

# Short Redactable Signatures Using Random Trees

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**Abstract.** A redactable signature scheme for a string of objects supports verification even if multiple substrings are removed from the original string. It is important that the redacted string and its signature do not reveal anything about the content of the removed substrings. Existing schemes completely or partially leak a piece of information: the lengths of the removed substrings. Such length information could be crucial for many applications, especially when the removed substring has low entropy. We propose a scheme that can hide the length. Our scheme consists of two components. The first component  $\mathcal{H}$ , which is a “collision resistant” hash, maps a string to an unordered set, whereby existing schemes on unordered sets can then be applied. However, a sequence of random numbers has to be explicitly stored and thus it produces a large signature of size at least  $(mk)$ -bits where  $m$  is the number of objects and  $k$  is the size of a key sufficiently large for cryptographic operations. The second component uses RGGM tree, a variant of GGM tree, to generate the pseudo random numbers from a short seed, expected to be of size  $O(k + tk \log m)$  where  $t$  is the number of removed substrings. Unlike GGM tree, the structure of the proposed RGGM tree is random. By an intriguing statistical property of the random tree, the redacted tree does not reveal the lengths of the substrings removed. The hash function  $\mathcal{H}$  and the RGGM tree can be of independent interests.

**Key words:** Redactable Signature Scheme, Random Tree, Privacy

## 1 Introduction

We are interested in a signature scheme for strings of objects whereby the authenticity of a string can be verified even if it is redacted, that is, some substrings have been removed. Let  $\mathbf{x} = x_1x_2 \dots x_m$  be a string, for example a text document where each object can be a character or a word, or an audio file where each object is a sample. The string  $\mathbf{x}$  is signed by the authority and both the string and its signature  $(\mathbf{x}, \mathbf{s})$  are passed to another party, say Alice. Alice wants to show Bob  $\mathbf{x}$  but Bob is not authorized to view certain parts of the string, say  $x_2x_3x_4$  and  $x_7$ . Thus, Alice shows Bob  $\tilde{\mathbf{x}} = x_1 \diamond x_5x_6 \diamond x_8 \dots x_m$  where each  $\diamond$  indicates the location of a removed substring. On the other hand, Bob may want to verify the authenticity of  $\tilde{\mathbf{x}}$ . A redactable signature scheme allows Alice to produce a valid signature  $\tilde{\mathbf{s}}$  for the redacted string  $\tilde{\mathbf{x}}$ , even if Alice does not

have the authority’s secret key. From the new signature  $\tilde{\mathbf{s}}$ , Bob can then verify that  $\tilde{\mathbf{x}}$  is indeed a redacted version of a string signed by the authority.

Unlike the usual signature schemes, redactable signature scheme has additional requirement on privacy: information of the removed strings should be hidden. In this paper, we consider the stringent requirement that, Bob could not obtain any information of any removed substring, except the fact that a non-empty substring has been removed at each location  $\diamond$ . This simple requirement turns out to be difficult to achieve. Existing schemes are unable to completely hide a piece of usually important information: the length of each removed substring. Note that information on length could be crucial if the substring has low entropy. For example, if the substring is either “Approved” or “Not Approved”, then its length reveals everything. There are scenarios and applications where the lengths of the removed strings should not be hidden. As noted by Johnson et al. [JMSW02], semantic attack could be possible in some scenarios if the length information is hidden. On the other side, there are also applications where the fact that a string has been redacted must be hidden. Our scheme can be modified to cater for the above scenarios. The redactable signature scheme proposed by Johnson et al. [JMSW02] employs a Merkle tree [Mer80] and a GGM tree [GGM86] to generate a short signature. However, it is easy to derive the length from the structures of the redacted Merkle and GGM trees. A straightforward modification by introducing randomness into the tree structure also does not hide the length completely. Schemes by Johnson et al. [JMSW02] (set-homomorphic signatures) and Miyazaki et al. [MHI06] are designed for unordered sets and are not applicable for a string. A way to extend their schemes to strings is by assigning a sequence of increasing random numbers to the objects [MHI06]. However, this leads to large signatures since the random numbers have to be explicitly stored, and more importantly, it is insecure since the gaps in the sequence reveal some information about the number of removed objects.

In this paper, we propose a scheme that can hide the lengths of the removed substrings. Our scheme incorporates two components: a hash, and a random tree with a hiding property. We first give a scheme  $\mathcal{RSS}$  using the first component, and then another scheme  $\mathcal{SRSS}$  with both components. The first component hashes a string of objects to an unordered set. For the unordered set, existing redactable schemes [MHI06, JMSW02] on unordered sets can be applied. The scheme  $\mathcal{RSS}$  satisfies the requirements on unforgeability and privacy preserving under reasonable cryptographic assumptions. However, it produces a large signature. Essentially, the main portion of the signature is a sequence of random numbers  $\langle r_1, r_2, \dots, r_m \rangle$ , where each  $r_i$  is associated with the  $i$ -th object in the string.

The goal of the second component is to reduce the signature size by replacing the  $r_i$ ’s with a sequence of pseudo random numbers, generated from a small seed  $t$ . If a substring is removed, the corresponding random numbers have to be removed accordingly. Thus, generating the random numbers iteratively starting from the seed violates privacy, since the seed  $t$  reveals all the random numbers.

We employ a variant of GGM binary tree to generate the  $r_i$ 's in a top-down manner, where the  $r_i$ 's are at the leaves, and the seed  $t$  is at the root. Unlike the GGM tree which is balanced, we use a random binary tree where the *structure* of the binary tree is random. After a substring is removed, the associated leaves and all their ancestors are to be removed, resulting in a collection of subtrees (Figure 1). The roots of the subtrees collectively form the new seed  $\tilde{t}$  for the redacted  $r_i$ 's. Note that from the structures of the subtrees, an adversary might still derive some information of the length of a removed substring. Our main observation is that, by choosing an appropriate tree generation algorithm, the structure of the subtrees reveals nothing about the size of the original tree. Consider a game between Alice and Bob. Suppose Alice randomly picks a binary tree and it is equal likely that the tree contains 1000 leaves or 9 leaves. Now Alice redacts the tree by removing one substring and only 8 leaves are left. From the structure of the remaining subtrees (for example Figure 1(b)), Bob tries to guess the size of the original tree. Now, if Alice employs a tree generation algorithm with the hiding property, Bob cannot succeed with probability more than 0.5. This hiding property is rather counter-intuitive. Since the size of the tree is involved in the tree generation and thus intuitively the information about the size of the tree is spread throughout the tree. Thus it is quite surprising that the global information on size can be completely removed by deleting some nodes.

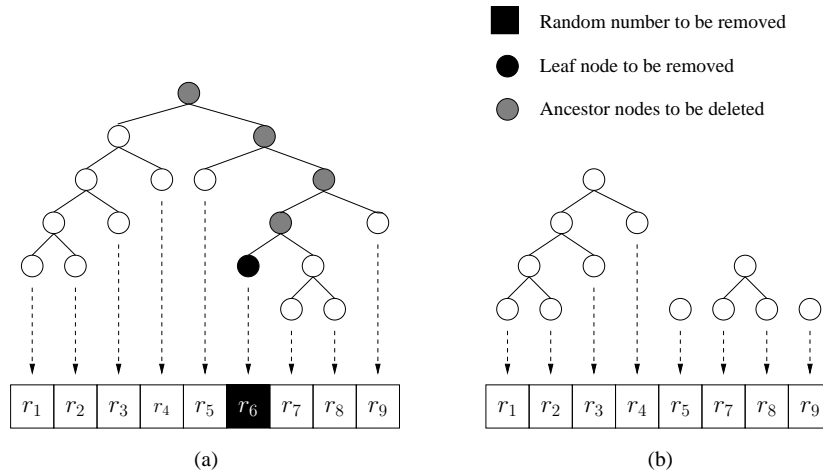


Fig. 1. Redacting the tree in (a) by removing  $r_6$ , gives rise to the redacted tree (b).

*Contribution and Organization.*

1. We propose a “collision resistant” hash  $\mathcal{H}$  that maps strings to unordered sets. From  $\mathcal{H}$  we obtain  $\mathcal{RSS}$ , a redactable signature scheme for strings. Unlike previously known methods,  $\mathcal{RSS}$  is able to hide the lengths of the removed substrings. We show that  $\mathcal{RSS}$  is secure against chosen message attack (Theorem 2) and

privacy preserving (Theorem 3) under reasonable assumptions, which are weaker than random oracle assumption. However, the signature size is large. It consists of  $km + kt + \kappa$  bits, where  $\kappa$  is the size of the signature produced by a known redactable signature scheme for unordered sets,  $m$  is the number of objects in the redacted string,  $t$  is the number of substrings removed, and  $k$  is a security parameter (e.g.  $k = 1024$ ).

2. We observe a hiding property of a random tree (Theorem 4). Based on the observation, we propose RGGM, a pseudo random number generator which can be viewed as a randomized version of GGM [GGM86]. If multiple substrings of pseudo random numbers are to be removed, we can efficiently find a new seed that generates the retained numbers, and yet it is computationally difficult to derive the content and length of each removed substring from the new seed, except the locations of the removed substrings.

3. We propose  $\mathcal{SRSS}$  by incorporating RGGM into  $\mathcal{RSS}$ . The expected size of the signature is in  $\kappa + O(k + kt \log m)$ .  $\mathcal{SRSS}$  is secure against chosen message attack (Corollary 5) and privacy preserving (Corollary 6).

## 2 Related Work

Johnson et al. [JMSW02] introduced redactable signature schemes which enable verification of a redacted signed document. Signature scheme with similar property has also been proposed for XML documents [SBZ01], where the redaction operation is to remove XML nodes. Redactable signatures are examples of homomorphic signatures which are introduced by Rivest in his talks on “Two New Signature Schemes” [Riv01] and formalized by Johnson et al. [JMSW02]. Micali et al. [MR02] gave a transitive signature scheme as the first construction of homomorphic signatures. They also asked for other possible “signature algebras”. The notions on homomorphic signatures can be traced back to incremental cryptography, introduced by Bellare, Goldreich and Goldwasser [BGG94,BGG95]. Recently, Ateniese et al. [ACMT05] introduced sanitizable signature scheme [KL06,IKTY05,STT06,MHI06,YSLM08] allowing a semi-trusted censor modifies the signed documents in a limited and controlled way.

The redactable signature scheme on strings is closely related to directed transitive signature scheme [MR02,Yi07]. It is possible to convert a directed transitive signature scheme to a redactable signature scheme on strings. However, existing directed transitive signature schemes do not provide privacy in the sense that the resulting signatures reveal some information about the removed substrings.

## 3 Formulation and Background

Johnson et al.[JMSW02] gave definitions on homomorphic signature schemes and their security for binary operators. It can be easily adapted for binary relations.

A string is a sequence of objects from an object space (or alphabet)  $\mathbb{O}$ . For example,  $\mathbb{O}$  can be the set of ASCII characters, collection of words, or audio samples, etc. After a few substrings are removed from  $\mathbf{x}$ , the string  $\mathbf{x}$  may break into

substrings, say  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_u$ . The redacted string  $(\tilde{\mathbf{x}}, \mathbf{e})$ , which we call *annotated string*<sup>1</sup>, is represented by the string  $\tilde{\mathbf{x}} = \mathbf{x}_1 \parallel \mathbf{x}_2 \parallel \dots \parallel \mathbf{x}_u$  and an *annotation*  $\mathbf{e} = \langle m, b_1, b_2, \dots, b_v \rangle$  where  $\parallel$  denotes concatenation,  $b_i$ 's is a strictly increasing sequence indicating the locations of the removed substrings,  $m$  is the number of objects in  $\tilde{\mathbf{x}}$ , and  $v \in \{u-1, u, u+1\}$ . For each  $i$ ,  $b_i$  indicates that a non-empty substring has been removed in between the  $b_i$ -th and  $(1+b_i)$ -th locations. If  $b_1 = 0$  or  $b_v = m$ , this indicates that a non-empty substring has been removed at the beginning or end of the string respectively. For example,  $(abcda, \langle 5, 0, 3 \rangle)$  is a redacted string of the original  $xxxcabcyyda$ . For convenient, we sometimes use a sequence of objects like  $\langle x_1, x_2, x_3, \dots, x_m \rangle$  and a string of objects like  $x_1x_2x_3 \dots x_m$  interchangeably.

Let us define a binary relation  $\succ$  between annotated strings. Given two annotated strings  $\mathcal{X}_1 = (\mathbf{x}_1, \mathbf{e}_1)$  and  $\mathcal{X}_2 = (\mathbf{x}_2, \mathbf{e}_2)$ , we say  $\mathcal{X}_1 \succ \mathcal{X}_2$ , if either  $\mathbf{x}_2$  can be obtained from  $\mathbf{x}_1$  by removing a non-empty substring in  $\mathbf{x}_1$ , and the  $\mathbf{e}_2$  is updated from  $\mathbf{e}_1$  accordingly, or there is a  $\mathcal{X}$  s.t.  $\mathcal{X}_1 \succ \mathcal{X}$  and  $\mathcal{X} \succ \mathcal{X}_2$ .

**DEFINITION 1 (REDACTABLE SIGNATURE SCHEME [JMSW02])** *A redactable signature scheme with respect to binary relation  $\vdash$ , is a tuple of probabilistic polynomial time algorithms  $(\text{KGen}, \text{Sign}, \text{Verify}, \text{Redact})$ , such that*

1. for any message  $x$ ,  $\sigma = \text{Sign}_{\text{SK}}(x) \Rightarrow \text{Verify}_{\text{PK}}(x, \sigma) = \text{TRUE}$ ;
2. for any messages  $x$  and  $y$ , such that  $x \vdash y$ ,

$$\text{Verify}_{\text{PK}}(x, \sigma) = \text{TRUE} \wedge \sigma' = \text{Redact}_{\text{PK}}(x, \sigma, y) \Rightarrow \text{Verify}_{\text{PK}}(y, \sigma') = \text{TRUE},$$

where  $(\text{PK}, \text{SK}) \leftarrow \text{KGen}(1^k)$  and  $k$  is the security parameter.

Both Johnson et al.[JMSW02] and Miyazaki et al.[MHI06] presented a redactable signature scheme w.r.t superset relation. Johnson et al.[JMSW02] also gave security definition for homomorphic signature schemes. We adapt their definition for redactable signature scheme. Let  $\vdash$  denote a binary relation. For any set  $S$ , let  $\text{span}_{\vdash}(S)$  denote the set  $\{x : \exists y \in S, \text{ s.t. } y \vdash x\}$ .

**DEFINITION 2 (UNFORGEABILITY OF REDACTABLE SIGNATURE SCHEME [JMSW02])**

*A redactable signature scheme  $(\text{KGen}, \text{Sign}, \text{Verify}, \text{Redact})$  is  $(t, q, \epsilon)$ -unforgeable against existential forgeries with respect to  $\vdash$  under adaptive chosen message attack, if any adversary  $\mathcal{A}$  that makes at most  $q$  chosen-message queries adaptively and runs in time at most  $t$ , has advantage  $\text{Adv}_{\mathcal{A}} \leq \epsilon$ . The advantage of an adversary  $\mathcal{A}$  is defined as the probability that, after queries on  $\ell$  ( $\ell \leq q$ ) messages  $x_1, x_2, \dots, x_\ell$ ,  $\mathcal{A}$  outputs a valid signature  $\sigma$  for some message  $x \notin \text{span}_{\vdash}(\{x_1, x_2, \dots, x_\ell\})$ . Formally,*

$$\text{Adv}_{\mathcal{A}} = \Pr \left[ \begin{array}{l} (\text{PK}, \text{SK}) \leftarrow \text{KGen}(1^k); \mathcal{A}^{\text{Sign}_{\text{SK}}} = (x, \sigma); \\ \text{Verify}_{\text{PK}}(x, \sigma) = \text{TRUE} \text{ and } x \notin \text{span}_{\vdash}(\{x_1, x_2, \dots, x_\ell\}) \end{array} \right],$$

where the probability is taken over the random coins used by  $\text{KGen}, \text{Sign}$  and  $\mathcal{A}$ .

<sup>1</sup> A string with an annotation which specifies the locations of redactions.

Redactable signature schemes have an additional security requirement on privacy [ACMT05]: the adversary should not be able to derive any information about the removed substrings from a redacted string and its signature.

**DEFINITION 3 (PRIVACY PRESERVING)** *A redactable signature scheme  $\langle \text{KGen}, \text{Sign}, \text{Verify}, \text{Redact} \rangle$  is privacy preserving if, given the public key  $\mathcal{PK}$  and any annotated strings  $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}$ , such that  $\mathcal{X}_1 \succ \mathcal{X}$  and  $\mathcal{X}_2 \succ \mathcal{X}$ , the following distributions  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are computationally indistinguishable:*

$$\begin{aligned}\mathcal{S}_1 &= \{\sigma : \sigma = \text{Redact}_{\mathcal{PK}}(\mathcal{X}_1, \text{Sign}_{\mathcal{SK}}(\mathcal{X}_1; \mathbf{r}_1), \mathcal{X}; \mathbf{r}_2)\}, \\ \mathcal{S}_2 &= \{\sigma : \sigma = \text{Redact}_{\mathcal{PK}}(\mathcal{X}_2, \text{Sign}_{\mathcal{SK}}(\mathcal{X}_2; \mathbf{r}_1), \mathcal{X}; \mathbf{r}_2)\},\end{aligned}$$

where  $\mathbf{r}_1$  and  $\mathbf{r}_2$  are random coins used by **Sign** and **Redact** respectively, and public/private key  $(\mathcal{PK}, \mathcal{SK})$  is generated by **KGen**.

## 4 **RSS**: Redactable Signature Scheme for Strings

We propose **RSS**, a redactable signature scheme for strings that is able to hide the lengths of the removed substrings. Our approach is as follows: we first propose a hash function  $\mathcal{H}$  that maps an annotated string  $\tilde{\mathbf{x}}$  and an auxiliary input  $\mathbf{y}$  to an unordered set. This hash is “collision resistant” and satisfies some properties on substring removal. Using  $\mathcal{H}$  and some known redactable signature schemes for unordered sets, we have a redactable signature scheme for strings.

### 4.1 Hashing strings to unordered sets

Let  $\mathcal{H}$  be a hash function that maps an annotated string  $\mathcal{X}$  and an auxiliary input  $\mathbf{y}$ , which could be a sequence of numbers from a finite field, to a (unordered) set of elements from some universe. In our construction (Table 1),  $\mathcal{H}$  maps the input to a set of 3-tuples from  $\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$ , where  $n$  is some chosen parameter.

**DEFINITION 4 (COLLISION RESISTANT)**  $\mathcal{H}$  is  $(t, \epsilon)$ -collision-resistant if, for any algorithm  $\mathcal{A}$  with running time at most  $t$ ,

$$\Pr[\mathcal{X}_1 \neq \mathcal{X}_2 \wedge \mathcal{H}(\mathcal{X}_2, \mathbf{y}_2) \subset \mathcal{H}(\mathcal{X}_1, \mathbf{y}_1)] \leq \epsilon,$$

where  $(\mathcal{X}_1, \mathcal{X}_2, \mathbf{y}_2)$  is the output of  $\mathcal{A}$  on input  $\mathbf{y}_1$ , and the probability is taken over uniformly randomly chosen  $\mathbf{y}_1$  and random coins used by  $\mathcal{A}$ .

To be used in constructing a secure string redactable signature scheme, besides collision resistance, the hash function  $\mathcal{H}$  is also required to be 1) “redactable”: given  $\mathcal{X}_1, \mathcal{X}_2$  and  $\mathbf{y}_1$ , such that  $\mathcal{X}_1 \succ \mathcal{X}_2$ , it is easy to find  $\mathbf{y}_2$  such that  $\mathcal{H}(\mathcal{X}_1, \mathbf{y}_1) \supset \mathcal{H}(\mathcal{X}_2, \mathbf{y}_2)$ ; 2) privacy preserving:  $\mathcal{H}(\mathcal{X}_2, \mathbf{y}_2)$  must not reveal any information about the removed substring. The property on privacy preserving is essential and used in the proof of Theorem 3. However, for simplicity, we will not explicitly formulate the requirement here.

## 4.2 Construction of $\mathcal{H}$

We present a hash function  $\mathcal{H}(\cdot, \cdot)$  in Table 1 based on some hash function  $h$ .

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Let  $n$  be a RSA modulus, and  $h : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  be a hash function. Given  $\mathbf{x} = x_1x_2 \dots x_m$  associated with annotation  $\mathbf{e}$ ,  $\mathbf{r} = r_1r_2r_3 \dots r_m$ , and  $\mathbf{w} = w_1w_2w_3 \dots w_m$ , where for each  $i$ ,  $x_i, r_i, w_i \in \mathbb{Z}_n$  (i.e.  $\mathbf{x}$ ,  $\mathbf{r}$  and  $\mathbf{w}$  are strings over alphabet  $\mathbb{Z}_n$ ), we define  $\mathcal{H}$  as

$$\mathcal{H}((\mathbf{x}, \mathbf{e}), (\mathbf{r}, \mathbf{w})) \triangleq \{t_i : t_i = (x_i, r_i, (w_i^{\prod_{j=1}^i h(r_j)} \bmod n)), 1 \leq i \leq m\}.$$


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**Table 1.** Definition of  $\mathcal{H}(\cdot, \cdot)$ .

*Remarks on the construction of  $\mathcal{H}$ .*

1. The witness  $\mathbf{w}$  should be consistent with the annotation  $\mathbf{e}$ , because  $\mathbf{w}$  is supposed to implicitly describe the locations of the removed substrings. For any  $1 < i \leq m$ ,  $w_{i-1} \neq w_i$  means a non-empty substring has been removed just between  $x_{i-1}$  and  $x_i$ . If<sup>2</sup>  $w_1 \neq g$ , this indicates that a non-empty substring is removed from the front of the string  $\mathbf{x}$ . We assume that there are no substrings removed from the end of a string. Because of this correspondence, we may omit the explicit annotation  $\mathbf{e}$  in the use of  $\mathcal{H}$  for simplicity.
2. It is crucial that the value of  $r_i$  is explicitly represented in  $t_i$  for each  $i$  (Table 1). If the  $r_i$ 's are omitted, then it is easy to find collisions.

LEMMA 1 *The hash function  $\mathcal{H}$  as defined in Table 1, is  $(poly_1(k), \frac{1}{poly_2(k)})$ -collision-resistant for any positive polynomials  $poly_1(\cdot)$  and  $poly_2(\cdot)$ , where  $k$  is the security parameter, i.e. the bit length of  $n$ , assuming that  $h$  is division intractable<sup>3</sup> and always outputs odd prime integers, and Strong RSA Problem is hard.*

Appendix A gives the proof sketch. Basically, the proof reduces Strong RSA Problem or Division Problem [GHR99] to the collision finding problem.

## 4.3 Construction of $\mathcal{RSS}$

We construct a redactable signature scheme  $\mathcal{RSS}$ , which consists of four algorithms **TGen**, **Sign**, **Verify**, and **Redact**, for strings with respect to binary relation  $\succ$  based on the hash function  $\mathcal{H}$  defined in Table 1 and some redactable signature scheme  $\mathcal{SSS}$  for sets with respect to superset relation  $\supseteq$ .

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<sup>2</sup> Here  $g$  is an element of large order from  $\mathbb{Z}_n^*$ . If there are no redaction at the front of  $\mathbf{x}$ ,  $w_1 = g$ . See the use of  $\mathcal{H}$  in Table 3.

<sup>3</sup> Division intractability [GHR99] implies collision resistance.

The signer chooses a RSA modulus  $n$  and an element  $g$  of large order in  $\mathbb{Z}_n^*$ . Both  $n$  and  $g$  are public. Let the object space be  $\mathbb{Z}_n$ , that is, a string is a sequence of integers from  $\mathbb{Z}_n$ . Let  $h : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  be a collision resistant hash which satisfies the division intractable property [GHR99] and always outputs odd prime numbers<sup>4</sup>. In practice, hash function like  $h(x) = \text{SHA1}(x) \parallel 1$  may be sufficient. Let  $\text{SSS} = (\text{keygen}, \text{sig}, \text{vrf}, \text{rec})$  be a redactable signature scheme for unordered sets w.r.t superset relation  $\supseteq$ . The signer also needs to choose the public and secret key pair  $(\mathcal{PK}, \mathcal{SK})$ , which is actually used by the underlying signature scheme SSS. The details of **KGen**, **Sign**, **Verify**, and **Redact** are presented in Table 2, Table 3, Table 4 and Table 5.

The final signature of a string  $x_1x_2 \dots x_m$  consists of  $m$  random numbers  $r_1, r_2, \dots, r_m$ , the *witnesses*  $w_1, w_2, \dots, w_m$  where  $r_i, w_i \in \mathbb{Z}_n$  for each  $i$ , and a signature  $\mathbf{s}$  constructed by SSS. Recall that witness  $w_i$ 's imply the annotation of string  $x_1x_2 \dots x_m$ .

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**KGen.** Given security parameter  $k$ .

1. Choose a RSA modulus  $n$ , and an element  $g$  of large order in  $\mathbb{Z}_n$ .
2. Run key generating algorithm **keygen** on input  $1^k$  to get key  $(\mathcal{PK}, \mathcal{SK})$ .
3. Output  $(n, g, \mathcal{PK})$  as public key and  $\mathcal{SK}$  as private key.

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**Table 2.**  $\mathcal{RSS}$ : **KGen**.

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**Sign.** Given  $\mathbf{x} = x_1x_2 \dots x_m$  and its associated annotation  $\mathbf{e} = \langle m \rangle$ .

1. Let  $w_i = g$  for each  $i$ . Choose  $m$  distinct random numbers  $r_1, r_2, \dots, r_m$ . Let  $\mathbf{r} = r_1r_2r_3 \dots r_m$  and  $\mathbf{w} = w_1w_2w_3 \dots w_m$ . Compute

$$\mathbf{t} = \mathcal{H}((\mathbf{x}, \mathbf{e}), (\mathbf{r}, \mathbf{w})).$$

2. Sign the set  $\mathbf{t}$  using SSS with the secret key  $\mathcal{SK}$  to obtain  $\mathbf{s}$ :

$$\mathbf{s} = \text{sig}_{\mathcal{SK}}(\mathbf{t}).$$

3. The final signature consists of the random numbers  $r_i$ 's, witnesses  $w_i$ 's, and the signature  $\mathbf{s}$ . That is,

$$(\mathbf{r}, \mathbf{w}, \mathbf{s}) \text{ or } (r_1, r_2, \dots, r_m; w_1, w_2, \dots, w_m; \mathbf{s})$$

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**Table 3.**  $\mathcal{RSS}$ : **Sign**.

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<sup>4</sup> Gennaro [GHR99] gave a way to construct a hash function that is division intractable and always outputs odd prime numbers.



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**Verify.** Given a string  $\mathbf{x} = x_1x_2 \dots x_m$  associated with annotation  $\mathbf{e}$ , its signature  $(\mathbf{r}, \mathbf{w}, \mathbf{s})$ , the public information  $n, g$ , and the public key  $\mathcal{PK}$  of SSS.

1. If  $\mathbf{e}$  and  $\mathbf{w}$  are not consistent, output FALSE.
2. Compute  $\mathbf{t} = \mathcal{H}(\mathbf{x}, (\mathbf{r}, \mathbf{w}))$ .
3.  $(\mathbf{r}, \mathbf{w}, \mathbf{s})$  is a valid signature of  $\mathbf{x}$  under  $\mathcal{RSS}$ , if and only if  $\mathbf{s}$  is a valid signature of  $\mathbf{t}$  under SSS, i.e.

$$\text{vrf}_{\mathcal{PK}}(\mathbf{t}, \mathbf{s}) = \text{TRUE}.$$

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**Table 4.**  $\mathcal{RSS}$ : Verify.

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**Redact.** Given a string  $\mathbf{x} = x_1x_2 \dots x_m$  associated with annotation  $\mathbf{e}$ , and its signature  $(\mathbf{r}, \mathbf{w}, \mathbf{s})$ , where  $\mathbf{r} = r_1r_2 \dots r_m$ ,  $\mathbf{w} = w_1w_2 \dots w_m$ , the public information  $n, g$ , public key  $\mathcal{PK}$  for SSS, and  $(i, j)$  the location of the string to be removed (that is  $x_ix_{i+1} \dots x_j$  is to be removed).

1. Update  $\mathbf{e}$  to obtain new annotation  $\hat{\mathbf{e}}$ . Compute  $u = \prod_{k=i}^j h(r_k)$ , to update the witnesses in the following way: for each  $\ell > j$ , update  $w_\ell$

$$\hat{w}_\ell \leftarrow w_\ell^u \pmod n.$$

2. Let  $\hat{\mathbf{x}} = x_1x_2 \dots x_{i-1}x_{j+1} \dots x_m$ ,  $\hat{\mathbf{r}} = r_1r_2 \dots r_{i-1}r_{j+1} \dots r_m$  and  $\hat{\mathbf{w}} = w_1w_2 \dots w_{i-1}\hat{w}_{j+1}\hat{w}_{j+2} \dots \hat{w}_m$ . Compute

$$\hat{\mathbf{t}} = \mathcal{H}((\hat{\mathbf{x}}, \hat{\mathbf{e}}), (\hat{\mathbf{r}}, \hat{\mathbf{w}})).$$

3. Compute

$$\hat{\mathbf{s}} = \text{rec}_{\mathcal{PK}}(\hat{\mathbf{t}}, \mathbf{s}, \hat{\mathbf{t}})$$

where  $\mathbf{t} = \mathcal{H}((\mathbf{x}, \mathbf{e}), (\mathbf{r}, \mathbf{w}))$ .

4. Output  $(\hat{\mathbf{r}}, \hat{\mathbf{w}}, \hat{\mathbf{s}})$  as the signature of  $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$ .

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**Table 5.**  $\mathcal{RSS}$ : Redact.

**THEOREM 2**  $\mathcal{RSS}$  is  $(t, q, \frac{\epsilon_1}{1-\epsilon_2})$ -unforgeable against existential forgeries with respect to relation  $\succ$ , if SSS is  $(t + qt_0, q, \epsilon_1)$ -unforgeable against existential forgeries with respect to superset relation  $\supseteq$ , and  $\mathcal{H}$  is  $(t + qt_1, \epsilon_2)$ -collision-resistant, where  $t_0$  is the running time of  $\mathcal{H}$  and  $t_1$  is the time needed by  $\mathcal{RSS}$  to sign a document.

The proof sketch is given in Appendix B. Our construction of  $\mathcal{H}$  (Table 1) is collision resistant (Lemma 1). Johnson et al.[JMSW02] showed their redactable signature scheme **Sig** (in Section 5 of [JMSW02]) is  $(t, q, \epsilon)$ -unforgeable under reasonable assumptions (see Theorem 1 in [JMSW02]), for some proper parameters  $t, q$  and  $\epsilon$ . Miyazaki et al.[MHI06] also showed a similar result on the unforgeability of the redactable signature scheme they proposed. Hence, conditions in Theorem 2 can be satisfied.

**THEOREM 3** *The redactable signature scheme  $\mathcal{RSS}$  is privacy preserving (as defined in Definition 3), assuming that hash function  $h$  satisfies the property: the two distributions  $\mathbf{X} = g^{h(U_1)h(U_2)} \pmod n$  and  $\mathbf{Y} = g^{h(U'_1)} \pmod n$  are computationally indistinguishable, where  $n$  is a RSA modulus,  $g$  is an element of large order in  $\mathbb{Z}_n^*$  and  $U_i$ 's and  $U'_j$ 's are all independent uniform random variables over  $\mathbb{Z}_n$ .*

The proof sketch of Theorem 3 is in Appendix C. Interestingly,  $\mathcal{RSS}$  is privacy preserving, without any assumptions on the privacy preserving property of  $\mathcal{SSS}$ . This is because the relevant information is already removed by  $\mathcal{H}$  during redaction.

#### 4.4 Efficiency

The size of  $\mathbf{s}$  depends on  $\mathcal{SSS}$ , and let us assume it requires  $\kappa$  bits. The number of distinct  $w_i$ 's is about the same as the number of redactions occurred. So  $w_i$ 's can be represented in  $t(k + \lceil \log m \rceil)$  bits, where  $t$  is the number of substrings removed, and  $k$  is the bit length of  $n$ . Thus the total number of bits required is at most  $k(m + t) + t\lceil \log m \rceil + \kappa$ . One may ignore the term  $t\lceil \log m \rceil$  since it is used to specify the locations of removed substrings, which could be treated as part of the message. Hence, the size of signature is essentially  $km + kt + \kappa$ . The dominant term is  $km$ , which is the total size of the random numbers  $r_i$ 's.

Disregarding the time taken by the scheme  $\mathcal{SSS}$ , and the time required to compute the hash  $h(\cdot)$ , during signing,  $O(m)$  of  $k$ -bits exponentiation operations are required. During redaction, if  $\ell$  consecutive objects are to be removed between position  $i$  and  $j$ , and  $t'$  number of redactions have been made after position  $j$ , then the number of  $k$ -bit exponentiation operations is at most  $\ell(t' + 1)$ , which is in  $O(\ell m)$ . During verification,  $O(tm)$  number of  $k$ -bits exponentiation operations are required. Hence, our scheme is suitable for small  $t$ , which is reasonable in practice. In sum, the main drawback of  $\mathcal{RSS}$  is the size of its signature. In the next section, we will reduce its size using a random tree.

## 5 RGGM: Random tree with Hiding property

We propose RGGM, a variant of GGM tree [GGM86] to generate a sequence of pseudo random numbers, where the structure of the tree is randomized. This generator provides us with the ability to remove multiple substrings<sup>5</sup> of pseudo random numbers, while still being able to generate the retained numbers from a short seed. The expected size of the new seed is in  $O(k + tk \log m)$  where  $t$  is the number of removed substrings,  $m$  is the number of pseudo random numbers, and  $k$  is a security parameter. More importantly, the new seed does not reveal any information about the size nor the content of the removed substrings.

<sup>5</sup> A substring is as a subsequence where all the elements are consecutive elements in the full sequence.

*Pseudo random number generation.* To generate  $m$  pseudo random numbers we employ a method similar to that in the redactable signature scheme proposed by Johnson et al. [JMSW02], which is based on the GGM tree [GGM86]. Let  $G : \mathcal{K} \rightarrow \mathcal{K} \times \mathcal{K}$  be a length-doubling pseudo random number generator. First pick an arbitrary binary tree  $T$  with  $m$  leaves, where all internal nodes of  $T$  have exactly two children, the left and right child. Next, pick a seed  $t \in \mathcal{K}$  uniformly at random, and associate it with the root. The pseudo random numbers  $r_1, r_2, \dots, r_m$  are then computed from  $t$  in the usual top-down manner along the binary tree.

*Hiding random numbers.* If  $r_i$  is to be removed, the associated leaf node and all its ancestors will be removed, as illustrated by the example in Figure 1(b). The values associated with the roots of the remaining subtrees, and a description of the structure of the subtrees, form the new seed, whereby the remaining random values  $r_j$ 's ( $j \neq i$ ) can be re-computed. By the property of  $G$ , it is computationally difficult to guess the removed value  $r_i$  from the new seed.

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**TreeGen:** Given  $m$ , output a binary tree  $T$  with  $m$  leaves:

1. Pick a  $p$  uniformly at random from  $\{1, 2, \dots, m - 1\}$ .
  2. Recursively generate a tree  $T_1$  with  $p$  leaves.
  3. Recursively generate a tree  $T_2$  with  $m - p$  leaves.
  4. Output a binary tree with  $T_1$  as the left subtree and  $T_2$  as the right subtree.
- 

**Table 6.** TreeGen: a random tree generation algorithm

Unlike the method proposed by Johnson et al. [JMSW02], our tree  $T$  is randomly generated. If the tree is known to be balanced (or known to be of some fixed structure), some information on the number of leaf nodes removed can be derived from the redacted tree. Our random trees are generated by the probabilistic algorithm TreeGen in Table 6. Note that descriptions of the structure of the tree are required for the regeneration of the random values  $r_i$ 's.

At the moment, for ease of presentation, the descriptions are stored together with the seed. This increases the size of the seed. To reduce the size, we can replace the description by another short random seed  $\hat{t}$ , which is assigned to the root. The random input required in Step 1 of the algorithm can be generated from  $\hat{t}$  using  $G$ . A difference between the two methods of storing the (redacted) tree structure information is that in the former, we will have an information theoretic security result, whereas in the later, the security depends on the security of  $G$ .

Our *main observation* is as follows: after a substring of leaves is removed from the random tree, the remaining subtrees do not reveal (information theoretically) anything about the number of leaves removed, except the fact that at least one leaf has been removed at that location.

*Notations.* Given a binary tree  $T$ , its leaf nodes can be listed from left to right to obtain a sequence. We call a subsequence of consecutive leaves a substring of leaves. After multiple substrings of leaves and all of their ancestor nodes are deleted, the remaining structures form a *redacted tree*<sup>6</sup> represented by two sequences,  $\mathbf{T} = \langle T_1, T_2, \dots, T_v \rangle$  and  $\mathbf{b} = \langle m, b_1, b_2, \dots, b_u \rangle$ . where  $T_i$ 's are the subtrees retained, and each  $b_i$  indicates that a substring was removed between the  $b_i$ -th and  $(b_i + 1)$ -th locations in the remaining sequence of leaf nodes. Let  $q_i$  be the number of leaves that were removed in this substring. We call the sequence  $\langle m, (b_1, q_1), (b_2, q_2), \dots, (b_u, q_u) \rangle$  the *original annotation* of  $\mathbf{b}$ . Thus, the total number of leaf nodes removed is  $\sum_{i=1}^u q_i$ .

Let us consider this process. Given an original annotation  $\mathbf{b}_1 = \langle m, (b_1, q_1), (b_2, q_2), \dots, (b_u, q_u) \rangle$ , for a string of size  $m$ , a random tree  $T$  of size  $m + \sum_{i=1}^u q_i$  is generated using **TreeGen**, and then redacted according to  $\mathbf{b}_1$ . Let  $\text{RED}(\mathbf{b}_1)$  be the redacted tree.

From an adversary's point of view, he has  $\text{RED}(\mathbf{b}_1)$ , represented as  $(\mathbf{T}, \mathbf{b})$ , and wants to guess the  $q_i$ 's in the original annotation  $\mathbf{b}_1$ , or the original total number of leaf nodes. We want to show that the additional knowledge of  $\mathbf{T}$  does not improve his chances, compared to another adversary who only has  $\mathbf{b}$ . It suffice to show that, given any  $\mathbf{b}$  and any two possible original annotations  $\mathbf{b}_1$  and  $\mathbf{b}_2$ , the conditional probabilities of obtaining  $(\mathbf{T}, \mathbf{b})$  are the same. That is,

**THEOREM 4** *For any redacted tree  $(\mathbf{T}, \mathbf{b})$ , any distribution  $\mathcal{B}$  on the original annotation, and  $\mathbf{b}_1 = \langle m, (b_1, q_1), (b_2, q_2), \dots, (b_u, q_u) \rangle$ ,  $\mathbf{b}_2 = \langle m, (b_1, q'_1), (b_2, q'_2), \dots, (b_u, q'_u) \rangle$ ,*

$$\text{Prob}(\text{RED}(\mathcal{B}) = (\mathbf{T}, \mathbf{b}) \mid \mathcal{B} = \mathbf{b}_1) = \text{Prob}(\text{RED}(\mathcal{B}) = (\mathbf{T}, \mathbf{b}) \mid \mathcal{B} = \mathbf{b}_2)$$

The proof is given in Appendix D.

## 6 **SRSS**: A Short Redactable Signature Scheme for Strings

**RSS** produces a large signature, whose main portion is a sequence of true random numbers  $r_i$ 's. We can combine **RGGM** with **RSS** to produce a short signature by replacing the  $r_i$ 's with pseudo random numbers generated by **RGGM**. Let us call this combined scheme **SRSS**, short redactable signature scheme for strings.

It is easy to show that **SRSS** is unforgeable and privacy preserving from Lemma 1, Theorem 2, Theorem 3, Theorem 4, and the fact that **RGGM** is a pseudo random number generator.

<sup>6</sup> Although strictly speaking it is a forest.

*Unforgeability.* From the definition of cryptographic secure pseudo random number generator and Theorem 2, we conclude that  $\mathcal{SRSS}$  is unforgeable.

**COROLLARY 5** *For any positive polynomials (in  $\kappa$ )  $t$  and  $q$ ,  $\mathcal{SRSS}$  is  $(t, q, \frac{\epsilon_1}{1-\epsilon_2})$ -unforgeable against existential forgeries with respect to  $\succ$ , if  $\mathcal{SSS}$  is  $(t+qt_0, q, \epsilon_1)$ -unforgeable against existential forgeries with respect to  $\supseteq$ ,  $\mathcal{H}$  is  $(t+qt_1, \epsilon_2)$ -collision-resistant, and  $G$  is a cryptographic secure pseudo random number generator, where  $t_0$  is the running time of  $\mathcal{H}$ ,  $t_1$  is the time needed by  $\mathcal{SRSS}$  to sign a document, and  $\kappa$  is the security parameter.*

*Privacy.* From the definition of cryptographic secure pseudo random number generator, Theorem 3 and Theorem 4, we conclude that  $\mathcal{SRSS}$  is privacy preserving.

**COROLLARY 6** *The redactable signature scheme  $\mathcal{SRSS}$  is privacy preserving (as defined in Definition 3), assuming that the hash function  $h$  satisfies the property: the two distributions  $X = g^{h(U_1)h(U_2)} \bmod n$  and  $Y = g^{h(U'_1)} \bmod n$  are computationally indistinguishable, and  $G$  is a cryptographic secure pseudo random number generator, where  $n$  is a RSA modulus,  $g$  is an element of large order in  $\mathbb{Z}_n^*$  and  $U_i$ 's and  $U'_j$ 's are all independent uniform random variables over  $\mathbb{Z}_n$ , and  $h(\cdot)$  is used to define  $\mathcal{H}$  in Table 1.*

*Efficiency.* The improvement of  $\mathcal{SRSS}$  is in signature size. Given the unredacted string, the size of the signature is  $\kappa + 2k$ , where  $\kappa$  is the signature size of  $\mathcal{SSS}$ , and  $k$  is the length of each seed. Recall that we need two seeds in RGM, one for the generation of the numbers, and the other for the tree structure.

If  $t$  substrings are removed, the signature size is  $\kappa + tk + O(kt \log m)$ , where the term  $tk$  is for the witness, and  $O(kt \log m)$  is required for the RGM.

## 7 Other variants

### 7.1 Allowing removal of empty substring

Both  $\mathcal{RSS}$  and  $\mathcal{SRSS}$  do not allow removal of empty substrings. In fact, it is considered to be a forgery if a censor declares that a substring has been removed but actually he/she doesnot remove anything.

It is desirable to allow removal of empty substrings in some application scenarios. This can be achieved by slight modifications to our schemes. To sign a string  $x_1x_2 \dots x_m$ , special symbol  $\natural$  is inserted to obtain the expanded string  $\tilde{x} = \natural x_1 \natural x_2 \natural \dots \natural x_m \natural$  which will be signed directly using  $\mathcal{RSS}$  or  $\mathcal{SRSS}$ . To remove a substring  $\mathbf{x}_0$ , the expanded substring of  $\mathbf{x}_0$  is actually removed. In the case where a substring has already being removed in front or at the end of  $\mathbf{x}_0$ , the  $\natural$  is not included at the front or the end accordingly. To remove an empty substring, simply remove the  $\natural$  at intended location.

## 7.2 Hiding the fact that the string is redacted

There is a question on whether one should hide the location of a removed substring or even the occurrence of redaction. This requirement is also known as *invisibility* or *transparency* [ACMT05,MHI06]. For a small object space, if invisibility is satisfied, a censor may take a long signed string, remove some substrings to form an arbitrary “authentic” short string. Nevertheless, some applications may need invisibility.

Here is a simple variation of  $\mathcal{RSS}$  that achieves this. To sign a string, simply add a special symbols  $\#$  in-between any two consecutive objects. Sign the expanded string and then immediately redact it by removing all  $\#$ 's. Redaction and verification is the same as before.

However, this variant produces a large signature even if we use  $\mathcal{SRSS}$ . Furthermore, the computation during verification is high. At least  $\Omega(m^2)$  exponentiation operations are required.

To reduce the size of signature, there is an alternative: sign all the pairs of objects. To sign the string  $\mathbf{x} = x_1x_2x_3\dots x_m$ , first generate random numbers  $r_1, r_2, \dots, r_m$  such that  $r_i \| x_i$ 's are distinct. Next, let  $\mathbf{t}$  be the set of all pairs  $\{(r_i \| x_i, r_j \| x_j)\}_{i < j}$  and employ SSS to sign  $\mathbf{t}$ . When an object  $x_i$  is to be removed, simply remove all the pairs that involve  $x_i$  from  $\mathbf{t}$ .

Since the role of  $r_i$  is to ensure that all elements are distinct, the size of each  $r_i$  can be smaller than the random numbers required by  $\mathcal{RSS}$ .

## 8 Discussion and Conclusion

We considered a simple but difficult requirement in redactable signature scheme: hiding the lengths of the removed substrings. We exploited an intriguing statistical property of random trees, and employed a hash from strings to unordered sets to achieve the requirement. Although the signature is short, its size still depends on the number of substrings removed and the length of the string. In contrast, there are known schemes for unordered sets, whose signature size is a constant. Intuitively, the larger signature size is the price that we need pay for the ordering information. To keep the same ratio of message space size to the signature space size, the signature size of string with length  $m$  should be about  $m \log m$  times larger than that for corresponding unordered set. The computation time required during verification is high for long strings when many substrings are removed. It is interesting to find a practical way to organize the string hierarchically, so as to achieve speedup.

The two main components, the hash  $\mathcal{H}$  and the RGGM tree, proposed in this paper, could be of independent interests. The hash function may play a role in the design of transitive signature with additional property on privacy preservation. Many secure outsourced database applications involve Merkle tree or GGM tree. The hiding property of the RGGM tree may be useful in those applications.

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## A Proof of Lemma 1

LEMMA 1. The hash function  $\mathcal{H}$  as defined in Table 1, is  $(poly_1(k), \frac{1}{poly_2(k)})$ -collision-resistant for any positive polynomials  $poly_1(\cdot)$  and  $poly_2(\cdot)$ , where  $k$  is the security parameter, i.e. the bit length of  $n$ , assuming that  $h$  is division intractable<sup>7</sup> and always outputs odd prime integers, and Strong RSA Problem is hard.

<sup>7</sup> Division intractability [GHR99] implies collision resistance.

*Proof.* (Sketch) Suppose there exists a probabilistic polynomial time algorithm  $\mathcal{A}$ , which takes as input  $(\mathbf{r}_1, \mathbf{w}_1)$ , and outputs  $(\mathbf{x}_1, \mathbf{e}_1), (\mathbf{x}_2, \mathbf{e}_2)$ , and  $(\mathbf{r}_2, \mathbf{w}_2)$ , such that  $((\mathbf{x}_2, \mathbf{e}_2), (\mathbf{r}_2, \mathbf{w}_2))$  is a collision with  $((\mathbf{x}_1, \mathbf{e}_1), (\mathbf{r}_1, \mathbf{w}_1))$  with noticeable probability.

We first consider the special case where  $\mathbf{w}_1 = \underbrace{\langle g, g, \dots, g \rangle}_{m \text{ } g\text{'s}}$  and  $\mathbf{e}_1 = \langle m \rangle$ .

Let  $\mathbf{x}_1 = x_{1,1}x_{1,2} \dots x_{1,m}, \mathbf{x}_2 = x_{2,1}x_{2,2} \dots x_{2,m'}, \mathbf{r}_1 = \langle r_{1,1}, r_{1,2}, \dots, r_{1,m} \rangle, \mathbf{r}_2 = \langle r_{2,1}, r_{2,2}, \dots, r_{2,m'} \rangle, \mathbf{w}_2 = \langle w_1, w_2, \dots, w_{m'} \rangle$  for some  $m' < m$ . By the definition of collision, we have

$$\begin{aligned} (\mathbf{x}_1, \mathbf{e}_1) &\not\asymp (\mathbf{x}_2, \mathbf{e}_2) \quad \wedge \\ \mathcal{H}((\mathbf{x}_2, \mathbf{e}_2), (\mathbf{r}_2, \mathbf{w}_2)) &= \{t_{2,1}, \dots, t_{2,m'}\} \subset \\ \mathcal{H}((\mathbf{x}_1, \mathbf{e}_1), (\mathbf{r}_1, \mathbf{w}_1)) &= \{t_{1,1}, \dots, t_{1,m}\}, \end{aligned}$$

where for each  $i$ ,  $t_{1,i} = (x_{1,i}, r_{1,i}, (g^{\prod_{j=1}^i h(r_{1,j})} \bmod n))$  and  $t_{2,i} = (x_{2,i}, r_{2,i}, (w_i^{\prod_{j=1}^i h(r_{2,j})} \bmod n))$ .

There are two different cases of collisions: 1)  $\mathbf{x}_2$  is not a substring of  $\mathbf{x}_1$ , e.g.  $\mathbf{x}_1 = abc, \mathbf{x}_2 = cb$ ; 2)  $\mathbf{x}_2$  is a substring of  $\mathbf{x}_1$  but their annotations  $\mathbf{e}_1$  and  $\mathbf{e}_2$  are incompatible, e.g.  $\mathbf{x}_1 = abc, \mathbf{w}_1 = ggg, \mathbf{e}_1 = \langle 3 \rangle$ , and  $\mathbf{x}_2 = ac, \mathbf{w}_2 = gg, \mathbf{e}_2 = \langle 2 \rangle$ .

*Case 1:  $\mathbf{x}_2$  is not a substring of  $\mathbf{x}_1$ .* the sequence  $\langle t_{2,1}, t_{2,2}, \dots, t_{2,m'} \rangle$  must be a re-permutation of some subsequence of the sequence  $\langle t_{1,1}, t_{1,2}, \dots, t_{1,m} \rangle$ . Thus, there must be a swap of positions, i.e. there are indices  $a, b, c$  and  $d$  such that  $a < b, c < d$ , and  $t_{1,a} = t_{2,d}$  and  $t_{1,b} = t_{2,c}$ . That means,

$$(x_{1,a}, r_{1,a}, g^A) = (x_{2,d}, r_{2,d}, w_d^{EF}), \quad \text{and} \quad (x_{1,b}, r_{1,b}, g^{AB}) = (x_{2,c}, r_{2,c}, w_c^E),$$

where  $A = \prod_{i=1}^a h(r_{1,i}), B = \prod_{i=a+1}^b h(r_{1,i}), E = \prod_{i=1}^c h(r_{2,i}), F = \prod_{i=c+1}^d h(r_{2,i})$ .

Next we can conduct a case analysis based on whether  $EF$  divides  $A$ , to find a contradiction with our assumptions. If  $EF$  does not divide  $A$  (with noticeable probability), we could solve the Strong RSA Problem based on algorithm  $\mathcal{A}$  using similar techniques in the proof of Theorem 5 in [GHR99]; otherwise we could solve the Division Problem [GHR99].

*Case 2:  $\mathbf{x}_2$  is a substring of  $\mathbf{x}_1$  but they have incompatible annotations.* Let  $\mathbf{e}_3$  be the annotation such that  $(\mathbf{x}_1, \mathbf{e}_1) \succ (\mathbf{x}_2, \mathbf{e}_3)$ . Let  $\mathbf{w}_3 = w_{3,1} \dots w_{3,m'}$  be the witness obtained from step 1 of Table 5 when redacting  $((\mathbf{x}_1, \mathbf{e}_1), (\mathbf{r}_1, \mathbf{w}_1))$  to produce a signature for the annotated substring  $(\mathbf{x}_2, \mathbf{e}_3)$ . We know

$$\mathcal{H}((\mathbf{x}_2, \mathbf{e}_3), (\mathbf{r}_2, \mathbf{w}_3)) \subset \mathcal{H}((\mathbf{x}_1, \mathbf{e}_1), (\mathbf{r}_1, \mathbf{w}_1))$$

and

$$\mathcal{H}((\mathbf{x}_2, \mathbf{e}_2), (\mathbf{r}_2, \mathbf{w}_2)) \subset \mathcal{H}((\mathbf{x}_1, \mathbf{e}_1), (\mathbf{r}_1, \mathbf{w}_1)).$$



Because distinct random numbers  $r_{1,i}$ 's and  $r_{2,j}$ 's are involved in the output of  $\mathcal{H}$ , we conclude that  $\mathcal{H}((\mathbf{x}_2, \mathbf{e}_3), (\mathbf{r}_2, \mathbf{w}_3)) = \mathcal{H}((\mathbf{x}_2, \mathbf{e}_2), (\mathbf{r}_2, \mathbf{w}_2))$ . We also know  $\mathbf{w}_2$  and  $\mathbf{w}_3$  implies different annotations  $\mathbf{e}_2$  and  $\mathbf{e}_3$ , so  $\mathbf{w}_2 \neq \mathbf{w}_3$ . That means there exists an index  $i, 1 \leq i \leq m'$ , such that

$$w_{2,i} \neq w_{3,i} \text{ and } w_{2,i}^Q = w_{3,i}^Q \pmod n, \quad (1)$$

where  $Q = \prod_{j=1}^i h(r_{2,j})$  is an odd number. Recall that  $n = pq, p = 2p' + 1, q = 2q' + 1$  and  $p, q, p', q'$  are all prime numbers. Let  $\lambda(n) = \text{lcm}(p-1, q-1) = 2p'q'$ . Suppose  $Q$  is coprime with  $\lambda(n)$ . Then  $w_{2,i}^Q = w_{3,i}^Q \pmod n$  implies  $w_{2,i} = w_{3,i}$ , which is a contradiction with Eq 1. So we conclude that the odd number  $Q$  and  $\lambda(n) = 2p'q'$  have common factors, i.e.,  $p'$  or  $q'$  divides  $Q$ . W.L.O.G., we assume  $p' | Q$ .

Because  $h(\cdot)$  outputs odd prime number only, there must exist  $\xi, 1 \leq \xi \leq m'$ ,  $h(r_{2,\xi}) = p'$ . Consequently,  $2h(r_{2,\xi}) + 1$  divides  $n$ , and we can factorize  $n$  efficiently by trying every  $h(r_{2,i}), 1 \leq i \leq m'$ .

In the beginning of the proof, we assume that  $\mathbf{w}_1 = \langle g, \dots, g \rangle$ . For case 1, it is easy to generalize the proof for general  $\mathbf{w}_1 = \langle w_{1,1}, w_{1,2}, \dots, w_{1,m} \rangle$  using similar techniques in the proof of Theorem 5 in [GHR99], by randomly guessing the index  $a$  and setting  $w_{1,a} = s$ ; for case 2, there is no need of changes in the proof.

## B Proof of Theorem 2

**THEOREM 2.** The scheme  $\mathcal{RSS}$  as constructed in Table 3, Table 4 and Table 5, is  $(t, q, \frac{\epsilon_1}{1-\epsilon_2})$ -secure against existential forgeries with respect to  $\succ$ , if  $\mathcal{SSS}$  is  $(t + qt_0, q, \epsilon_1)$ -secure against existential forgeries with respect to  $\supseteq$ , and  $\mathcal{H}$  is  $(t + qt_1, \epsilon_2)$ -collision-resistant, where  $t_0$  is the running time of  $\mathcal{H}$  and  $t_1$  is the time needed by  $\mathcal{RSS}$  to sign a document.

*Proof:* We prove this theorem using proof by contradiction. Let  $\mathcal{O}^{\mathcal{RSS}}$  and  $\mathcal{O}^{\mathcal{SSS}}$  denote the oracles for Sign algorithms of  $\mathcal{RSS}$  and  $\mathcal{SSS}$  respectively, and both use the same public/private key  $(\mathcal{PK}, \mathcal{SK})$ . Suppose there exists an adversary  $\mathcal{A}$  against  $\mathcal{RSS}$ , such that  $\mathcal{A}$  runs in time at most  $t$ , makes at most  $q$  chosen-message queries to  $\mathcal{O}^{\mathcal{RSS}}$ , and has advantage  $\text{Adv}_{\mathcal{A}} > \epsilon_3 = \frac{\epsilon_1}{1-\epsilon_2}$ . Let  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_\ell$  denote the  $\ell (\leq q)$  messages (annotated strings of objects) chosen by  $\mathcal{A}$ , and  $(\sigma_i, \mathbf{r}_i)$  be the reply from  $\mathcal{O}^{\mathcal{RSS}}$  for query  $\mathbf{x}_i$ . Let  $(\mathbf{x}, (\sigma, \mathbf{r}))$  denote the output of  $\mathcal{A}$ .

$$\text{Adv}_{\mathcal{A}} = \Pr(\text{Verify}_{\mathcal{PK}}(\mathbf{x}, (\mathbf{r}, \sigma)) = \text{TRUE} \wedge \mathbf{x} \notin \text{span}_{\succ}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_\ell)) > \epsilon_3.$$

Now we construct an adversary  $\mathcal{B}$  against  $\mathcal{SSS}$  invoking  $\mathcal{A}$  as a subroutine and simulating  $\mathcal{O}^{\mathcal{RSS}}$  with  $\mathcal{O}^{\mathcal{SSS}}$ :

1. Run algorithm  $\mathcal{A}$ .
2. Receive query  $\mathbf{x}_i$  from  $\mathcal{A}$ .

3. Choose a random coin  $\mathbf{r}_i$ , and compute  $\mathcal{H}(\mathbf{x}_i, \mathbf{r}_i)$ . Make query  $\mathcal{H}(\mathbf{x}_i, \mathbf{r}_i)$  to  $\mathcal{O}^{\text{SSS}}$  and get reply  $\sigma_i$ .
4. Reply  $(\sigma_i, \mathbf{r}_i)$  to  $\mathcal{A}$  corresponding to the query  $\mathbf{x}_i$ .
5. If receiving another query from  $\mathcal{A}$ , go to Step 2.
6. Receive output  $(\mathbf{x}, (\sigma, \mathbf{r}))$  from  $\mathcal{A}$ .
7. Output  $(\mathcal{H}(\mathbf{x}, \mathbf{r}), \sigma)$ .

The number of queries made to  $\mathcal{O}^{\text{SSS}}$  by  $\mathcal{B}$  equals to the number of queries made to (simulated)  $\mathcal{O}^{\text{RSS}}$  by  $\mathcal{A}$ . The running time of  $\mathcal{B}$  is at most  $t + qt_0$ .

With probability larger than  $\epsilon_3$ ,  $(\mathbf{x}, (\sigma, \mathbf{r}))$  is an existential forgery w.r.t to  $\text{RSS}$ , formally

$$\Pr(\text{Verify}_{\mathcal{PK}}(\mathbf{x}, (\sigma, \mathbf{r})) = \text{TRUE} \wedge \mathbf{x} \notin \text{span}_{\succ}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_\ell)) > \epsilon_3.$$

Suppose  $\mathbf{x}_i \neq \mathbf{x}$  for any  $i$ . Because  $\mathcal{H}$  is  $(t + qt_1, \epsilon_2)$ -collision-resistant and real running time (taking the time for signing into account) of  $\mathcal{A}$  is at most  $t + qt_1$ , we have

$$\Pr(\bigvee_{i=1}^{\ell} (\mathcal{H}(\mathbf{x}, \mathbf{r}) \subseteq \mathcal{H}(\mathbf{x}_i, \mathbf{r}_i))) \leq \epsilon_2.$$

Hence,

$$\Pr\left(\begin{array}{c} \text{vrf}_{\mathcal{PK}}(\mathcal{H}(\mathbf{x}, \mathbf{r}), \sigma) = \text{TRUE} \wedge \\ \mathcal{H}(\mathbf{x}, \mathbf{r}) \notin \text{span}_{\supseteq}(\mathcal{H}(\mathbf{x}_1, \mathbf{r}_1), \mathcal{H}(\mathbf{x}_2, \mathbf{r}_2), \dots, \mathcal{H}(\mathbf{x}_\ell, \mathbf{r}_\ell)) \end{array}\right) > \epsilon_3(1 - \epsilon_2) = \epsilon_1.$$

That means  $\mathcal{B}$  is an existential forger against  $\text{SSS}$ :  $\mathcal{B}$  runs in time at most  $t + qt_0$ , makes at most  $q$  queries to  $\mathcal{O}^{\text{SSS}}$ , and has advantage larger than  $\epsilon_1$ . This is a contradiction with our assumption that  $\text{SSS}$  is  $(t + qt_0, q, \epsilon_1)$ -secure against existential forgeries.

Hence, our hypothesis that there exists such adversary  $\mathcal{A}$  is wrong. In other words,  $\text{RSS}$  is  $(t, q, \frac{\epsilon_1}{1 - \epsilon_2})$ -secure against existential forgeries with respect to  $\succ$ .  $\square$

## C Proof of Theorem 3

**Lemma 1.** *If the distributions of  $X_1$  and  $X_2$  are computationally indistinguishable, then for any constant integers  $\ell, m > 1$ , the distributions of  $X_\ell$  and  $X_m$  are computationally indistinguishable, where for any integer  $N > 1$ ,  $X_N$  represents the random variable  $g^{h(U_1)h(U_2)h(U_3)\dots h(U_N)} \bmod n$ ,  $n$  is a RSA modulus,  $g$  is an element of large order in  $\mathbb{Z}_n^*$  and  $U_i$ 's are all independent uniform random variables over  $\mathbb{Z}_n$ .*

*Proof:* (Sketch) Let  $X \sim Y$  denote that the two distributions  $X$  and  $Y$  are computationally indistinguishable. For any  $\ell \geq 2$ ,  $X_1 \sim X_2$  implies  $X_\ell \sim X_{\ell+1}$ . As a consequence, for any constant  $\ell < m$ , using the hybrid technique, we have

$$X_\ell \sim X_{\ell+1} \sim X_{\ell+2} \dots \sim X_{m-1} \sim X_m.$$

Hence,

$$\mathbf{X}_\ell \sim \mathbf{X}_m.$$

□

**THEOREM 3.** The redactable signature scheme  $\mathcal{RSS}$  is privacy preserving (as defined in Definition 3), assuming that hash function  $h$  satisfies the property: the two distributions  $\mathbf{X} = g^{h(U_1)h(U_2)} \pmod n$  and  $\mathbf{Y} = g^{h(U'_1)} \pmod n$  are computationally indistinguishable, where  $n$  is a RSA modulus,  $g$  is an element of large order in  $\mathbb{Z}_n^*$  and  $U_i$ 's and  $U'_j$ 's are all independent uniform random variables over  $\mathbb{Z}_n$ .

*Proof:* W.L.O.G, we represent the three strings  $\mathbf{x}_1$ ,  $\mathbf{x}_2$  and  $\mathbf{x}$  as stated in Definition 3 in following way:

$$\begin{aligned} \mathbf{x}_1 &= x_1x_2x_3 \dots x_{a_1}\mathbf{y}_1x_{a_1+1} \dots x_{a_2}\mathbf{y}_2x_{a_2+1} \dots x_{a_\ell}\mathbf{y}_\ell x_{a_\ell+1} \dots x_m \\ \mathbf{x}_2 &= x_1x_2x_3 \dots x_{a_1}\mathbf{z}_1x_{a_1+1} \dots x_{a_2}\mathbf{z}_2x_{a_2+1} \dots x_{a_\ell}\mathbf{z}_\ell x_{a_\ell+1} \dots x_m \\ \mathbf{x} &= x_1x_2x_3 \dots x_{a_1} \diamond x_{a_1+1} \dots x_{a_2} \diamond x_{a_2+1} \dots x_{a_\ell} \diamond x_{a_\ell+1} \dots x_m \end{aligned}$$

$\mathbf{x}$  is a string of size  $m$  with  $\ell$  substrings being redacted at locations  $a_1, a_2, a_3, \dots, a_\ell$ .  $\mathbf{x}$  can be obtained from  $\mathbf{x}_1$  by removing non-empty substrings  $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_\ell$ , or from  $\mathbf{x}_2$  by removing non-empty substrings  $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_\ell$ .

For any sequence  $\mathbf{b} = \langle b_1, b_2, \dots, b_m \rangle$ , define  $h(\mathbf{b})$  as

$$h(\mathbf{b}) = \prod_{i=1}^m h(b_i).$$

Denote with  $\mathbf{r}_{y,i}$  (or  $\mathbf{r}_{z,i}$ ) the sequence of random numbers associated with substring  $\mathbf{y}_i$  (or  $\mathbf{z}_i$ ) respectively. Denote with  $\mathbf{r}_1 = \langle r_{1,1}, r_{1,2}, \dots, r_{1,m} \rangle$  the sequence of random numbers, which are associated with  $x_i$ 's and chosen when signing  $\mathbf{x}_1$ . Denote with  $\mathbf{r}_2 = \langle r_{2,1}, r_{2,2}, \dots, r_{2,m} \rangle$  the sequence of random numbers, which are associated with  $x_i$ 's and chosen when signing  $\mathbf{x}_2$ . All  $r_{1,i}$  and  $r_{2,i}$  are independent uniform random numbers over the set of odd numbers in  $\mathbb{Z}_n$ .

The signature of  $\mathbf{x}$  under  $\mathcal{RSS}$  by redacting  $\mathbf{x}_1$  is

$$\begin{aligned} \sigma_1 &= \text{Redact}_{\mathcal{PK}}(\mathbf{x}_1, \text{Sign}_{\mathcal{SK}}(\mathbf{x}_1), \mathbf{x}) \\ &= (\delta_1, \mathbf{r}_1 = \langle r_{1,1}, r_{1,2}, r_{1,3}, \dots, r_{1,m} \rangle, \mathbf{w}_1 = \langle w_{1,1}, w_{1,2}, w_{1,3}, \dots, w_{1,m} \rangle) \end{aligned}$$

where for each  $0 \leq i \leq \ell$ , assuming  $a_0 = 0, a_{\ell+1} = m + 1, h(\mathbf{r}_{y,0}) = 1$ ,

$$w_{1,a_i+1} = w_{1,a_i+2} = \dots = w_{1,a_{i+1}} = g^{\prod_{j=0}^i h(\mathbf{r}_{y,j})} \pmod n \quad (2)$$

and  $\delta_1$  is the signature under SSS of the following set  $H_1$

$$H_1 = \mathcal{H}(\mathbf{x}, (\mathbf{r}_1, \mathbf{w}_1)) = \left\{ s : s = \left( x_i, r_{1,i}, \left( w_{1,i}^{\prod_{j=1}^i h(r_{1,j})} \pmod n \right) \right) \right\}.$$

Similarly, the signature of  $\mathbf{x}$  under  $\mathcal{RSS}$  by redacting  $\mathbf{x}_2$  is

$$\begin{aligned}\sigma_2 &= \text{Redact}_{\mathcal{PK}}(\mathbf{x}_2, \text{Sign}_{\mathcal{SK}}(\mathbf{x}_2), \mathbf{x}) \\ &= (\delta_2, \mathbf{r}_2 = \langle r_{2,1}, r_{2,2}, r_{2,3}, \dots, r_{2,m} \rangle, \mathbf{w}_2 = \langle w_{2,1}, w_{2,2}, w_{2,3}, \dots, w_{2,m} \rangle)\end{aligned}$$

where for each  $0 \leq i \leq \ell$ , assuming  $a_0 = 0, a_{\ell+1} = m + 1, h(\mathbf{r}_{z,0}) = 1,$

$$w_{2,a_i+1} = w_{2,a_i+2} = \dots = w_{2,a_{i+1}} = g^{\prod_{j=0}^i h(\mathbf{r}_{z,j})} \pmod n \quad (3)$$

and  $\delta_2$  is the signature under  $\mathcal{SSS}$  of the following set  $H_2$

$$H_2 = \mathcal{H}(\mathbf{x}, (\mathbf{r}_2, \mathbf{w}_2)) = \left\{ s : s = \left( x_i, r_{2,i}, \left( w_{2,i}^{\prod_{j=1}^i h(r_{2,j})} \pmod n \right) \right) \right\}.$$

Note  $w_{1,i}$ 's and  $w_{2,i}$ 's can be written in form of  $g^{\prod h(\cdot)} \pmod n$  as in Equation 2 and 3. So  $\mathbf{w}_1$  and  $\mathbf{w}_2$  are indistinguishable. Furthermore,  $w_{1,i}^{\prod_{j=1}^i h(r_{1,j})} \pmod n$  and  $w_{2,i}^{\prod_{j=1}^i h(r_{2,j})} \pmod n$  are indistinguishable for each  $i$ . As a result  $H_1$  is indistinguishable from  $H_2$ , and  $\delta_1$  is indistinguishable from  $\delta_2$ .

Hence,  $\sigma_1$  and  $\sigma_2$  are computationally indistinguishable.  $\square$

## D Proof of Theorem 4

**THEOREM 4.** For any redacted tree  $(\mathbf{T}, \mathbf{b})$ , any distribution  $\mathcal{B}$  on the original annotation, and  $\mathbf{b}_1 = \langle m, (b_1, q_1), (b_2, q_2), \dots, (b_u, q_u) \rangle,$   
 $\mathbf{b}_2 = \langle m, (b_1, q'_1), (b_2, q'_2), \dots, (b_u, q'_u) \rangle,$

$$\text{Prob}(\text{RED}(\mathcal{B}) = (\mathbf{T}, \mathbf{b}) \mid \mathcal{B} = \mathbf{b}_1) = \text{Prob}(\text{RED}(\mathcal{B}) = (\mathbf{T}, \mathbf{b}) \mid \mathcal{B} = \mathbf{b}_2)$$

*Proof:* (Sketch)

1. Consider the random tree generation algorithm **TreePermGen** in Table 7, which takes in random permutations as the source of randomness. The distribution of the random trees generated by **TreePermGen** is the same as the distribution by **TreeGen** (Table 6). To see that, consider the root and its left subtree. The distribution of the size of the left subtree is the same for both algorithms. This observation can be generalized to the distribution of the whole tree.

Given a permutation  $\pi$  and an original annotation  $\mathbf{b}'$ , let us write  $\mathcal{A}(\pi)$  as the tree generated by **TreePermGen**. Recall that the tree is to be redacted by removing some leaves and all their ancestors, and the leaves to be removed are specified by an original annotation  $\mathbf{b}'$ . Let us write  $\mathcal{A}(\pi, \mathbf{b}')$  as the redacted tree.

2. We now analyze original annotations of these special forms:

$$\begin{aligned}\mathbf{b}_1 &= \langle M, (b_1, q_1), (b_2, q_2), \dots, (b_{i_0-1}, q_{i_0-1}), (b_{i_0}, k), (b_{i_0+1}, q_{i_0+1}) \dots, (b_u, q_u) \rangle \\ \mathbf{b}_2 &= \langle M, (b_1, q_1), (b_2, q_2), \dots, (b_{i_0-1}, q_{i_0-1}), (b_{i_0}, k + 1), (b_{i_0+1}, q_{i_0+1}) \dots, (b_u, q_u) \rangle\end{aligned}$$

---

**TreePermGen:** Given  $m$ , output a binary tree  $T$  with  $m$  leaves. During the execution of this algorithm, each node in  $T$  is associated with a subset of  $\{1, 2, \dots, m\}$ . Denote by  $\pi_m$  a permutation on  $m$  elements  $\{1, 2, \dots, m\}$ .

1. Randomly pick a  $\pi_{m-1}$ .
2. Let  $\mathcal{S}$  be a collection of sets, which is initially empty. Let  $T$  be an empty tree.
3. Let  $A = \{1, 2, 3, \dots, m\}$  and associate it to a new node  $p$ . Add  $A$  into  $\mathcal{S}$ , and insert  $p$  into  $T$  as the root node.
4. For  $i = 1$  to  $m - 1$ ,
  - (a) Find the set  $A \in \mathcal{S}$  s.t.  $\pi_{m-1}(i) \in A$ . Let  $p$  be the node that is associated with  $A$ . Remove  $A$  from  $\mathcal{S}$ .
  - (b) Divide  $A$  into two sets  $A_{\text{left}}$  and  $A_{\text{right}}$ , where

$$A_{\text{left}} = A \cap \{1, 2, \dots, \pi_{m-1}(i)\}, \text{ and}$$

$$A_{\text{right}} = A \cap \{\pi_{m-1}(i) + 1, \dots, m\}.$$

- (c) Create two children for  $p$  and insert them into  $T$ . Associate  $A_{\text{left}}$  and  $A_{\text{right}}$  to the left and right children respectively. Add  $A_{\text{left}}$  and  $A_{\text{right}}$  into  $\mathcal{S}$ .
  5. Output the final tree  $T$ , without the sets associated to its nodes.
- 

**Table 7.** **TreePermGen**, a tree generation algorithm using random permutation.

for some  $i_0$  and  $u$ . Under annotation  $\mathbf{b}_1$ , the  $i_0$ -th substring that is being removed has  $k$  items. Let the last item in this  $i_0$ -th removed substring be the  $v$ -th item in the original string, that is,  $v = b_{i_0} + k + \sum_{j=1}^{i_0-1} q_j$ . In contrast, under annotation  $\mathbf{b}_2$  the  $i_0$ -th substring that is removed has  $k + 1$  items. Note that the tree on which  $\mathbf{b}_1$  is applied has  $m$  leaf nodes, and the tree on which  $\mathbf{b}_2$  is applied has  $m + 1$  leaf nodes, where  $m = M + \sum_{j=1}^u q_j$  and  $q_{i_0} = k$ .

We want to show that, for any redacted tree, represented as  $(\mathbf{T}, \mathbf{b})$ , and any  $\mathbf{b}_1$  and  $\mathbf{b}_2$  as defined above, we have

$$m |\{\pi_{m-1} : \mathcal{A}(\pi_{m-1}, \mathbf{b}_1) = (\mathbf{T}, \mathbf{b})\}| = |\{\pi_m : \mathcal{A}(\pi_m, \mathbf{b}_2) = (\mathbf{T}, \mathbf{b})\}| \quad (4)$$

Since a permutation is uniformly chosen in Step 1 of **TreePermGen**, the conditional probability of obtaining  $(\mathbf{T}, \mathbf{b})$  given  $\mathbf{b}_1$  is

$$\frac{1}{(m-1)!} |\{\pi_{m-1} : \mathcal{A}(\pi_{m-1}, \mathbf{b}_1) = (\mathbf{T}, \mathbf{b})\}| \quad (5)$$

and the conditional probability given  $\mathbf{b}_2$  is

$$\frac{1}{m!} |\{\pi_m : \mathcal{A}(\pi_m, \mathbf{b}_2) = (\mathbf{T}, \mathbf{b})\}| \quad (6)$$

From (4), we have (5)=(6), which is the result of this theorem for  $\mathbf{b}_1$  and  $\mathbf{b}_2$ .

3. Given a  $\pi_{m-1}$ , we associate it with  $m$  permutations of  $m$  elements. For each  $1 \leq j \leq m$ , let:

$$\pi_{m,j}(i) = \begin{cases} \text{incr}(\pi_{m-1}(i)) & \text{if } i < j, \\ v & \text{if } i = j, \\ \text{incr}(\pi_{m-1}(i-1)) & \text{otherwise,} \end{cases} \quad (7)$$

$$\text{where } \text{incr}(x) = \begin{cases} x & \text{if } x < v, \\ x + 1 & \text{otherwise.} \end{cases} \quad (8)$$

In other words, we can associate  $\pi_{m-1}$  to  $m$  permutations of  $m$  elements by inserting  $v$  into  $m$  possible places in  $\pi_{m-1}$ , and incrementing items larger than  $v$  by one. For example, given the permutation  $\langle 3, 2, 1, 4 \rangle$  which generates a tree with the  $2^{\text{nd}}$  and  $3^{\text{rd}}$  leaf nodes removed,  $v = 3$ . Upon inserting  $v$  in between 2 and 1 in the permutation, the resultant permutation is  $\langle 4, 2, 3, 1, 5 \rangle$ .

4. We claim that  $\pi_{m-1}$  and  $\pi_{m,j}$ , for each  $j$ , all lead to the same redacted tree. That is, for any  $j$

$$\mathcal{A}(\pi_{m-1}, \mathbf{b}_1) = \mathcal{A}(\pi_{m,j}, \mathbf{b}_2) \quad (9)$$

5. There are  $m$  ways to insert  $v$  into  $\pi_{m-1}$ . Thus, for any  $\pi_{m-1}$ , there are  $m$  permutation  $\pi_m$  that leads to the same redacted tree. For a fixed  $1 \leq v \leq m$ , (5) and (6) together maps a  $(\pi_m, v, j)$  for any  $1 \leq j \leq m$ , to exactly one  $\pi_{m-1}$ , and every  $\pi_{m-1}$  is mapped to one  $(\pi_m, v, j)$  for some  $j$ . Thus (5) and (6) together is a bijective mapping and this process will generate all possible  $\pi_m$  from all possible  $\pi_{m-1}$ , and we have the equality (4) and the result for  $\mathbf{b}_1$  and  $\mathbf{b}_2$ . We can generalize the claim to any  $\mathbf{b}_1$  and  $\mathbf{b}_2$  that differs in more than one removed substrings and by more than one item each, by repeatedly applying the argument here on just one removed substring and holding the other removed substrings constant.

□