -1$ .

Case	$a_{i-2}$	$a_{i-1}$	$a_i$	$a_{i+1}$	$a_{i+2}$	$e_{i+2}$	$e_{i+1}$
I	$\alpha$	$\alpha$	-1	$\alpha$	$\alpha$	$-\Sigma_0(\alpha) + \alpha$	$1 + a_{i-3}$
II(a)	0	0	-1	0	-1	-1	$1 + a_{i-3}$
II(b)	-1	-1	-1	-1	0	1	$1 + a_{i-3}$
III(a)	-1	-1	-1	0	0	0	$2 + a_{i-3}$
III(b)	0	0	-1	-1	-1	1	$a_{i-3}$
IV(a)	-1	-1	-1	0	-1	-1	$2 + a_{i-3}$
IV(b)	0	0	-1	-1	0	2	$a_{i-3}$
V(a)	-1	0	-1	0	0	-1	$2 + a_{i-3}$
V(b)	0	-1	-1	-1	-1	1	$1 + a_{i-3}$
VI(a)	-1	0	-1	0	-1	-2	$2 + a_{i-3}$
VI(b)	0	-1	-1	-1	0	2	$1 + a_{i-3}$
VII(a)	-1	0	-1	-1	-1	0	$1 + a_{i-3}$
VII(b)	0	-1	-1	0	0	1	$2 + a_{i-3}$
VIII(a)	-1	0	-1	-1	0	1	$1 + a_{i-3}$
VIII(b)	0	-1	-1	0	-1	-1	$2 + a_{i-3}$

### C.3 Simplifying $\delta W_{i+4}$ to $\delta W_{i+7}$

The expression for  $\delta W_{i+4}$  involves  $\delta \Sigma_1^{i+3}(z)$  and  $\delta f_{IF}^{i+3}(z, y, x)$ . By imposing certain conditions, it is possible to simplify both these expressions.

Joint simplification of the above two quantities is possible by ensuring that both  $e_{i+3}$  and  $e_{i+3} + z$  are either 0 or  $-1$ . If  $z = 0$ , then  $e_{i+3}$  can be either 0 or  $-1$ . If  $z = -w$ , then we choose  $e_{i+3} = 0$  if  $w = 1$ ; and  $e_{i+3} = -1$  if  $w = -1$ . Similarly, simplification of  $\delta W_{i+5}$  is possible by ensuring that both  $w$  and  $e_{i+4} + w$  are either 0 or  $-1$ . For  $\delta W_{i+6}$  and  $\delta W_{i+7}$  we respectively ensure that  $e_{i+5}$  and  $e_{i+6}$  are either 0 or  $-1$ . The effect of these simplifications is summarized in Table 14. In particular, the simplifying conditions and the resulting values of the respective  $\delta W$ s are shown.

**Table 14.** Summary of simplifying conditions for  $\delta W_{i+4}$  to  $\delta W_{i+7}$ . The simplifications for  $\delta W_{i+4}$  and  $\delta W_{i+5}$  require Rules 1 and 2, whereas the simplifications for  $\delta W_{i+6}$  and  $\delta W_{i+7}$  do not require these rules.

$\delta W$	Condition(s)	Value of $\delta W$
$\delta W_{i+4}$	$z = 0, e_{i+3} = 0$	$-w - x$
	$z = 0, e_{i+3} = -1$	$-w - y$
	$w = 1, z = -w, e_{i+3} = 0$	$e_{i+1} - e_{i+2} + y$
$\delta W_{i+5}$	$w = 1, e_{i+4} = -1$	$-w - x - y + e_{i+3} - e_{i+2}$
$\delta W_{i+6}$	$e_{i+5} = 0$	$-y - z$
	$e_{i+5} = -1$	$-y - w$
$\delta W_{i+7}$	$e_{i+6} = 0$	$-w - z$
	$e_{i+6} = -1$	$-z$

## D Details of up to 22-Round Collisions

### D.1 20-Round Collisions [5,10]

Deterministic 20-round collisions can be obtained by setting  $i = 5$  (i.e., the local collision spans from  $i = 5$  to  $i + 8 = 13$ ) and ensuring  $\delta W_{i+4} = \delta W_{i+5} = \delta W_{i+6} = \delta W_{i+7} = 0$ . The conditions for individually setting any of these to 0 are given in Table 14.

In the present case, we need to consider how to simultaneously set all of these to 0. In this situation, some conditions become infeasible. More precisely, certain conditions for obtaining  $\delta W_{i+4} = 0$  are incompatible with certain conditions for obtaining  $\delta W_{i+5} = 0$ . The possible conditions for ensuring these two  $\delta W$ s to be zero are given in Table 15. In particular, we see that  $z = 0$  in all cases.

**Table 15.** Conditions for setting  $\delta W_{i+4} = \delta W_{i+5} = 0$ .

Case	$w$	$x$	$y$	$z$	$e_{i+2}$	$e_{i+3}$	$e_{i+4}$	Extra Condition
A	1	-1	0	0	0	0	-1	Case I
B	1	-1	-1	0	1	0	-1	Case III (b)
C	1	-2	-1	0	1	0	-1	Case VII (b)
D	1	-1	-1	0	0	-1	-1	Case III (a)
E	1	-2	0	0	1	-1	-1	Case V (b)
F	1	-2	-1	0	1	-1	-1	Case VII (b)

The conditions for setting  $\delta W_{i+6} = 0$  and  $\delta W_{i+7} = 0$  do not cause any conflict with other conditions. The set of conditions required for setting  $\delta W_{i+4} = \delta W_{i+5} = \delta W_{i+6} = \delta W_{i+7} = 0$  are summarized in Table 16.

**Table 16.** Conditions for setting  $\delta W_{i+4} = \delta W_{i+5} = \delta W_{i+6} = \delta W_{i+7} = 0$ .

A row of Table 15 AND ( $e_{i+5} = 0$ and $y = -z$ ) or ( $e_{i+5} = -1$ and $y = -w$ ) AND $e_{i+6} = -1$ .
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**Note.** Tables 15 and 16 show that it is possible to deterministically set all the four  $\delta W$ s to zero using the Nikolić-Biryukov local collision. Consequently, it is possible to obtain *deterministic* 20-round collision using this local collision. This was not done in [5] but mentioned in [10] later.

### D.2 21-Round Collisions [9]

We set  $i = 6$ , i.e., the local collision spans from  $i = 6$  to  $i + 8 = 14$ . As in the case of 20-round collision, we set  $\delta W_{i+4} = \delta W_{i+5} = \delta W_{i+6} = \delta W_{i+7} = 0$  by a suitable set of conditions given by Table 16.

Let  $\delta\sigma_1(\delta W_i)$  denote  $\sigma_1(W_i + \delta W_i) - \sigma_1(W_i)$ . We have  $\delta W_{14} = \delta W_{i+8} = -w$  and so

$$\delta W_{16} = \delta\{\sigma_1(W_{14}) + W_9\} = \delta\sigma_1(\delta W_{14}) + \delta W_9.$$

We now consider  $\delta W_9 = W_{i+3}$  which by the differential path is equal to  $z - \delta \Sigma_1^{i+2}(y) - \delta f_{IF}^{i+2}(y, x, w)$ . To simplify this, we choose rows from Table 15 such that both  $e_{i+2}$  and  $e_{i+2} + y$  are either 0 or  $-1$ . These are rows A and D. In both cases, we have  $\delta W_9 = e_7 - e_6$ .

Since  $i = 6$ , rows of Tables 13 corresponding to rows A and D of Table 15 ensure that  $a_4, a_5, a_6, a_7, a_8$  and  $e_8$  are all fixed to particular values. Due to CDE, we can now use  $a_3$  to set  $e_7$  to any specific value and then use  $a_2$  to set  $e_6$  to any specific value.

Now, the following strategy is used to ensure that  $\delta W_{16} = 0$ . Choose an arbitrary value for  $W_{14}$  and compute  $\delta$  to be

$$\begin{aligned}\delta &= \delta \sigma_1(\delta W_{14}) \\ &= \sigma_1(W_{14} + \delta W_{14}) - \sigma_1(W_{14}) \\ &= \sigma_1(W_{14} - w) - \sigma_1(W_{14}).\end{aligned}$$

Choose  $W_2$  and  $W_3$  to set  $a_2$  and  $a_3$  such that  $e_7 - e_6 = -\delta$ . This ensures that  $\delta W_{16} = 0$  and hence, provides a deterministic 21-round collision.

It is possible to obtain deterministic 21-round collision by placing the SS local collision from Steps 7 to 15. Set  $i = 7$  so that the local collision spans steps  $i = 7$  to  $i + 8 = 15$ . In this case, set  $\delta W_{i+4} = \delta W_{i+5} = \delta W_{i+6} = 0$  the sufficient condition for this being any row of Table 15 AND ( $(e_{i+5} = 0, y = -z)$  or  $(e_{i+5} = -1, y = -w)$ ). This ensures  $\delta W_{11} = \delta W_{12} = \delta W_{13} = 0$ . Now

$$\begin{aligned}\delta W_{16} &= \sigma_1(\delta W_{14}) + \delta W_9, \\ \delta W_{17} &= \sigma_1(\delta W_{15}) + \delta W_{10}.\end{aligned}$$

We have  $\delta W_{15} = -w$  and by setting  $e_{i+6} = 0$ , we also have  $\delta W_{14} = -w$ . Also,

$$\begin{aligned}W_9 &= W_{i+2} = y - \delta \Sigma_1^{i+1}(x) - \delta f_{IF}^{i+1}(x, w, 0) \\ W_{10} &= W_{i+3} = -\delta \Sigma_1^{i+2}(y) - \delta f_{IF}^{i+2}(y, x, w).\end{aligned}$$

To simplify  $\delta W_{10} = \delta W_{i+3}$  we choose rows from Table 15 such that both  $e_{i+2}$  and  $e_{i+2} + y$  are either 0 or  $-1$ . These are rows A and D. Similarly, to simplify  $\delta W_9 = \delta W_{i+2}$ , in row A we choose  $e_{i+1} = 0$  and in row D, we choose  $e_{i+1} = -1$ .

The overall strategy is now the following. Choose arbitrary values for  $W_{14}$  and  $W_{15}$  and compute  $\delta_1 = \delta \sigma_1(\delta W_{14})$  and  $\delta_2 = \delta \sigma_1(\delta W_{15})$ , where  $\delta W_{14} = \delta W_{15} = -w$ . Now set  $\delta W_9 = -\delta_1$  and  $W_{10} = -\delta_2$  using  $W_3$  and  $W_4$  to set  $a_3$  and  $a_4$  and hence, using CDE to set  $e_7$  and  $e_8$  to desired values. This can be done deterministically.

### D.3 22-Round Collisions [6]

Set  $i = 7$  so that the local collision spans from  $i = 7$  to  $i + 8 = 15$ . Use sufficient conditions from Table 16 to ensure that  $\delta W_{i+4} = \delta W_{i+5} = \delta W_{i+6} = \delta W_{i+7} = 0$ . This ensures that  $\delta W_j = 0$  for  $j = 18, 19, 20, 21$  provided  $\delta W_{16} = \delta W_{17} = 0$ . To ensure  $\delta W_{16} = 0$ , we need to set  $\delta W_9 = \delta W_{i+2} = 0$  and  $\delta W_{17} = 0$  if we can ensure  $\delta \sigma_1(\delta W_{15}) + \delta W_{10} = 0$ .

So, apart from the conditions required to set  $\delta W_{i+4} = \delta W_{i+5} = \delta W_{i+6} = \delta W_{i+7} = 0$ , we need sufficient conditions to set  $\delta W_9 = \delta W_{i+2} = 0$  and to set  $\delta \sigma_1(\delta W_{15}) + \delta W_{10} = 0$ .

To simplify  $\delta W_{10} = \delta W_{i+3}$  we need to choose both  $e_{i+2}$  and  $e_{i+2} + y$  to be 0 or  $-1$ . These imply that we have to use either row E or row H of Table 15. The rest of the strategy is similar to the previously described collisions.

Choose an arbitrary value for  $W_{15}$  and set  $\delta_1 = \delta\sigma_1(\delta W_{15})$  where  $\delta W_{15} = -w$ . Then use  $W_4$  to set  $a_4$  such that due to CDE,  $e_8$  gets set to a particular value required to ensure that  $\delta W_{10} = -\delta_1$ . Similarly, use  $W_3$  to set  $a_3$  such that due to CDE,  $e_7$  gets set to a particular value required to ensure that  $\delta W_9 = 0$ . Both of these can be done deterministically, giving rise to deterministic 22-round collisions.

## E Guess-Then-Verify Algorithm for Solving (11)

For the ease of notation, in this section we will use  $W$  instead of  $W_1$ . Consider Table 1 where the structure of  $W$  and  $\sigma_0(W)$  is shown. We have  $-W + \sigma_0(W) = D$ , where  $D = (d_{31}, \dots, d_0)$  is a 32-bit constant. For  $31 \geq k \geq l \geq 0$ , we will use the notation  $X[k, l]$  to denote bits  $x_k, \dots, x_l$  of the 32-bit quantity  $X$ .

We explain how the guess-then-verify attack proceeds. Suppose that we guess  $W[14, 0]$ . Let  $X = D + W$  and  $Y = (W[14, 0] \gg 3) \oplus (W[14, 0] \gg 7)$ . Then  $W[25, 18] = (X \oplus Y) \& (\mathbf{ff})$ . Having determined  $W[25, 18]$  we next determine  $W[29, 26]$  using positions 22 to 19 of Table 1. This time, however, there may have been a possible carry into the 19th bit and we need to account for that. Let  $c_0$  be a bit. Define  $X = (D \gg 19) + (W[25, 18] \gg 1) + c_0$  and  $Y = (W[14, 0] \gg 5) \oplus (W[25, 18] \gg 4)$ . Then  $W[29, 26] = (X \oplus Y) \& (\mathbf{f})$ . This illustrates the general idea and can be extended to determine the other bits. Once the entire  $W$  has been determined we need to verify whether  $-W + \sigma_0(W) = D$ . The entire algorithm is shown in Figure 2. This algorithm involves guessing  $W[14, 0]$  and bits  $c_0, c_1, c_2$ , which is

**Fig. 1.** Structure of  $W$  and  $\sigma_0(W)$ .

$W$	$w_{31}$	$w_{30}$	$w_{29}$	$w_{28}$	$w_{27}$	$w_{26}$	$w_{25}$	$w_{24}$	$w_{23}$	$w_{22}$	$w_{21}$	$w_{20}$	$w_{19}$	$w_{18}$	$w_{17}$	$w_{16}$
$W \gg 3$	0	0	0	$w_{31}$	$w_{30}$	$w_{29}$	$w_{28}$	$w_{27}$	$w_{26}$	$w_{25}$	$w_{24}$	$w_{23}$	$w_{22}$	$w_{21}$	$w_{20}$	$w_{19}$
$W \gg 7$	$w_6$	$w_5$	$w_4$	$w_3$	$w_2$	$w_1$	$w_0$	$w_{31}$	$w_{30}$	$w_{29}$	$w_{28}$	$w_{27}$	$w_{26}$	$w_{25}$	$w_{24}$	$w_{23}$
$W \gg 18$	$w_{17}$	$w_{16}$	$w_{15}$	$w_{14}$	$w_{13}$	$w_{12}$	$w_{11}$	$w_{10}$	$w_9$	$w_8$	$w_7$	$w_6$	$w_5$	$w_4$	$w_3$	$w_2$
$W$	$w_{15}$	$w_{14}$	$w_{13}$	$w_{12}$	$w_{11}$	$w_{10}$	$w_9$	$w_8$	$w_7$	$w_6$	$w_5$	$w_4$	$w_3$	$w_2$	$w_1$	$w_0$
$W \gg 3$	$w_{18}$	$w_{17}$	$w_{16}$	$w_{15}$	$w_{14}$	$w_{13}$	$w_{12}$	$w_{11}$	$w_{10}$	$w_9$	$w_8$	$w_7$	$w_6$	$w_5$	$w_4$	$w_3$
$W \gg 7$	$w_{22}$	$w_{21}$	$w_{20}$	$w_{19}$	$w_{18}$	$w_{17}$	$w_{16}$	$w_{15}$	$w_{14}$	$w_{13}$	$w_{12}$	$w_{11}$	$w_{10}$	$w_9$	$w_8$	$w_7$
$W \gg 18$	$w_1$	$w_0$	$w_{31}$	$w_{30}$	$w_{29}$	$w_{28}$	$w_{27}$	$w_{26}$	$w_{25}$	$w_{24}$	$w_{23}$	$w_{22}$	$w_{21}$	$w_{20}$	$w_{19}$	$w_{18}$

**Fig. 2.** A guess-then-verify algorithm for solving  $D = -W + \sigma_0(W)$ .

1. Guess  $W[14, 0]$ .
2. Let  $X = D + W$  and  $Y = (W[14, 0] \gg 3) \oplus (W[14, 0] \gg 7)$  and set  $W[25, 18] = (X \oplus Y) \& (\mathbf{ff})$ .
3. Guess  $c_0$ .
4. Let  $X = (D \gg 19) + (W[25, 18] \gg 1) + c_0$  and  $Y = (W[14, 0] \gg 5) \oplus (W[25, 18] \gg 4)$  and set  $W[29, 26] = (X \oplus Y) \& (\mathbf{f})$ .
5. Guess  $c_1$ .
6. Let  $X = (D \gg 23) + (W[25, 18] \gg 6) + c_1$  and  $Y = (W[14, 0] \gg 9) \oplus (W[29, 26] \gg 4)$  and set  $W[31, 20] = (X \oplus Y) \& (\mathbf{3})$ .
7. Guess  $c_2$ .
8. Let  $X = (D \gg 8) + (W[14, 0] \gg 8) + c_2$  and  $Y = (W[14, 0] \gg 11) \oplus (W[29, 26])$  and set  $W[31, 20] = (X \oplus Y) \& (\mathbf{7})$ .
9. If  $-W + \sigma_0(W) = D$ , then output  $W$  as one solution.

a total of 18 bits. If the equation  $D = -W + \sigma_0(W)$  does not have any solution, then none will be

returned by this algorithm; on the other hand, if there is a solution or there are more than one solutions, then all solutions will be returned. A total of  $2^{18}$  operations are required. The time for each operation is significantly less than the time for a single SHA-256 step and by our assessment it is about  $2^{-4}$  times the time for a single SHA-256 step.

## F Details of 27-Step SHA-256 Collisions

We have  $(w, x, y, z) = (1, -1, -1, 0)$  and the local collision is placed from Step 13 to 21. From Table 4, this fixes the values of  $e_{11}$  to  $e_{19}$  and the values of  $a_{11}$  to  $a_{15}$ . Using CDE, the values of  $e_{14}$  down to  $e_{11}$  is achieved by fixing the values of  $a_{10}$  down to  $a_7$ . Consequently, the values of  $a_7$  to  $a_{15}$  are fixed as are the values of  $e_{11}$  to  $e_{19}$ .

Below we recall the values of  $W_{14}$  to  $W_{16}$  from (6) and (5) and also present the values of  $W_{17}$  to  $W_{19}$ .

$$\left. \begin{aligned} W_{14} &= e_{14} - \Sigma_1(e_{13}) - f_{IF}(e_{13}, e_{12}, e_{11}) - a_{10} - e_{10} - K_{14} \\ W_{15} &= e_{15} - \Sigma_1(e_{14}) - f_{IF}(e_{14}, e_{13}, e_{12}) - a_{11} - e_{11} - K_{15} \\ W_{16} &= e_{16} - \Sigma_1(e_{15}) - f_{IF}(e_{15}, e_{14}, e_{13}) - a_{12} - e_{12} - K_{16} \\ W_{17} &= e_{17} - \Sigma_1(e_{16}) - f_{IF}(e_{16}, e_{15}, e_{14}) - a_{13} - e_{13} - K_{17} \\ W_{18} &= e_{18} - \Sigma_1(e_{17}) - f_{IF}(e_{17}, e_{16}, e_{15}) - a_{14} - e_{14} - K_{18} \\ W_{19} &= e_{19} - \Sigma_1(e_{18}) - f_{IF}(e_{18}, e_{17}, e_{16}) - a_{15} - e_{15} - K_{19}. \end{aligned} \right\} \quad (15)$$

The values of  $e_{11}$  onward and the values of  $a_{10}$  onwards are fixed. Consequently, the values of  $W_{15}$  to  $W_{19}$  are also fixed. The value of  $W_{14}$  is determined by the value of  $e_{10}$  which is not fixed. As mentioned above, the values of  $a_7$  onward are already fixed. So, if we randomly choose and fix a value for  $a_6$ , then due to CDE, the value of  $e_{10}$  gets fixed and so the value of  $W_{14}$  also gets fixed.

Using message expansion, we can write the following equations.

$$\left. \begin{aligned} W_{16} - \sigma_1(W_{14}) &= C_4 + f_{MAJ}(a_4, a_3, a_2) - \Phi_0 + W_0 - W_1 + \sigma_0(W_1) \\ W_{17} - \sigma_1(W_{15}) &= C_5 + f_{MAJ}(a_5, a_4, a_3) - \Phi_1 + W_1 - W_2 + \sigma_0(W_2) \\ W_{18} - \sigma_1(W_{16}) &= C_6 + f_{MAJ}(a_6, a_5, a_4) - \Phi_2 + W_2 - W_3 + \sigma_0(W_3) \\ W_{19} - \sigma_1(W_{17}) &= C_7 + f_{MAJ}(a_7, a_6, a_5) - \Phi_3 + W_3 - W_4 + \sigma_0(W_4) \\ W_{20} - \sigma_1(W_{18}) &= C_8 + f_{MAJ}(a_8, a_7, a_6) - \Phi_4 + W_4 - W_5 + \sigma_0(W_5) \\ W_{21} - \sigma_1(W_{19}) &= C_9 + f_{MAJ}(a_9, a_8, a_7) - \Phi_5 + W_5 - W_6 + \sigma_0(W_6) \end{aligned} \right\} \quad (16)$$

In (16),  $W_0$  to  $W_6$  are free. The strategy for solving (16) for  $W_0$  to  $W_6$  is similar to that for 24 and 25-step collisions described earlier.

Choose a random value for  $a_6$ . This fixes the value of  $e_{10}$  and consequently that of  $W_{14}$ . Randomly choose  $W_0$  and  $a_2$  to  $a_5$ ; run SHA-256 for one step to define  $\Phi_0$ ; then the first equation (involving  $W_{16}$ ) reduces to the form (11) which as observed earlier can be solved with probability  $\approx 1$  using a look-up table with  $2^{32}$  entries where each entry consists of 8 bytes.

At this point,  $W_0$  and  $a_2$  to  $a_6$  has been chosen and  $W_1$  has been determined so as to satisfy (11). There is no more freedom left in the system of equations. The equations involving  $W_{17}$  to  $W_{19}$  has to hold and the values of  $W_{20}$  and  $W_{21}$  has to be such that the conditions on  $\delta_1$  and  $\delta_2$  are satisfied. As explained earlier, the probability that the values of  $W_{20}$  and  $W_{21}$  are jointly proper is about  $2^{-19}$ ; whereas the probability that the equations involving  $W_{17}$  to  $W_{19}$  hold is  $2^{-96}$ . The overall probability that all the conditions are satisfied is about  $2^{-115}$  and so, after about  $2^{115}$  trials we should obtain a solution. The cost of each trial is about 6 steps of SHA-256. Hence, the overall effort will be about  $2^{112}$  27-step SHA-256 computations.