Ideal Hierarchical Secret Sharing Schemes *

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Abstract

The search of efficient constructions of ideal secret sharing schemes for families of non-threshold access structures that may have useful applications has attracted a lot of attention. Several proposals have been made for access structures with hierarchical properties, in which the participants are distributed into levels that are hierarchically ordered.

Here, we study hierarchical secret sharing in all generality by providing a natural definition for the family of the hierarchical access structures. Specifically, an access structure is said to be hierarchical if every two participants can be compared according to the following natural hierarchical order: whenever a participant in a qualified subset is substituted by a hierarchically superior participant, the new subset is still qualified.

We present a complete characterization of the ideal hierarchical access structures, that is, the ones admitting an ideal secret sharing scheme. We use the well known connection between ideal secret sharing and matroids and, in particular, the fact the every ideal access structure is a matroid port. In addition, we use recent results on ideal multipartite access structures and the connection between multipartite matroids and discrete polymatroids. We prove that every ideal hierarchical access structure is the port of a representable matroid and, more specifically, we prove that every ideal structure in this family admits ideal *linear* secret sharing schemes over fields of all characteristics. This generalizes previous results on weighted threshold access structures. Finally, we use our results to find a new characterization of the ideal weighted threshold access structures that is more precise than the existing one.

Key words. Secret sharing, Ideal secret sharing schemes, Hierarchical secret sharing, Weighted threshold secret sharing, Multipartite secret sharing, Multipartite matroids, Discrete polymatroids.

1 Introduction

A *secret sharing scheme* is a method to distribute *shares* of a *secret value* among a set of *participants*. Only the *qualified* subsets of participants can recover the secret value from their shares, while the *unqualified* subsets do not obtain any information about the secret value. The qualified subsets form the *access structure* of the scheme, which is a monotone increasing family of subsets of participants. Only *unconditionally secure perfect* secret sharing schemes are considered in this paper.

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Secret sharing was independently introduced by Shamir [32] and Blakley [5] in 1979. They presented two different methods to construct secret sharing schemes for *threshold access structures*, whose qualified subsets are those with at least some given number of participants. These schemes are *ideal*, that is, the length of every share is the same as the length of the secret, which is the best possible situation [17].

There exist scenarios in which non-threshold secret sharing schemes are required because, for instance, some participants should be more powerful than others. The first attempt to overcome the limitation of threshold access structures was made by Shamir in his seminal work [32] by proposing a simple modification of the threshold scheme. Namely, every participant receives as its share a certain number of shares from a threshold scheme, according to its position in the hierarchy. In this way a scheme for a *weighted threshold access structure* is obtained. That is, every participant has a weight (a positive integer) and a set is qualified if and only if its weight sum is at least a given threshold. This new scheme is not ideal because the shares have in general greater length than the secret.

Every access structure admits a secret sharing scheme [3, 15], but in general the shares must be longer than the secret [8, 10]. Very little is known about the optimal length of the shares in secret sharing schemes for general access structures, and there is a wide gap between the best known general lower and upper bounds.

Because of that, the construction of ideal secret sharing schemes for families of access structures that may have interesting applications is worth considering. This line of work was initiated by Simmons [33], who proposed two families of access structures, the *multilevel* and the *compartmented* ones, and conjectured them to admit ideal secret sharing schemes. Brickell [6] proposed a general method, based on linear algebra, to construct ideal secret sharing schemes for access structures that are not necessarily threshold, and he applied it to the construction of particular ideal secret sharing schemes proving the conjecture by Simmons. The multilevel and compartmented access structures are *multipartite*, which means that the participants are divided into several parts (levels or compartments) and all participants in the same part play an equivalent role in the structure. By using different kinds of polynomial interpolation, Tassa [35], and Tassa and Dyn [36] proposed constructions of ideal secret sharing schemes for several families of multipartite access structures, some of them with hierarchical properties. These constructions are based on the general linear algebra method by Brickell [6], but they provide schemes for the multilevel and compartmented access structures that are simpler and more efficient than the particular ones proposed in [6] for those structures. Other constructions of ideal multipartite secret sharing schemes have been presented in [25]. All these constructions require that the set of possible values of the secret is of a certain size. The optimization of this parameter have been considered in [4, 12] for very particular families of hierarchical access structures.

Another line of work is the characterization of the *ideal access structures*, that is, the ones admitting an ideal secret sharing scheme. This is an important and long-standing open problem in secret sharing. Brickell and Davenport [7] proved that every ideal secret sharing scheme defines a matroid. Actually, this matroid is univocally determined by the access structure of the scheme. This implies a necessary condition for an access structure to be ideal. Namely, every ideal access structure is a *matroid port*. A sufficient condition is obtained from the method to construct ideal secret sharing schemes by Brickell [6]: the ports of representable matroids are ideal access structures. Seymour [31] proved that the necessary condition is not sufficient, while the sufficient condition is not necessary because of the counterexample given by Simonis and Ashikhmin [34]. The results in [7] have been generalized in [19] by proving that, if all shares in a secret sharing scheme are shorter than 3/2 times the secret value, then its access structure is a matroid port. At this point, the remaining open question about the characterization of ideal access structures is determining the matroids that can be defined from ideal secret sharing schemes. Some important results, ideas and techniques to solve this question have been given by Matúš [21, 22].

In addition to the search of general results, several authors studied this open problem for particular families of access structures. Some of them deal with families of multipartite access structures. Beimel, Tassa and Weinreb [1] presented a characterization of the ideal weighted threshold access structures that generalizes the partial results in [23, 28]. Another important result about weighted threshold access structures have been obtained recently by Beimel and Weinreb [2]. They prove that all such access structures admit secret sharing schemes in which the size of the shares is quasi-polynomial in the number of users. A complete characterization of the ideal bipartite access structures was given in [28], and related results were given independently in [24, 26]. Partial results on the characterization of the ideal tripartite access structures appeared in [9, 13], and this question was solved in [11]. In every one of these families, all matroid ports are ports of representable matroids, and hence, all ideal access structures are *vector space access structures*, that is, they admit an ideal linear secret sharing scheme constructed by the method proposed by Brickell [6].

The characterization of the ideal tripartite access structures in [11] was obtained actually from the much more general results about ideal multipartite access structures in that paper. Pointing out the close connection between multipartite matroids and *discrete polymatroids* (a combinatorial object introduced by Herzog and Hibi [14]), and the use for the first time in secret sharing of these concepts are among the main contributions in [11]. The basic definitions and facts about discrete polymatroids and the main results in [11] are recalled in Section 6.

In this paper we continue and, in some way, culminate the line of research of those previous works by answering to the following question: what hierarchical access structures admit an ideal secret sharing scheme?

First of all, we formalize the concept of *hierarchical access structure* by introducing in Section 3 a natural definition for it. Basically, if a participant in a qualified subset is substituted by a *hierarchically superior* participant, the new subset must be still qualified. An access structure is *hierarchical* if, for any two given participants, one of them is hierarchically superior to the other. According to this definition, the family of the hierarchical access structures contains the multilevel access structures [6, 33], the hierarchical threshold access structures studied by Tassa [35] and by Tassa and Dyn [36], and also the weighted threshold access structures that were first considered by Shamir [32] and studied in [1, 2, 23, 28].

Duality and minors of access structures are fundamental concepts in secret sharing, as they are in matroid theory. Several important classes of access structures are closed by duality and minors, as for instance, matroid ports or $\mathbb{K}$-vector space access structures. Similarly to multipartite and weighted threshold access structures, the family of the hierarchical access structures is closed by duality and minors.

Our main result is Theorem 9.2, which provides a complete characterization of the ideal hierarchical access structures. In particular, we prove that all hierarchical matroid ports are ports of representable matroids. By combining this with the results in [19], we obtain the following theorem.

Theorem 1.1. *Let Γ be a hierarchical access structure. The following properties are equivalent:*

1. *Γ admits a vector space secret sharing scheme over every large enough finite field.*

2. Γ *is ideal.*

3. Γ *admits a secret sharing scheme in which the length of every share is less than* $3/2$ *times the length of the secret value.*

4. Γ *is a matroid port.*

This generalizes the analogous statement that holds for weighted threshold access structures as a consequence of the results in [1, 19]. Actually, as an application of our results, we present in Section 10 a new characterization of the ideal weighted threshold access structure that is more precise than the one given by Beimel, Tassa and Weinreb [1]. In addition, our results make it possible to solve an open problem proposed by Tassa [35].

The proofs of our results strongly rely on the connection between matroids and ideal secret sharing schemes discovered by Brickell and Davenport [7]. Moreover, since hierarchical access structures are in particular multipartite, the results and techniques in [11] about the characterization of ideal multipartite access structures are extremely useful in achieving our results. In particular, discrete polymatroids play a fundamental role in our proofs. Another important tool is the geometric representation introduced in [11, 28] for multipartite access structures, which is adapted in Section 4 to the hierarchical case by introducing the notion of access structures that are *stable* under some set of translations. By using this representation, every hierarchical access structure with m levels can be determined from the partition into levels and a set of points in $\mathbb{Z}^m$, the so-called H-minimal points of the structure. Our characterization of the ideal hierarchical access structures is given in terms of some properties of the H-minimal points that can be efficiently checked. By using our results, given a hierarchical access structure that is described by its H-minimal points, one can efficiently determine whether it is ideal or not. If the access structure is described by its minimal qualified subsets, it is easy to determine the H-minimal points. If the access structure is described in another way, one has to find the H-minimal points, but this can be done efficiently most of the times. This is the case, for instance, of weighted threshold access structures that are determined by the weights and the threshold.

2 Ideal Secret Sharing Schemes and Matroids

We recall in this section some facts about the connection between ideal secret sharing schemes and matroids that is derived from the results by Brickell [6] and by Brickell and Davenport [7]. See [19], for instance, for more information on these topics.

We begin by describing the method by Brickell [6] to construct ideal secret sharing schemes. Let C be an $[n+1, k]$-linear code over a finite field $\mathbb{K}$ and let M be a generator matrix of C, that is, a $k \times (n+1)$ matrix over $\mathbb{K}$ whose rows span C. Such a code defines an ideal secret sharing scheme on a set $P = \{p_1, \ldots, p_n\}$ of participants. Specifically, every random choice of a codeword $(s_0, s_1, \ldots, s_n) \in C$ corresponds to a distribution of shares for the secret value $s_0 \in \mathbb{K}$, in which $s_i \in \mathbb{K}$ is the share of the participant p_i. Such an ideal scheme is called a $\mathbb{K}$-*vector space secret sharing scheme* and its access structures is called a $\mathbb{K}$-*vector space access structure*. It is easy to check that a set $A \subseteq P$ is in the access structure Γ of this scheme if and only if the column of M with index 0 is a linear combination of the columns whose indices correspond to the players in A. Therefore, if $Q = P \cup \{p_0\}$ and $\mathcal{M}$ is the representable matroid with ground set Q and rank function r that is defined by the columns of the matrix M, then

$$\Gamma = \Gamma_{p_0}(\mathcal{M}) = \{A \subseteq P \,:\, r(A \cup \{p_0\}) = r(A)\}.$$

That is, Γ is the *port of the matroid* $\mathcal{M}$ *at the point* p_0. Consequently, a sufficient condition for an access structure to be ideal is obtained. Namely, the ports of representable matroids are ideal access structures. Actually, they coincide with the vector space access structures.

Brickell and Davenport [7] proved that this sufficient condition is not very far from being necessary. Specifically, they proved that every ideal secret sharing scheme on a set P of participants determines a matroid $\mathcal{M}$ with ground set $Q = P \cup \{p_0\}$ such that the access structure of the scheme is $\Gamma_{p_0}(\mathcal{M})$. Therefore, a necessary condition for an access structure to be ideal is obtained: every ideal access structure is a matroid port.

With a slightly different definition, matroid ports were introduced by Lehman [18] to solve the *Shannon switching game* in 1964, much before secret sharing was invented by Shamir [32] and Blakley [5] in 1979. A forbidden minor characterization of matroid ports was given by Seymour [30]. Even though the results in [6, 7] deal with matroid ports, this terminology was not used in those and many other subsequent works on secret sharing. The old results on matroid ports in [18, 30] were rediscovered for secret sharing by Martí-Farré and Padró [19], who used them to generalize the result by Brickell and Davenport by proving that, if all shares in a secret sharing scheme are shorter than 3/2 times the secret, then its access structure is a matroid port.

3 Hierarchical Access Structures

We present here a natural definition for the family of the *hierarchical access structures*, which embraces all possible situations in which there is a hierarchy on the set of participants. For instance, the weighted threshold access structures and the hierarchical threshold access structures [35] are contained in this new family. Hierarchical access structures are in particular multipartite. Therefore, we can take advantage of the results and techniques in [11] about the characterization of ideal multipartite access structures.

Let Γ be an access structure on a set P of participants. We say that the participant $p \in P$ is *hierarchically superior* to the participant $q \in P$, and we write $q \preceq p$, if $A \cup \{p\} \in \Gamma$ for every subset $A \subseteq P \smallsetminus \{p, q\}$ with $A \cup \{q\} \in \Gamma$. An access structure is said to be *hierarchical* if all participants are hierarchically related, that is, for every pair of participants $p, q \in P$, either $q \preceq p$ or $p \preceq q$. If $p \preceq q$ and $q \preceq p$, we say that these two participants are *hierarchically equivalent*. Clearly, this is an equivalence relation, and the hierarchical relation $\preceq$ induces an order on the set of the equivalence classes. Observe that an access structure is hierarchical if and only if this is a total order.

For a set P, a sequence $\Pi = (P_1, \dots, P_m)$ of subsets of P is called here a *partition* of P if $P = P_1 \cup \cdots \cup P_m$ and $P_i \cap P_j = \emptyset$ whenever $i \neq j$. Observe that some of the parts may be empty. An access structure Γ is said to be Π-*partite* if every pair of participants in the same part P_i are hierarchically equivalent. A different but equivalent definition for this concept is given in [11]. If m is the number of parts in Π, such structures are called m-*partite access structures*. The participants that are not in any minimal qualified subset are called *redundant*. An m-partite access structure is said to be *strictly* m-*partite* if there are no redundant participants, all parts are nonempty, and participants in different parts are not hierarchically equivalent.

A Π-partite access structure is said to be Π-*hierarchical* if $q \preceq p$ for every pair of participants $p \in P_i$ and $q \in P_j$ with $i < j$. That is, the participants in the first level are hierarchically superior to those in the second level and so on. Obviously, an access structure is hierarchical if and only if it is Π-hierarchical for some partition Π of the set of participants. The term *m-hierarchical access structure* applies to every Π-hierarchical access structure with $|\Pi| = m$.

4 A Geometric Representation of Hierarchical Access Structures

In this section we recall the geometric representation for multipartite access structures that was introduced in [11, 28]. This representation is adapted to hierarchical access structures by introducing the new concept of *stabilizers* of multipartite access structures.

We notate $\mathbb{Z}_+$ and $\mathbb{Z}_-$ for the sets of the non-negative and the non-positive integers, respectively, while $\mathbb{Z}_+^*$ and $\mathbb{Z}_-^*$ denote, respectively, the sets of the positive and the negative integers. For any $u \in \mathbb{Z}^m$, we write u_i for its i-th coordinate, that is, $u = (u_1, \dots, u_m)$. If $u, v \in \mathbb{Z}^m$, we write $u \le v$ if $u_i \le v_i$ for every $1 \le i \le m$, and we write $u < v$ if $u \le v$ and $u \ne v$.

For each partition $\Pi = (P_1, \dots, P_m)$ of the set P, we consider a mapping $\Pi\colon \mathcal{P}(P) \to \mathbb{Z}_+^m$ defined by $\Pi(A) = (|A \cap P_1|, \dots, |A \cap P_m|) \in \mathbb{Z}_+^m$. We write $\mathbf{p} = \Pi(P) = (|P_1|, \dots, |P_m|)$ and

$$\mathbf{P} = \Pi(\mathcal{P}(P)) = \{u \in \mathbb{Z}_+^m \,:\, u \le \mathbf{p}\}.$$

For a Π-partite access structure $\Gamma \subseteq \mathcal{P}(P)$, consider $\Pi(\Gamma) = \{\Pi(A) \,:\, A \in \Gamma\} \subseteq \mathbf{P}$. Observe that $A \in \Gamma$ if and only if $\Pi(A) \in \Pi(\Gamma)$, so Γ is univocally represented by the set of points $\Pi(\Gamma) \subseteq \mathbf{P}$. By an abuse of notation, we will use Γ to denote both a Π-partite access structure on P and the corresponding set $\Pi(\Gamma)$ of points in $\mathbf{P}$.

Let Γ be a Π-partite access structure on P. If two points $u, v \in \mathbf{P}$ are such that $u \le v$ and $u \in \Gamma$, then $v \in \Gamma$. This is due to the fact that Γ is a monotone increasing family of subsets. Therefore, $\Gamma \subseteq \mathbf{P}$ is determined by the family $\min \Gamma \subseteq \mathbf{P}$ of its minimal points. We are using here an abuse of notation as well, because $\min \Gamma$ denotes also the family of minimal subsets of the access structure Γ.

A set $V \subseteq \mathbb{Z}^m$ is called a *stabilizer* if V is closed by sums, and $\mathbb{Z}_+^m \subseteq V$, and $V \cap \mathbb{Z}_-^m = \{0\}$. For a stabilizer $V \subseteq \mathbb{Z}^m$, we define the binary relation $\le_V$ in $\mathbb{Z}^m$ by $u \le_V v$ if and only if $v - u \in V$. Since $0 \in V$ and V is closed by sums, this binary relation is reflexive and transitive. It is an order if and only if $V \cap (-V) = \{0\}$.

For a stabilizer $V \subseteq \mathbb{Z}^m$ and an m-partite access structure $\Gamma \subseteq \mathbf{P} \subseteq \mathbb{Z}_+^m$, we say that Γ is *V-stable* if $(\Gamma + V) \cap \mathbf{P} = \Gamma$. If $\le_V$ is an order, that is, if $V \cap (-V) = \{0\}$, we can consider the minimal points in Γ according to the order $\le_V$, which are called the *V-minimal points of* Γ. Clearly, if $V \cap (-V) = \{0\}$, a V-stable multipartite access structure is completely determined by its V-minimal points.

Obviously, every m-partite access structure is $\mathbb{Z}_+^m$-stable. In the following we present a stabilizer that characterizes the m-hierarchical access structures. For $i = 1, \dots, m$, we notate $\mathbf{e}^i$ for the i-th vector of the canonical basis of $\mathbb{R}^m$, and we take $\mathbf{v}^i = \mathbf{e}^i - \mathbf{e}^{i+1}$ for $i = 1, \dots, m-1$ and $\mathbf{v}^m = \mathbf{e}^m$. For a vector $u = (u_1, \dots, u_m) \in \mathbb{R}^m$, we notate $\hat{u}_i = \sum_{j=1}^{i} u_j$ for $i = 1, \dots, m$. Clearly, $(\mathbf{v}^1, \dots, \mathbf{v}^m)$ is a basis of $\mathbb{R}^m$ and $(\hat{u}_1, \dots, \hat{u}_m)$ are the components of the vector $u \in \mathbb{R}^m$ in this basis, that is, $u = \sum_{i=1}^{m} \hat{u}_i \mathbf{v}^i$. Consider

$$H_m = \{u \in \mathbb{Z}^m \,:\, \hat{u}_i \ge 0 \text{ for every } i = 1, \dots, m\} \subseteq \mathbb{Z}^m$$

and also the subset $G_m \subseteq H_m$ of the vectors $u \in H_m$ such that $\hat{u}_m = 0$. Usually the subindex in H_m or G_m is clear from the context and it will be omitted. Clearly, H is a stabilizer and

$H \cap (-H) = \{0\}$. Observe that $v \leq_H u$ if and only if $\hat{v}_i \leq \hat{u}_i$ for every $i = 1, \ldots, m$. In addition, an m-partite access structure is m-hierarchical if and only if it is H-stable. Consequently, every hierarchical access structure is determined by its family of H-minimal points, that we call $\min_H \Gamma$. Lemma 4.1 will be very useful in our study of hierarchical access structures.

Lemma 4.1. *If $x, y \in \mathbb{Z}_+^m$ are such that $y - x \in H$, then there exists $v \in G$ such that $0 \leq x + v \leq y$. In particular, if Γ is an m-hierarchical access structure and $y \in \min \Gamma$, then there exists $x \in \min_H \Gamma$ such that $y - x \in G$.*

Proof. The proof is by induction on m. The result is trivial for $m = 1$. Assume that $m > 1$. For a vector $x \in \mathbb{Z}^m$, we notate $x = (\widetilde{x}, x_m)$ with $\widetilde{x} \in \mathbb{Z}^{m-1}$. If $x, y \in \mathbb{Z}_+^m$ are such that $y - x \in H$, then $\widetilde{y} - \widetilde{x} \in H_{m-1}$. By the induction hypothesis, there exists $\widetilde{v} \in G_{m-1}$ such that $0 \leq \widetilde{x} + \widetilde{v} \leq \widetilde{y}$. If $x_m \leq y_m$, the vector $v = (\widetilde{v}, 0) \in G_m$ is such that $0 \leq x + v \leq y$. If $x_m > y_m$, then the vector $w = (0, \ldots, 0, x_m - y_m, y_m - x_m)$ is such that $w \in G_m$ and $x' = x + w \geq 0$. Since $y - x' \in H_m$ and $x'_m = y_m$, there exists $v' \in G_m$ such that $0 \leq x' + v' \leq y$. Finally, $v = v' + w \in G_m$ is such that $0 \leq x + v \leq y$. If Γ is an m-hierarchical access structure and $y \in \min \Gamma$, there exists an H-minimal point $x \in \min_H \Gamma$ such that $x \leq_H y$. Then there exists $v \in G$ such that $0 \leq x + v \leq y$, and hence $x + v \in \mathbf{P}$. Since $x + v \in \Gamma$ and y is a minimal point of Γ, we have that $y = x + v$. $\square$

For a vector $w \in \mathbb{R}_+^m \setminus \{0\}$ such that $w_1 \geq w_2 \geq \cdots \geq w_m$, consider the stabilizer

$$W(w) = \{v \in \mathbb{Z}^m \,:\, w \cdot v \geq 0\}.$$

If a Π-partite access structure Γ is $W(w)$-stable, then Γ is the weighted threshold access structure defined by the weights $(w_1, \ldots, w_m)$ and the threshold $t = \min\{w \cdot u \,:\, u \in \Gamma\}$. Moreover, Γ is Π-hierarchical. This implies that every weighted threshold access structure is hierarchical.

Example 4.2. Brickell [6] showed how to construct ideal secret sharing schemes for the multilevel structures proposed by Simmons [33]. These structures are of the form

$$\Gamma = \{A \subseteq P \,:\, |A \cap (\cup_{j=1}^{i} P_j)| \geq t_i \text{ for every } i = 1, \ldots, m\}.$$

for some monotone increasing sequence of integers $0 < t_1 < \ldots < t_m$. Clearly, if the number of participants in every level is large enough, Γ is a Π-hierarchical access structure with only one H-minimal point: $(t_1, t_2 - t_1, \ldots, t_m - t_{m-1})$.

Example 4.3. Another hierarchical threshold access structure was proposed by Tassa [35]. Given integers $0 < t_1 < \ldots < t_m$, the access structure is defined as

$$\Gamma = \{A \subseteq P \,:\, |A \cap (\cup_{j=1}^{i} P_j)| \geq t_i \text{ for some } i = 1, \ldots, m\}.$$

In this case, if the number of participants in each level is large enough, the access structure Γ is Π-hierarchical and its family of H-minimal points is $\min_H \Gamma = \{t_1 \mathbf{e}^1, \ldots, t_m \mathbf{e}^m\}$.

5 Operations on Hierarchical Access Structures

Duality and minors of access structures are fundamental concepts in secret sharing, as they are in matroid theory. Several important classes of access structures are closed by duality and minors, as for instance, matroid ports or $\mathbb{K}$-vector space access structures. The *dual* of an access structure Γ on a set P is the access structure on the same set defined by $\Gamma^* = \{A \subseteq P : P \smallsetminus A \notin \Gamma\}$. It is not difficult to prove that Γ is Π-partite if and only if Γ^* is so. For a subset $B \subseteq P$, we define the access structures $\Gamma \backslash B$ and Γ / B on the set $P \smallsetminus B$ by $\Gamma \backslash B = \{A \subseteq P \smallsetminus B : A \in \Gamma\}$ and $\Gamma / B = \{A \subseteq P \smallsetminus B : A \cup B \in \Gamma\}$. Every access structure that can be obtained from Γ by repeatedly applying the operations $\backslash$ and $/$ is called a *minor* of Γ. If $\Pi = (P_1, \ldots, P_m)$ is a partition of P and Γ is a Π-partite access structure, then the minors $\Gamma \backslash B$ and Γ / B are $(\Pi \backslash B)$-partite access structures, where $\Pi \backslash B = (P_1 \smallsetminus B, \ldots, P_m \smallsetminus B)$, a partition of $P \smallsetminus B$. If $\Pi(B) = b$, then the geometric representations of these access structures are $\Gamma \backslash B = \{x \leq \mathbf{p} - b : x \in \Gamma\}$ and $\Gamma / B = \{x \leq \mathbf{p} - b : x + b \in \Gamma\}$.

Proposition 5.1. *Let $V \subseteq \mathbb{Z}^m$ be a stabilizer. Then the class of the V-stable m-partite access structures is minor-closed and duality-closed. In particular, this holds for the classes of the hierarchical and the weighted threshold access structures.*

Proof. Let Γ be a V-stable m-partite access structure. Consider a point $u \in \mathbf{P}$ with $u \in \Gamma^*$ and a vector $v \in V$ such that $u + v \in \mathbf{P}$. Then $\mathbf{p} - u \notin \Gamma$, and hence $\mathbf{p} - u - v = \mathbf{p} - (u + v) \notin \Gamma$ because Γ is V-stable. This implies that $u + v \in \Gamma^*$.

Consider now the minors $\Gamma \backslash B$ and Γ / B for some $B \subseteq P$, and take $b = \Pi(B)$. Consider vectors $0 \leq u \leq \mathbf{p} - b$ and $v \in V$ such that $0 \leq u + v \leq \mathbf{p} - b$. If $u \in \Gamma \backslash B$, then $u \in \Gamma$. This implies that $u + v \in \Gamma$ and hence $u + v \in \Gamma \backslash B$. If $u \in \Gamma / B$, then $u + b \in \Gamma$ and hence $u + v + b \in \Gamma$. Therefore, $u + v \in \Gamma / B$. $\square$

Let P' and P'' be two disjoint sets and let Γ' and Γ'' be access structures on P' and P'', respectively. The *composition of* Γ' *and* Γ'' *over* $p \in P'$ is denoted by $\Gamma'[\Gamma''; p]$ and is defined as the access structure on the set of participants $P = P' \cup P'' \smallsetminus \{p\}$ that is formed by all subsets $A \subseteq P$ such that $A \cap P' \in \Gamma'$ and all subsets $A \subseteq P$ such that $(A \cup \{p\}) \cap P' \in \Gamma'$ and $A \cap P'' \in \Gamma''$. The composition of matroid ports is a matroid port, and the same applies to $\mathbb{K}$-vector space access structures. A proof for these facts can be found in [20]. The access structures that can be expressed as the composition of two access structures on sets with at least two participants are called *decomposable*.

Suppose that Γ' is $(P_1, \ldots, P_r)$-partite and Γ'' is $(P_{r+1}, \ldots, P_{r+s})$-partite, and take $p \in P_r$. Then the composition $\Gamma'[\Gamma''; p]$ is $(P'_1, \ldots, P'_{r+s})$-partite, where $P'_r = P_r \smallsetminus \{p\}$ and $P'_i = P_i$ for $i \neq r$. If Γ' and Γ'' are hierarchical and $p \in P_r$ then $\Gamma'[\Gamma''; p]$ is also hierarchical. Observe that the composition is made over a participant in the lowest level of Γ'.

6 Multipartite Matroid Ports and Discrete Polymatroids

The aim of this and the following sections is to present and to prove our main result, Theorem 9.2, which is a complete characterization of the ideal hierarchical access structures in terms of the properties of their H-minimal points. First we recall here some facts about discrete polymatroids and we show the connection between these combinatorial objects and multipartite matroids and their ports. Since all ideal access structures are matroid ports, we obtain in this way some necessary conditions for a hierarchical access structure to be ideal in Section 7. Finally, in Sections 8 and 9 we show that these necessary conditions are also sufficient.

Multipartite matroid ports are ports of *multipartite matroids*, and those matroids are closely related to *discrete polymatroids*, a combinatorial object that was introduced by Herzog and Hibi [14] to study some problems in commutative algebra. We recall here some definitions and basic facts about discrete polymatroids and multipartite matroids, the relation between these two combinatorial objects, and their connections to the characterization of multipartite access structures. More information about these concepts can be found in [11, 14].

We need to introduce some notation before defining discrete polymatroids. Consider a finite set J. For every two points $u = (u_i)_{i \in J}$ and $v = (v_i)_{i \in J}$ in $\mathbb{Z}^J$, the point $w = u \vee v$ is defined by $w_i = \max\{u_i, v_i\}$ for every $i \in J$. As before, we write $u \leq v$ if $u_i \leq v_i$ for every $i \in J$. The *modulus* of a point $u \in \mathbb{Z}^J$ is $|u| = \sum_{i \in J} u_i$. For every subset $X \subseteq J$, we notate $u(X) = (u_i)_{i \in X} \in \mathbb{Z}^X$ and $|u(X)| = \sum_{i \in X} u_i$. A *discrete polymatroid* with *ground set* J is a nonempty finite set of points $\mathcal{D} \subseteq \mathbb{Z}_+^J$ satisfying the following properties.

1. If $u \in \mathcal{D}$ and $v \in \mathbb{Z}_+^J$ is such that $v \leq u$, then $v \in \mathcal{D}$, and
2. for every pair of points $u, v \in \mathcal{D}$ with $|u| < |v|$, there exists $w \in \mathcal{D}$ with $u < w \leq u \vee v$.

A *basis* of a discrete polymatroid $\mathcal{D}$ is a maximal element in $\mathcal{D}$, that is, a point $u \in \mathcal{D}$ such that there does not exist any $v \in \mathcal{D}$ with $u < v$. Similarly to matroids, all bases have the same modulus, and discrete polymatroids are completely determined by their bases. Moreover, a nonempty set $\mathcal{B} \subseteq \mathbb{Z}_+^J$ is the family of bases of a discrete polymatroid if and only if it satisfies the following *exchange condition*.

- For every $u \in \mathcal{B}$ and $v \in \mathcal{B}$ with $u_i > v_i$, there exists $j \in J$ such that $u_j < v_j$ and $u - \mathbf{e}^i + \mathbf{e}^j \in \mathcal{B}$, where $\mathbf{e}^i \in \mathbb{Z}^J$ is such that $\mathbf{e}^i_k = 0$ if $i \neq k$ and $\mathbf{e}^i_i = 1$.

The mapping $h \colon \mathcal{P}(J) \to \mathbb{Z}$ defined by $h(X) = \max\{|u(X)| : u \in \mathcal{D}\}$ for every $X \subseteq J$ is called the *rank function* of the discrete polymatroid $\mathcal{D}$. A discrete polymatroid is completely determined by its rank function. So we will write $\mathcal{D} = (J, h)$ to denote the discrete polymatroid with ground set J and rank function h. A mapping $h \colon \mathcal{P}(J) \to \mathbb{Z}$ is the rank function of a discrete polymatroid with ground set J if and only if

1. $h(\emptyset) = 0$, and
2. h is *monotone increasing*: if $X \subseteq Y \subseteq J$, then $h(X) \leq h(Y)$, and
3. h is *submodular*: if $X, Y \subseteq J$, then $h(X \cup Y) + h(X \cap Y) \leq h(X) + h(Y)$.

We say that a discrete polymatroid $\mathcal{D}' = (J', h')$ is an extension of a discrete polymatroid $\mathcal{D} = (J, h)$ if $J \subseteq J'$ and $h'(A) = h(A)$ for all $A \subseteq J$. Since h' is an extension of h, both will be usually denoted by h. For a discrete polymatroid $\mathcal{D}$ with ground set J and a subset $X \subseteq J$, we define the discrete polymatroid $\mathcal{D}(X)$ with ground set X by $\mathcal{D}(X) = \{u(X) : u \in \mathcal{D}\} \subseteq \mathbb{Z}_+^X$. We consider the set of points $\mathcal{B}(\mathcal{D}, X) \subseteq \mathbb{Z}_+^J$ such that $u \in \mathcal{B}(\mathcal{D}, X)$ if and only if $u(X)$ is a basis of $\mathcal{D}(X)$ and $u_i = 0$ for every $i \in J \smallsetminus X$. Observe that $\mathcal{D}$ is an extension of $\mathcal{D}(X)$ for all $X \subseteq J$.

For a partition $\Pi = (Q_1, \ldots, Q_m)$ of the ground set Q, a matroid $\mathcal{M} = (Q, r)$ is said to be Π-*partite* if every permutation σ on Q such that $\sigma(Q_i) = Q_i$ for $i = 1, \ldots, m$ is an automorphism of $\mathcal{M}$. From now on, we notate $J_m = \{1, \ldots, m\}$ and $J'_m = \{0, 1, \ldots, m\}$ for every positive integer m. Then the function $h \colon \mathcal{P}(J_m) \to \mathbb{Z}$ defined by $h(X) = r(\bigcup_{i \in X} Q_i)$ is the rank function of a discrete polymatroid $\mathcal{D}(\mathcal{M}) = (J_m, h)$. Reciprocally, for every discrete polymatroid $\mathcal{D} = (J_m, h)$ with $h(\{i\}) \le |Q_i|$ for $i \in J_m$, there exists a unique Π-partite matroid $\mathcal{M}$ with $\mathcal{D}(\mathcal{M}) = \mathcal{D}$.

Consider a partition $\Pi = (P_1, \dots, P_m)$ of a set P and the partition $\Pi_0 = (\{p_0\}, P_1, \dots, P_m)$ of the set $Q = P \cup \{p_0\}$. A connected matroid port $\Gamma = \Gamma_{p_0}(\mathcal{M})$ on P is Π-partite if and only if the matroid $\mathcal{M}$ is Π_0-partite. Therefore, multipartite matroids, and hence discrete polymatroids, are fundamental in the characterization of ideal multipartite access structures. These connections are in the core of the results in [11]. In particular, we present next a characterization of multipartite matroid ports in terms of discrete polymatroids that was proved in [11] and will be extremely useful for our purposes.

Consider a Π-partite matroid port $\Gamma = \Gamma_{p_0}(\mathcal{M})$ and the associated discrete polymatroid $\mathcal{D}' = \mathcal{D}(\mathcal{M}) = (J'_m, h)$. The Π-partite matroid port Γ is completely determined by the partition Π and the discrete polymatroid $\mathcal{D}'$ and we write $\Gamma = \Gamma_0(\mathcal{D}')$. As a consequence of this fact, the following characterization of multipartite matroid ports is proved in [11].

Theorem 6.1 ([11])**.** *Let $\Pi = (P_1, \dots, P_m)$ be a partition of a set P and let Γ be an Π-partite access structure on P. Then Γ is a matroid port if and only if there exists a discrete polymatroid $\mathcal{D}' = (J'_m, h)$ with $h(\{0\}) = 1$ and $h(\{i\}) \le |P_i|$ such that*

$$\min \Gamma = \min \{u \in \mathcal{B}(\mathcal{D}, X) \,:\, X \subseteq J_m \textit{ is such that } h(X) = h(X \cup \{0\})\},$$

where $\mathcal{D} = \mathcal{D}'(J_m) = (J_m, h)$.

Since every ideal access structure is a matroid port, Theorem 6.1 provides a necessary condition for a multipartite access structure to be ideal. Several necessary conditions for a hierarchical access structure to be ideal will be deduced from this result in Section 7.

On the other hand, sufficient conditions can be obtained from the fact that the ports of linearly representable matroids are ideal access structures. We present in Theorem 6.2 an interesting result from [11] connecting the linear representations of multipartite matroids to the ones of discrete polymatroids. This result is used in Section 8 to find sufficient conditions for a hierarchical access structure to be ideal.

Let E be a vector space with finite dimension over a finite field $\mathbb{K}$ and, for every $i \in J$, consider a vector subspace $V_i \subseteq E$. It is not difficult to check that the mapping $h \colon \mathcal{P}(J) \to \mathbb{Z}$ defined by $h(X) = \dim(\sum_{i \in X} V_i)$ is the rank function of a discrete polymatroid $\mathcal{D} = (J, h)$. The discrete polymatroids that can be defined in this way are said to be $\mathbb{K}$*-linearly representable*.

Theorem 6.2 ([11])**.** *For every large enough field $\mathbb{K}$, an m-partite matroid $\mathcal{M}$ is $\mathbb{K}$-linearly representable if and only if its associated discrete polymatroid $\mathcal{D}(\mathcal{M}) = (J_m, h)$ is $\mathbb{K}$-linearly representable.*

7 Hierarchical Matroid Ports

In this section, we use the connection between discrete polymatroids and multipartite matroid ports that is discussed in Section 6 to find necessary conditions for hierarchical access structures to be matroid ports. We prove first some technical lemmas that apply to every discrete polymatroid. Specifical results on discrete polymatroids associated to hierarchical matroid ports will be given afterwards.

Lemma 7.1. *Consider a discrete polymatroid $\mathcal{D} = (J_m, h)$, a subset $A \subseteq J_m$, and a point $y \in \mathbb{Z}_+^m$ that is H-minimal in $\mathcal{B}(\mathcal{D}, A)$. Then y is the H-minimum point of $\mathcal{B}(\mathcal{D}, A)$, that is, $y \leq_H x$ for every $x \in \mathcal{B}(\mathcal{D}, A)$.*

Proof. We prove that $\mathcal{B}(\mathcal{D}, A) \subseteq y + H$. Suppose that, on the contrary, $R = \mathcal{B}(\mathcal{D}, A) \smallsetminus (y + H) \neq \emptyset$ and consider a point $x \in R$ that is H-minimal in R. Let $i \in A$ be the smallest index with $x_i \neq y_i$. If $x_i < y_i$, there exists $j \in A$ with $j > i$ such that $x_j > y_j$ and $z = y + \mathbf{e}^j - \mathbf{e}^i \in \mathcal{B}(\mathcal{D}, A)$. Observe that $y - z \in G \smallsetminus \{0\}$, a contradiction with the fact that y is H-minimal in $\mathcal{B}(\mathcal{D}, A)$. If $x_i > y_i$, there exists $j \in A$ with $j > i$ such that $x_j < y_j$ and $u = x + \mathbf{e}^j - \mathbf{e}^i \in \mathcal{B}(\mathcal{D}, A)$. Then $u \notin R$ because x is H-minimal in R, and hence $u \in y + G$. This implies that $x - y = (x - u) + (u - y) \in G$, a contradiction. $\square$

For every $i, j \in \mathbb{Z}$ we notate $[i, j] = \{i, i+1, \ldots, j\}$ if $i < j$, while $[i, i] = \{i\}$ and $[i, j] = \emptyset$ if $i > j$. Let $\mathcal{D} = (J_m, h)$ be a discrete polymatroid. For every $i \in J_m$, consider the point $y^i = y^i(\mathcal{D}) \in \mathbb{Z}_+^m$ defined by $y_j^i = h([j, i]) - h([j + 1, i])$. Observe that $\sum_{j=s}^{i} y_j^i = h([s, i])$ for every $s \in [1, i]$.

Lemma 7.2. *For every $i = 1, \ldots, m$, the point $y^i(\mathcal{D})$ is the H-minimum of $\mathcal{B}(\mathcal{D}, [1, i])$.*

Proof. By Lemma 7.1, it is enough to prove that $y^i(\mathcal{D})$ is an H-minimal point of $\mathcal{B}(\mathcal{D}, [1, i])$. We prove first that $y^i = y^i(\mathcal{D}) \in \mathcal{B}(\mathcal{D}, [1, i])$. Take $A \subseteq [1, i]$ and, for $j \in [1, i]$, consider $A_j = A \cap [j, i]$. Then

$$|y^i(A)| = \sum_{j \in A} y_j^i = \sum_{j \in A} (h([j, i]) - h([j + 1, i])) \leq \sum_{j \in A} (h(A_j) - h(A_{j+1})) = h(A).$$

The inequality holds because $A_{j+1} = A_j \cap [j + 1, i]$ and $[j, i] = A_j \cup [j + 1, i]$. Since $y_j^i = 0$ for all $j > i$, this implies that $y^i \in \mathcal{D}$ for all $i \in J_m$. Moreover, $y^i \in \mathcal{B}(\mathcal{D}, [1, i])$ because $|y^i| = h([1, i])$. We prove next that y^i is H-minimal in $\mathcal{B}(\mathcal{D}, [1, i])$. If not, there exists $v \in G \smallsetminus \{0\}$ such that $u = y^i - v \in \mathcal{B}(\mathcal{D}, [1, i])$. Observe that $v_j = 0$ or all $j > i$. Clearly, there exists $s \in [1, i]$ for which $\sum_{j=1}^{s-1} v_j > 0$, and hence $\sum_{j=s}^{i} v_j < 0$. Then $|u([s, i])| = \sum_{j=s}^{i} u_j > \sum_{j=s}^{i} y_j^i = h([s, i])$, a contradiction with the assumption that $u \in \mathcal{B}(\mathcal{D}, [1, i])$. $\square$

Lemma 7.3. *If $1 \leq j \leq i < m$, then $y_j^i \geq y_j^{i+1}$.*

Proof. Since h is submodular, $y_j^{i+1} = h([j, i+1]) - h([j+1, i+1]) \leq h([j, i]) - h([j+1, i]) = y_j^i$. $\square$

For the remaining of this section, we assume that Γ is a Π-hierarchical matroid port, where $\Pi = (P_1, \ldots, P_m)$ is an m-partition of the set of participants P. Recall that we notate $\mathbf{p} = \Pi(P)$ and $\mathbf{P} = \Pi(\mathcal{P}(P)) \subseteq \mathbb{Z}_+^m$. In addition, we assume that the access structure Γ is *connected*, that is, that every participant is in a minimal qualified subset or, equivalently, for every $i \in J_m$, there is a minimal point $x \in \min \Gamma$ such that $x_i > 0$. Consider the discrete polymatroid $\mathcal{D}' = (J'_m, h)$ such that $\Gamma = \Gamma_0(\mathcal{D}')$, and the discrete polymatroid $\mathcal{D} = \mathcal{D}'(J_m) = (J_m, h)$. Since Γ is connected, $h(\{i\}) > 0$ for all $i \in J_m$, and hence $y_i^i > 0$. Consider $\Delta(\Gamma) = \{\mathrm{supp}(x) : x \in \Gamma\} \subseteq \mathcal{P}(J_m)$. Observe that $\Delta(\Gamma) = \{A \subseteq J_m : h(A \cup \{0\}) = h(A)\}$ by Theorem 6.1. For every $x \in \mathbb{Z}_+^m$, we notate $\mathrm{supp}(x) = \{i \in J_m : x_i \neq 0\} \subseteq J_m$. Take $m(x) = \max(\mathrm{supp}(x))$ and $M(x) = \{1, \ldots, m(x)\}$.

Lemma 7.4. *If $x \in \mathbf{P}$ is a minimal point of Γ, then $x \in \mathcal{B}(\mathcal{D}, M(x))$.*

Proof. From Theorem 6.1, $x \in \mathcal{B}(\mathcal{D}, A)$ for some subset $A \subseteq M(x)$. We are going to prove that $x \in \mathcal{B}(\mathcal{D}, M(x))$ by checking that $h(A) = h(M(x))$. Specifically, we prove that $h(A \cup \{j\}) = h(A)$ for every $j \in M(x) \setminus A$. Consider $j \in M(x) \setminus A$ and the point $x' = x + \mathbf{e}^j - \mathbf{e}^{m(x)} \in \mathbf{P}$. Observe that $x' \in \Gamma$ because $x' - x \in H$. Applying Theorem 6.1 again, there exist $C \subseteq A \cup \{j\}$ with $C \in \Delta(\Gamma)$ and a point $u \in \mathcal{B}(\mathcal{D}, C)$ such that $x' \geq u$. If $u_j = 0$, then $u < x$, but this is not possible because $x \in \min \Gamma$. Thus, $u_j = 1$ and $j \in C$. Since h is submodular, $h(A \cup \{j\}) + h(C \setminus \{j\}) \leq h(A) + h(C)$. Therefore, $h(A \cup \{j\}) = h(A)$ if $h(C) = h(C \setminus \{j\})$. Suppose now that $h(C \setminus \{j\}) \leq h(C) - 1$. Observe that $h(C \setminus \{j\}) \geq |u(C \setminus \{j\})| = |u(C)| - 1 = h(C) - 1$ because $u \in \mathcal{B}(\mathcal{D}, C)$. Hence, $h(C \setminus \{j\}) = h(C) - 1$ and $u - \mathbf{e}^j \in \mathcal{B}(\mathcal{D}, C \setminus \{j\})$. Observe that $u - \mathbf{e}^j \notin \Gamma$ because $u - \mathbf{e}^j < x$ and $x \in \min \Gamma$. Thus, $C \setminus \{j\} \notin \Delta(\Gamma)$ and $h((C \setminus \{j\}) \cup \{0\}) = h(C \setminus \{j\}) + 1 = h(C)$. The submodularity of h implies that

$$h(A \cup \{j, 0\}) + h(C) = h(A \cup \{j, 0\}) + h((C \setminus \{j\}) \cup \{0\}) \leq h(A \cup \{0\}) + h(C \cup \{0\}) = h(A) + h(C).$$

Therefore, $h(A \cup \{j\}) = h(A)$. □

Lemma 7.5. *If $x \in \mathbf{P}$ is an H-minimal point of Γ, then $x = y^{m(x)}(\mathcal{D})$.*

Proof. From Lemma 7.4, $x \in \mathcal{B}(\mathcal{D}, M(x))$ and, since $\mathcal{B}(\mathcal{D}, M(x)) \subseteq \Gamma$ by Theorem 6.1, x is H-minimal in $\mathcal{B}(\mathcal{D}, M(x))$. By Lemmas 7.1 and 7.2, this implies that $x = y^{m(x)}(\mathcal{D})$. □

Lemma 7.6. *If $x, y \in \mathbf{P}$ are two different H-minimal points of Γ, then $m(x) \neq m(y)$. Moreover, if $m(x) < m(y)$, then $|x| < |y|$.*

Proof. It is obvious from Lemma 7.5 that $m(x) \neq m(y)$ if $x \neq y$. Observe that $|x| = h(M(x))$ and $|y| = h(M(y))$, and hence $|x| \leq |y|$ if $m(x) < m(y)$. If $|x| = |y|$, then $x \in \mathcal{B}(\mathcal{D}, M(y)) \subseteq y + H$ and $x - y \in H$, a contradiction. □

Lemma 7.7. *If $x, y \in \min_H \Gamma$ are such that $m(x) < m(y)$, then $x_i \geq y_i$ for all $i = 1, \ldots, m(x)$.*

Proof. A direct consequence of Lemmas 7.3 and 7.5. □

Lemma 7.8. *Let $x, y \in \mathbf{P}$ be two different H-minimal points of Γ with $m(x) < m(y)$ such that there is not any H-minimal point z with $m(x) < m(z) < m(y)$. If $x_i > y_i$ for some $i \in [1, m(x)-1]$, then $|P_j| = x_j$ for all $j \in [i+1, m(x)]$.*

Proof. Suppose that $x_i > y_i$ and $x_j < |P_j|$ for some i, j with $1 \leq i < j \leq m(x)$. Since $y_k \leq x_k$ for all $k = 1, \ldots, m(x)$ and $|y| > |x|$, there exists a point $y' \in (y + G) \cap \mathbf{P}$ such that

- $y'_k = y_k$ for all $1 \leq k < j$, and
- $y'_j = x_j + 1$, and
- $y'_k = x_k$ for all $j < k \leq m(x)$.

Clearly $y' \in \Gamma$, but $y' \notin \min \Gamma$ because $|y([j, m(x)])| > |x([j, m(x)])| = h([j, m(x)])$, and hence $y' \notin \mathcal{D}$. Therefore, there exists $z' \in \min \Gamma$ such that $z' < y'$, and by Lemma 4.1 there exists $z \in \min_H \Gamma$ such that $z' - z \in G$. By Lemma 7.6, $m(z) < m(y)$ because $|z| = |z'| < |y'| = |y|$. Clearly, $m(z) \geq i$ because $z < y$ if $m(z) < i$. If $m(z) \leq m(x)$, then $z_k \geq x_k$ for all $k = 1, \ldots, m(z)$ by Lemma 7.7, a contradiction with $z_i \leq y'_i = y_i < x_i$. Therefore, there exists an H-minimal point z such that $m(x) < m(z) < m(y)$. □

8 A Family of Ideal Hierarchical Access Structures

Observe that Lemmas 7.6, 7.7, and 7.8 in previous section provide necessary conditions for a Π-hierarchical access structure to be a matroid port, and hence to be ideal, in terms of the properties of its H-minimal points. A sufficient condition is given in this section by constructing a new family of hierarchical vector space secret sharing schemes. Specifically, we present a family of linearly representable discrete polymatroids and we prove that the multipartite access structures that are obtained from them are actually hierarchical. In addition, they are vector space access structures by Theorem 6.2.

Consider a finite field $\mathbb{K}$ and a pair of integer vectors $\mathbf{a} = (a_0, \ldots, a_m) \in \mathbb{Z}_+^{m+1}$ and $\mathbf{b} = (b_0, \ldots, b_m) \in \mathbb{Z}_+^{m+1}$ such that

- $a_0 = a_1 = b_0 = 1$, and
- $a_i \leq a_{i+1} \leq b_i \leq b_{i+1}$ for every $i = 0, \ldots m-1$.

Take $d = b_m$ and consider a basis $\{e^1, \ldots, e^d\}$ of $\mathbb{K}^d$ and, for every $i = 1, \ldots, m$, consider the subspace $V_i = \langle e^{a_i}, \ldots, e^{b_i} \rangle \subseteq \mathbb{K}^d$. Let $\mathcal{D}' = \mathcal{D}'(\mathbf{a}, \mathbf{b}) = (J'_m, h)$ be the discrete polymatroid that is linearly represented by the subspaces $V_0, V_1, \ldots, V_m$. Observe that the rank function h of $\mathcal{D}'$ is such that $h(A) = |\cup_{i \in A} [a_i, b_i]|$ for all $A \subseteq J'_m$. In particular, $h([j,i]) = |[a_j, b_i]| = b_i - a_j + 1$ whenever $0 \leq j \leq i \leq m$, and hence $h(\{0\}) = 1$. Therefore, for every set of players P and for every m-partition $\Pi = (P_1, \ldots, P_m)$ of P such that $|P_i| \geq h(\{i\}) = b_i - a_i + 1$, we can consider the Π-partite matroid port $\Gamma = \Gamma_0(\mathcal{D}')$ that is determined as in Theorem 6.1. Since $\mathcal{D}'$ is $\mathbb{K}$-linearly representable for every finite field $\mathbb{K}$, we have from Theorem 6.2 that Γ is a $\mathbb{K}$-vector space access structure for every large enough finite field $\mathbb{K}$. We prove in the following that Γ is actually a Π-hierarchical access structure.

Consider the discrete polymatroid $\mathcal{D} = \mathcal{D}(\mathbf{a}, \mathbf{b}) = \mathcal{D}'(J_m) = (J_m, h)$ and, for $i = 1, \ldots, m$, the points $y^i = y^i(\mathcal{D}) \in \mathbb{Z}_+^m$. Observe that $y^i_j = h([j,i]) - h([j+1,i]) = a_{j+1} - a_j$ if $j < i$ while $y^i_i = b_i - a_i + 1$. Therefore,

$$y^i = (a_2 - a_1, \ldots, a_i - a_{i-1}, b_i - a_i + 1, 0, \ldots, 0).$$

In the following lemma, we present a characterization of the families of points $(y^i(\mathcal{D}))_{1 \leq i \leq m}$ corresponding to discrete polymatroids of the form $\mathcal{D} = \mathcal{D}(\mathbf{a}, \mathbf{b})$. This and the following lemmas in this section are proved in Appendix **??**.

Lemma 8.1. *The points $y^1, \ldots, y^m \in \mathbb{Z}_+^m$ are of the form $y^i = y^i(\mathcal{D}(\mathbf{a}, \mathbf{b}))$ for some $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^{m+1}$ in the above conditions if and only if*

- *$m(y^i) = i$ for every $i = 1, \ldots, m$, and*
- *$|y^i| \leq |y^{i+1}|$ and $y^i_i > y^{i+1}_i$ for every $i = 1, \ldots, m-1$, and*
- *$y^i_j = y^{i+1}_j$ if $1 \leq j < i \leq m-1$.*

Proof. Clearly, the points of the form $y^i = y^i(\mathcal{D}(\mathbf{a}, \mathbf{b}))$ satisfy the required conditions. We prove now the converse. Given points $y^1, \ldots, y^m \in \mathbb{Z}_+^m$ satisfying the conditions in the statement, consider $\mathbf{a} = (a_0, \ldots, a_m)$ and $\mathbf{b} = (b_0, \ldots, b_m)$ defined as follows:

- $a_0 = a_1 = b_0 = 1$,
- $a_i = \sum_{j=1}^{i-1} y_j^i + 1$ for all $i = 1, \dots, m$,
- $b_i = \sum_{j=1}^{i} y_j^i$ for all $i = 1, \dots, m$.

Clearly $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^{m+1}$, and $a_{i+1} - a_i = y_i^{i+1} \geq 0$ and $b_i = |y^i| \leq |y^{i+1}| = b_{i+1}$. In addition, $b_i - a_{i+1} = y_i^i - y_i^{i+1} - 1 \geq 0$. Finally observe that $y^i = (a_2 - a_1, \dots, a_i - a_{i-1}, b_i - a_i + 1, 0, \dots, 0)$ for all $i = 1, \dots, m$. $\square$

Lemma 8.2. *If $h(A) < h([\min(A), \max(A)])$, then there exists $s \in [\min(A), \max(A)] \smallsetminus A$ such that $h(A) = h(A \cap [1, s]) + h(A \cap [s+1, m])$.*

Proof. Consider $s \in [\min(A), \max(A)] \smallsetminus A$ such that $h(A \cup \{s\}) > h(A)$ and define $A_1 = A \cap [1, s]$, and $A_2 = A \cap [s+1, m]$, and $B = \cup_{i \in A} [a_i, b_i]$. Then there exists $t \in [a_s, b_s]$ such that $t \notin B$, and hence $h(A) = |B \cap [1, t-1]| + |B \cap [t+1, m]| = h(A_1) + h(A_2)$. $\square$

Lemma 8.3. *If $x \in \min \Gamma$, then $x \in \mathcal{B}(\mathcal{D}, M(x))$.*

Proof. Take $A = \operatorname{supp}(x)$. Clearly, $x \in \mathcal{B}(\mathcal{D}, M(x))$ if $h(A) = h(M(x))$. Suppose that $h(A) < h(M(x))$. Observe that $h(A \cup \{0\}) = h(A)$ because $A \in \Delta(\Gamma)$, and hence $a_{\min(A)} = 1$. Then the subset $A' = A \cup [1, \min(A)]$ is such that $h(A') = h(A)$. By applying Lemma 8.2 to A', there exists $s \in [1, m(x)] \smallsetminus A'$ such that $h(A') = h(A' \cap [1, s]) + h(A' \cap [s+1, m])$. Consider $A_1 = A' \cap [1, s]$. Since $|x(B)| \leq h(B)$ for all $B \subseteq J_m$ and $|x| = h(A) = h(A')$, we have that $|x(A_1)| = h(A_1)$, and hence $x' = \sum_{i \in A_1} x_i \mathbf{e}^i \in \mathcal{B}(\mathcal{D}, A_1)$. Then $x' \in \Gamma$ because $A_1 \in \Delta(\Gamma)$, a contradiction with $x \in \min \Gamma$. $\square$

Lemma 8.4. *The access structure Γ is Π-hierarchical.*

Proof. It is enough to prove that $x + \mathbf{v}^i \in \Gamma$ if $x \in \Gamma$ and $x + \mathbf{v}^i \in \mathbf{P}$ (recall that, for $i = 1, \dots, m-1$, we notate $\mathbf{v}^i = \mathbf{e}^i - \mathbf{e}^{i+1} \in G$). First, we argue that we can assume $x \in \min \Gamma$. Consider $z \in \min \Gamma$ with $z \leq x$. If $z_{i+1} = 0$ and $x + \mathbf{v}^i \in \mathbf{P}$, then $z \leq x + \mathbf{v}^i$, and hence $x + \mathbf{v}^i \in \Gamma$. If $z_{i+1} > 0$, then $z + \mathbf{v}^i \in \mathbf{P}$, and $x + \mathbf{v}^i \in \Gamma$ if $z + \mathbf{v}^i \in \Gamma$ because $z + \mathbf{v}^i \leq x + \mathbf{v}^i$.

Let $x \in \min \Gamma$ be such that $y = x + \mathbf{v}^i \in \mathbf{P}$. Then $s = m(x) > i$ and $x \in \mathcal{B}(\mathcal{D}, [1, s])$. Clearly, $y \in \Gamma$ if $y \in \mathcal{B}(\mathcal{D}, [1, s])$. Suppose that $y \notin \mathcal{B}(\mathcal{D}, [1, s])$. We assert that, in this situation, there exists $t \in [1, i]$ such that $\sum_{j=t}^{i} y_j > h([t, i])$. Since $y \notin \mathcal{B}(\mathcal{D}, M(x))$, there exists $A \subseteq [1, s]$ such that $|y(A)| > h(A)$ and that is minimal with this property. It is clear that $i \in A$ and $i + 1 \notin A$. Take $t = \min(A)$ and $t' = \max(A)$. If $h(A) < h([t, t'])$, there exists by Lemma 8.2 a value $k \in [t, t'] \smallsetminus A$ such that $h(A) = h(A_1) + h(A_2)$, where $A_1 = A \cap [t, k]$ and $A_2 = A \cap [k+1, t']$. Then, $|y(A_\ell)| > h(A_\ell)$ if $i \in A_\ell$, a contradiction with the election of A. Therefore, $h(A) = h([t, t'])$ and $t' = i$ because $|y([t, t'])| > h([t, t'])$. This proves our assertion.

Observe that

$$h([1, i]) = \sum_{j=1}^{i} y_j^i = \sum_{j=1}^{t-1} y_j^i + h([t, i]) = \sum_{j=1}^{t-1} y_j^s + h([t, i]).$$

In addition, $\sum_{j=1}^{t-1} (x_j - y_j^s) \geq 0$ because $x \in \mathcal{B}(\mathcal{D}, [1, s]) \subseteq y^s + G$. Therefore,

$$h([1, i]) \leq \sum_{j=1}^{t-1} x_j + h([t, i]) < \sum_{j=1}^{t-1} y_j + \sum_{j=t}^{i} y_j = |y([1, i])|.$$

Clearly, this implies that $|y([1,i])| = h([1,i]) + 1$. Then $|x([1,i])| = |y([1,i])| - 1 = h([1,i])$, and hence $x' = \sum_{j=1}^{i} x_j \mathbf{e}^j \in \mathcal{B}(\mathcal{D}, [1,i])$ and $x' \in \Gamma$. But this is a contradiction with the fact that $x \in \min \Gamma$. Therefore, $y \in \mathcal{B}(\mathcal{D}, [1,s])$ and $y \in \Gamma$. $\square$

Lemma 8.5. *A point $x \in \mathbf{P}$ is H-minimal in Γ if and only if $x = y^i$ with $i = m$ or $i < m$ and $|y^i| < |y^{i+1}|$.*

Proof. From Lemma 7.5, $\min_H \Gamma \subseteq \{y^1, \dots, y^m\}$, and hence $\min_H \Gamma = \min_H \{y^1, \dots, y^m\}$. Take $i, j \in J_m$ with $i < j$. Then $\hat{y}^i_k - \hat{y}^j_k = 0$ if $1 \le k < i$, while $\hat{y}^i_i - \hat{y}^j_i = y^i_i - y^j_i > 0$, and $\hat{y}^i_k - \hat{y}^j_k \ge |y^i| - |y^j| = \hat{y}^i_m - \hat{y}^j_m$ if $i + 1 \le k \le m$. Therefore, $y^j - y^i \notin H$ while $y^i - y^j \in H$ if and only if $|y^i| = |y^j|$. $\square$

The next proposition summarizes the results in this section.

Proposition 8.6. *Let $\Pi = (P_1, \dots, P_m)$ be an m-partition of a set P and let Γ be a Π-hierarchical access structure on P. Let $x^1, \dots, x^r \in \mathbb{Z}_+^m$ be the H-minimal points of Γ and define $m_i = \max(\mathrm{supp}(x^i))$. Suppose that the following properties are satisfied.*

1. *If $i < j$, then $m_i < m_j$ and $x^i_k = x^j_k$ for all $k = 1, \dots, m_i - 1$.*
2. *If $m_{j-1} < i \le m_j$, then $|P_i| \ge \sum_{\ell=i}^{m_j} x^j_\ell$.*

Then Γ is ideal and, moreover, it admits a $\mathbb{K}$-vector space secret sharing scheme for every large enough finite field $\mathbb{K}$.

Proof. Consider the points $y^1, \dots, y^m \in \mathbf{P}$ defined as follows: if $m_{j-1} < i \le m_j$, then

- $y^i_k = x^j_k$ for every $k = 1, \dots, i$, and
- $y^i_i = \sum_{\ell=i}^{m_j} x^j_\ell$, and
- $y^i_k = 0$ for every $k = i+1, \dots, m$.

Observe that $x^j_{m_j} > x^{j+1}_{m_j}$ because $x^j \le x^{j+1}$ otherwise. With that in mind, it is not difficult to check that the points $y^1, \dots, y^m \in \mathbb{Z}_+^m$ satisfy the conditions in Lemma 8.1, and hence there exists a discrete polymatroid of the form $\mathcal{D} = \mathcal{D}(\mathbf{a}, \mathbf{b})$ such that $y^i = y^i(\mathcal{D})$ for every $i = 1, \dots, m$. In addition, from the previous results, $\Gamma_0(\mathcal{D})$ is a Π-hierarchical access structure with $\min_H \Gamma_0(\mathcal{D}) = \min_H \{y^1, \dots, y^m\} = \{x^1, \dots, x^r\}$. Therefore, $\Gamma = \Gamma_0(\mathcal{D})$ and, since $\mathcal{D}$ is linearly representable over every finite field, Γ is a $\mathbb{K}$-vector space access structure if $\mathbb{K}$ is large enough. $\square$

9 A Characterization of Ideal Hierarchical Access Structures

By using the results in Sections 7 and 8, we present here a complete characterization of ideal hierarchical access structures. Moreover, we prove that every ideal hierarchical access structure is a $\mathbb{K}$-vector space access structure for every large enough finite field $\mathbb{K}$. The next result is a consequence of Proposition 8.6 and the necessary conditions for a hierarchical access structure to be ideal given in Section 7. It provides a characterization of hierarchical access structures in which the number of participants in every hierarchical level is large enough in relation to the H-minimal points.

Theorem 9.1. *Let $\Pi = (P_1, \ldots, P_m)$ be an m-partition of a set P and let Γ be a Π-hierarchical access structure on P with $\min_H \Gamma = \{x^1, \ldots, x^r\}$. For $j = 1, \ldots, r$, consider $m_j = \max(\operatorname{supp}(x^j))$ and suppose that $|P_{m_j}| > x^j_{m_j}$. Then Γ is ideal if and only if*

1. *$m_i \neq m_j$ if $i \neq j$, and*

2. *if $m_i < m_j$, then $x^i_k = x^j_k$ for all $k = 1, \ldots, m_i - 1$.*

Moreover, in this situation Γ is a $\mathbb{K}$-vector space access structure for every large enough field $\mathbb{K}$.

Proof. The conditions are necessary because of the results in Section 7. We prove now that they are also sufficient. Suppose that the H-minimal points of Γ are ordered in such a way that $m_i < m_j$ if $i < j$. Consider a set $\widetilde{P} \supseteq P$ and an m-partition $\widetilde{\Pi} = (\widetilde{P}_1, \ldots, \widetilde{P}_m)$ of $\widetilde{P}$ such that $\widetilde{P}_i \supseteq P_i$ for all $i = 1, \ldots, m$ and $|\widetilde{P}_i| \geq \sum_{\ell=i}^{m_j} x^j_\ell$ if $m_{j-1} < i \leq m_j$. Let $\widetilde{\Gamma}$ be the $\widetilde{\Pi}$-hierarchical access structure with $\min_H \widetilde{\Gamma} = \{x^1, \ldots, x^r\}$. By Proposition 8.6, $\widetilde{\Gamma}$ is a $\mathbb{K}$-vector space access structure for every large enough field $\mathbb{K}$. Observe that $((x^j + H) \cap \widetilde{\mathbf{P}}) \cap \mathbf{P} = (x^j + H) \cap \mathbf{P}$ for every $j = 1, \ldots, r$. This implies that the access structure Γ is a minor of $\widetilde{\Gamma}$. Specifically, $\Gamma = \widetilde{\Gamma} \backslash (\widetilde{P} \smallsetminus P)$. $\square$

Finally, we present our complete characterization of ideal hierarchical access structures in terms of the properties of the H-minimal points. Actually, we prove that a hierarchical access structure is ideal if and only if it is a minor of an access structure in the family that is presented in Section 8. Therefore every ideal hierarchical access structure is a $\mathbb{K}$-vector access structure for all large enough finite fields $\mathbb{K}$, and this proves Theorem 1.1.

Theorem 9.2. *Let $\Pi = (P_1, \ldots, P_m)$ be an m-partition of a set P and let Γ be a Π-hierarchical access structure on P with $\min_H \Gamma = \{x^1, \ldots, x^r\}$. Consider $m_j = \max(\operatorname{supp}(x^j))$ and suppose that the H-minimal points are ordered in such a way that $m_j \leq m_{j+1}$. Then Γ is ideal if and only if*

1. *$m_j < m_{j+1}$ and $|x^j| < |x^{j+1}|$ for all $j = 1, \ldots, r-1$, and*

2. *$x^j_i \geq x^{j+1}_i$ if $1 \leq j \leq r-1$ and $1 \leq i \leq m_j$, and*

3. *if $x^j_i > x^r_i$ for some $1 \leq j < r$ and $1 \leq i < m_j$, then $|P_k| = x^j_k$ for all $k = i+1, \ldots, m_j$.*

Proof. As before, the results in Section 7 imply that the given conditions are necessary. Suppose that the conditions are satisfied. Take $\widetilde{x}^r = x^r$, and for $j = 1, \ldots, r-1$ consider the point $\widetilde{x}^j \in \mathbb{Z}_+^m$ defined by

- $\widetilde{x}^j_i = x^r_i$ if $1 \leq i \leq m_j - 1$, and
- $\widetilde{x}^j_{m_j} = x^j_{m_j} + \sum_{k=1}^{m_j-1} (x^j_k - x^r_k)$, and
- $\widetilde{x}^j_i = 0$ if $m_j + 1 \leq i \leq m$.

As we did in the proof of Theorem 9.1, we extend the set P of participants to a larger one. Consider a set $\widetilde{P} \supseteq P$ and an m-partition $\widetilde{\Pi} = (\widetilde{P}_1, \ldots, \widetilde{P}_m)$ of $\widetilde{P}$ such that $\widetilde{P}_i \supseteq P_i$ for all $i = 1, \ldots, m$ and $|\widetilde{P}_i| \geq \sum_{\ell=i}^{m_j} \widetilde{x}^j_\ell$ if $m_{j-1} < i \leq m_j$. Let $\widetilde{\Gamma}$ be the $\widetilde{\Pi}$-hierarchical access structure on $\widetilde{P}$ with $\min_H \widetilde{\Gamma} = \{\widetilde{x}^1, \ldots, \widetilde{x}^r\}$. It is not difficult to check that $\widetilde{\Gamma}$ satisfies the conditions in Proposition 8.6, and hence it is a $\mathbb{K}$-vector space access structure for every large enough field $\mathbb{K}$. Consider the discrete polymatroid $\widetilde{\mathcal{D}}' = (J'_m, \widetilde{h})$ associated to $\widetilde{\Gamma}$ and take $\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}'(J_m) = (J_m, \widetilde{h})$.

The proof is concluded by checking that Γ is a minor of $\widetilde{\Gamma}$. Specifically, we prove that

$$\Gamma = (\{x^1, \dots, x^r\} + H) \cap \mathbf{P} = (\{\widetilde{x}^1, \dots, \widetilde{x}^r\} + H) \cap \mathbf{P} = \widetilde{\Gamma} \cap \mathbf{P},$$

which implies that $\Gamma = \widetilde{\Gamma} \backslash (\widetilde{P} \smallsetminus P)$. Observe that $x^j - \widetilde{x}^j \in G$, and hence $\Gamma \subseteq \widetilde{\Gamma} \cap \mathbf{P}$. For $j = 1, \dots, r$, consider $A_j = (\widetilde{x}^j + G) \cap \mathbf{P}$. Clearly, it is enough to prove that $A_j \subseteq \Gamma$ for all $j = 1, \dots, r$. Suppose that, on the contrary, there exists $j = 1, \dots, r$ such that $A_j \not\subseteq \Gamma$ while $A_k \subseteq \Gamma$ for all $k = 1, \dots, j-1$.

Suppose that $x^j \notin \mathcal{B}(\widetilde{\mathcal{D}}, [1, m_j])$. Then $x^j \notin \min \widetilde{\Gamma}$ and, since $x^j \in \widetilde{\Gamma}$, there exists $z \in \min \widetilde{\Gamma}$ with $z < x^j$. By Lemma 4.1, there exists an H-minimal point x of $\widetilde{\Gamma}$ such that $z - x \in G$, and hence $|x| = |z| < |x^j|$. This is impossible if $j = 1$. If $j > 1$, then $x = \widetilde{x}^k$ for some $k < j$, and hence $z \in A_k \subseteq \Gamma$. Clearly, $z \in \min \Gamma$ and, by applying Lemma 4.1 again, $z - x^k \in G$. This implies that $x^j - x^k = (x^j - z) + (z - x^k) \in H$, a contradiction. Therefore, $x^j \in \mathcal{B}(\widetilde{\mathcal{D}}, [1, m_j])$.

Consider $R = A_j \smallsetminus \Gamma$ and consider a point $y \in R$ that is H-minimal in R. We assert that $y \in \mathcal{B}(\widetilde{\mathcal{D}}, [1, m_j])$. If not, $y \in \widetilde{\Gamma}$ but $y \notin \min \widetilde{\Gamma}$. By repeating the previous argument, $j > 1$ and $y - x^k \in H$ for some $k < j$. Since $y \notin \Gamma$, we reached a contradiction that proves our assertion.

Let $i \in J_m$ be the smallest value such that $y_i \neq x^j_i$. If $y_i < x^j_i$, there exists ℓ with $i+1 \leq \ell \leq m_j$ such that $y_\ell > x^j_\ell$. Since $y - \widetilde{x}^j \in G$, it follows that $|\widetilde{x}^j([1,i])| \leq |y([1,i])| < |x^j([1,i])|$, and hence $x^r_s = \widetilde{x}^j_s < x^j_s$ for some s with $1 \leq s \leq i$. This implies that $x^j_\ell = |P_\ell|$ and $y_\ell \leq x^j_\ell$ because $y \in \mathbf{P}$, a contradiction. If $y_i > x^j_i$, then $y_\ell < x^j_\ell$ and $y' = y - \mathbf{e}^i + \mathbf{e}^\ell \in \mathcal{B}(\widetilde{\mathcal{D}}, [1, m_j]) \cap \mathbf{P}$ for some ℓ with $i+1 \leq \ell \leq m_j$. Since $y - y' \in G$ and and y is an H-minimal point in R, it follows that $y' \notin R$, and hence $y' \in \Gamma$, a contradiction with $y \notin \Gamma$. □

Example 9.3. Let Γ be the weighted theshold access structure defined by the vector of weights $w = (7,5,4,3)$ and the threshold $T = 13$ on the set of participants $P = P_1 \cup P_2 \cup P_3 \cup P_4$ with $|P_i| = 4$ for all $i = 1, \dots, 4$. The H-minimal points of Γ are $x^1 = (2,0,0,0)$, $x^2 = (0,1,2,0)$, and $x^3 = (0,0,1,3)$. Since $x^2_2 > x^3_2$ and $|P_3| > x^2_3$, it follows from Theorem 9.2 that Γ is not ideal.

Example 9.4. Let $P = P_1 \cup P_2 \cup P_3 \cup P_4$ be a set of participants and $t_1 < t_2 < t_3 < t_4$ some positive integers. Consider a 4-partite hierarchical scheme on P in which all authorized subsets must have at least one participant from P_1, and also must have t_1 participants in P_1, or t_2 in $P_1 \cup P_2$, or t_3 in $P_1 \cup P_2 \cup P_3$, or t_4 in P. The access structure of this scheme, Γ, is a minor of Γ', the access structure whose H-minimal points are $(1,0,0,t_4)$, $(1,0,t_3,0)$, $(1,t_2,0,0)$ and $(t_1,0,0,0)$. Since Γ' is ideal by Proposition 8.6, Γ is ideal.

The access structures described in Example 4.2 with and Example 4.3 are ideal. If Γ is a hierarchical access structure with just one H-minimal point $(t_1, t_2 - t_1, \dots, t_m - t_{m-1})$, it is ideal by Proposition 8.6. The vector subspaces $V_0, \dots, V_m$ that represent the polymatroid associated to Γ satisfy $V_m \subseteq \ldots \subseteq V_1$, $V_0 \subseteq V_1$, and $V_0 \not\subseteq V_i$ for $i = 2, \dots, m$. If Γ is a hierarchical access structure with $\min_H \Gamma = \{t_1 \mathbf{e}^1, \dots, t_m \mathbf{e}^m\}$, then Γ is also ideal and the vector subspaces $V_0, \dots, V_m$ satisfy $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_m$.

Tassa [35] proposed an open problem on hierarchical access structures that can be solved by using our results. For a set of participants $P = P_1 \cup \cdots \cup P_m$, he asked for which sequence of integers $0 < k_1 < \cdots < k_m$ and for which $\ell \in \{1, \dots, m\}$, the access structure defined as follows is ideal

$$\Gamma_\ell = \bigcup_{A \in \{1,\dots,m\},\, |A| = \ell} \left\{ x \in \mathbf{P} \,:\, \sum_{j=1}^{i} x_j \geq k_i \text{ for all } i \in A \right\}$$

We assume that the access structure is strictly m-partite. In particular, we assume that $\sum_{j=1}^{i} |P_i| \geq k_i$ for all $i = 1, \ldots, m$.

Corollary 9.5. *The access structure Γ_ℓ is ideal if and only if $\ell = 1$ or $\ell = m$.*

Proof. Let $\Pi = (P'_1, \ldots, P'_m)$ be a partition of a set $P' \supset P$ with $P'_i \supset P_i$ and $|P'_i| > k_i$ for all $i = 1, \ldots, m$. For every subset $A = \{i_1, \ldots, i_\ell\} \subset [1, m]$ with $i_j < i_{j+1}$ for $j = 1, \ldots, \ell - 1$, consider the vector v^A defined as

- $v^A_{i_1} = k_{i_1}$
- $v^A_{i_j} = k_{i_j} - k_{i_{j-1}}$ for $j = 2, \ldots, \ell$
- $v^A_j = 0$ for all $j \notin A$

Define w^A as the H-minimal point of $(v^A + H) \cap \mathbf{P}$, which is not empty by hypothesis, that satisfies $m(w^A) = i_\ell$. Let Γ'_ℓ be the Π-partite access structure whose set of H-minimal points is $\{v^A : A \subset [1, m] \text{ and } |A| = \ell\}$. Observe that $\Gamma_\ell = \Gamma'_\ell \cap \mathbf{P} = (\{w^A : A \subset [1, m] \text{ and } |A| = \ell\} + H) \cap \mathbf{P}$. By Theorem 9.1, if $\ell = 1$ or $\ell = m$ then Γ'_ℓ is ideal and hence Γ_ℓ is so.

Suppose that $\ell \neq 1, m$. If there exist two subsets A, A' of size ℓ with $w^A \neq w^{A'}$ but $m(w^A) = m(w^{A'})$, then Γ_ℓ is not ideal by Theorem 9.2. If not, then we claim that Γ is not strictly m-partite. Define $\tilde{w}^t = w^{[m-\ell-t+1, m-t]}$ for every $t = 0, \ldots, m - \ell$. Taking into account that for every $1 \leq i \leq m-\ell-t$, $\tilde{w}^t = w^A$ for $A = [m-\ell-t+1, m-t] \cup \{i\} \setminus \{m-t-1\}$, it follows $\sum_{j=1}^{i} \tilde{w}^t_j = k_i$ for $t = 0, \ldots, m - \ell$ and $i = 1, \ldots, m - t$. Hence $\tilde{w}^t = v^{[1, \ldots, m-t]}$. Since $\tilde{w}^t \geq \tilde{w}^{t-1}$ for $t = 1, \ldots, m - \ell$, then $\min_H \Gamma = \{\tilde{w}^{m-\ell+1}\}$, and so the participants in the parts $m - \ell + 2, \ldots, m$ are not relevant in the structure. □

10 Ideal Weighted Threshold Access Structures

By using our characterization of ideal hierarchical access structures, we present in this section a characterization of ideal weighted threshold access structures that is more precise than the one given by Beimel, Tassa and Weinreb [1]. As was noticed in [1], such an ideal structure can be the composition smaller ideal weighted threshold access structures. Because of that, we focus on the indecomposable structures in this family.

First, we describe several families of ideal weighted threshold access structures, and then we prove in Theorem 10.1 that every indecomposable ideal weighted threshold access structure must be in one of these families.

The (t, n)-threshold access structures form the first of those families. Of course, they are ideal weighted threshold access structures. We consider as well three families of ideal bipartite hierarchical access structures, that is, ideal Π-hierarchical access structures for some partition $\Pi = (P_1, P_2)$ of the set of participants.

$\mathbf{B}_1$ This family consists of the access structures with $\min_H \Gamma = \{(x_1, x_2)\}$, where $0 < x_1 < |P_1|$ and $0 < x_2 = |P_2| - 1$. We affirm that every member of $\mathbf{B}_1$ is a weighted threshold access structure with weight vector

$$w = (w_1, w_2) = \left(1 + \frac{1}{x_1 + x_2}, 1 - \frac{x_1}{x_2(x_1 + x_2)}\right)$$

and threshold $T = x_1 + x_2$. Observe that the H-maximal non-authorized points of $\Gamma \in \mathbf{B}$ are $u = (x_1 - 1, x_2 + 1)$ and $u' = (t, x_2 + x_1 - 1 - t)$, where $t = \min\{|P_1|, x_2 + x_1 - 1\}$. Our affirmation is proved by checking that $(x_1, x_2) \cdot w \geq T$ while $u \cdot w < T$ and $u' \cdot w < T$.

$\mathbf{B}_2$ The family $\mathbf{B}_2$ is formed by the access structures with $\min_H(\Gamma) = \{(x_1, 0), (0, x_1{+}1)\}$ for some integer $x_1 > 1$. Those structures are defined by the weights $w = (w_1, w_2) = (1, 1 - 1/(x_1 + 1))$ and the threshold $T = x_1$, because $u = (x_1 - 1, 1)$ is the only H-maximal non-authorized point of Γ, and $x \cdot w \geq T$ for every $x \in \min_H \Gamma$ while $u \cdot w < T$

$\mathbf{B}_3$ This is the family of the access structures with $\min_H \Gamma = \{(y_1 + y_2 - 1, 0), (y_1, y_2)\}$, where $y_1 > 0$, $y_2 > 2$, and $|P_2| \leq y_2 \leq |P_2| + 1$. In this case we have weighted threshold access structures with $w = (w_1, w_2) = (1, 1 - 1/y_2)$ and $T = y_1 + y_2 - 1$. This is proved as before by taking into account the H-maximal non-authorized points of Γ are $u = (y_1 + y_2 - 2, 1)$ and $u' = (y_1 - 1, y_2 + 1)$ (the second point only if $|P_2| = y_2 + 1$).

In addition we consider three families of ideal tripartite hierarchical access structures.

$\mathbf{T}_1$ This family consists of the structures with $\min_H \Gamma = \{(x_1, 0, 0), (0, y_2, y_3)\}$, where $0 < y_2 < |P_2|$ and $1 < y_3 = |P_3| - 1$, and $x_1 = y_2 + y_3 - 1$. By taking into account that the H-maximal non-authorized points of Γ are $u = (x_1 - 1, 1, 0)$ and $u' = (y_2 - 1, 0, y_3 + 1)$, one can prove that every $\Gamma \in \mathbf{T}_1$ is a weighted threshold access structure with

$$w = \left(1, 1 - \frac{1}{(y_3 + 1)(y_2 + y_3)}, 1 - \frac{1}{y_3} + \frac{y_2}{y_3(y_3 + 1)(y_2 + y_3)}\right)$$

and $T = x_1$.

$\mathbf{T}_2$ We consider in this case the structures such that $\min_H \Gamma = \{(x_1, 0, 0), (y_1, y_2, y_3)\}$ with $0 < y_2 = |P_2|$ and $1 < y_3 = |P_3| - 1$, and $x_1 = y_1 + y_2 + y_3 - 1$. The H-maximal non-qualified points of those access structures are $u = (x_1 - 1, 1, 0)$ and $u' = (y_1 + y_2 - 1, 0, y_3 + 1)$. As before, we can check that the weights

$$w = \left(1, 1 - \frac{1}{(y_3 + 1)(y_1 + y_2 + y_3)}, 1 - \frac{1}{y_3} + \frac{y_1 + y_2}{y_3(y_3 + 1)(y_1 + y_2 + y_3)}\right)$$

and the threshold $T = x_1$ determine Γ.

$\mathbf{T}_3$ Finally, the family $\mathbf{T}_3$ contains the access structures with $\min_H \Gamma = \{(x_1, x_2, 0), (y_1, y_2, y_3)\}$, where $0 < y_1 < x_1$, and $1 < y_3 = |P_3|$, and $0 < x_2 = y_2{+}1 = |P_2|$, and $x_1{+}x_2 = y_1{+}y_2{+}y_3{-}1$. In this case we can consider the threshold $T = x_1 + x_2$ and the weight vector

$$w = \left(1 + \frac{1}{(x_1 + x_2)^2}, 1 - \frac{x_1}{x_2(x_1 + x_2)^2}, 1 - \frac{1}{x_1 - y_1 + 2}\left(1 + \frac{x_2 y_1 - x_1(x_2 - 1)}{x_2(x_1 + x_2)^2}\right)\right).$$

Observe that the H-maximal non-authorized points of Γ are $u = (x_1 + x_2 - 1, 0, 1)$ and $u' = (y_1 - 1, x_2, x_1 - y_1 + 2)$.

At this point, we can state the result that provides our characterization of the ideal weighted threshold access structures.

Theorem 10.1. *A weighted threshold access structure is ideal if and only if*

1. *it is a threshold access structure, or*
2. *it is a bipartite access structure in one of the families* $\mathbf{B}_1$, $\mathbf{B}_2$ *or* $\mathbf{B}_3$, *or*
3. *it is a tripartite access structure in one of the families* $\mathbf{T}_1$, $\mathbf{T}_2$ *or* $\mathbf{T}_3$, *or*
4. *it is a composition of smaller ideal weighted threshold access structures.*

The rest of this section is devoted to the proof this theorem, which is divided into several partial results. We assume that Γ is an ideal Π-hierarchical access structure for some partition $\Pi = (P_1, \ldots, P_m)$ of the set P of participants. Consider the set $\min_H \Gamma = \{x^1, \ldots, x^r\}$ of the H-minimal points of Γ. As before, we assume that $m_j < m_{j+1}$, where $m_j = \max(\mathrm{supp}(x^j))$. We begin by proving some technical lemmas.

Lemma 10.2. *If there exists* $i \in J_m$ *such that* $x_i^j = 0$ *for all* $j = 1, \ldots, r$, *then* Γ *is not strictly* m*-partite.*

Proof. If $i = m$, it is clear that the participants in P_m are redundant. If $i < m$ it is enough to prove that the participants in P_i are equivalent to the ones in P_{i+1}. Consider $x \in \min \Gamma$ such that $x' = x - \mathbf{e}^i + \mathbf{e}^{i+1} = x - \mathbf{v}^i \in \mathbf{P}$. Consider an H-minimal point y with $u = x - y \in G$. Then $\hat{u}_i - \hat{u}_{i-1} = u_i = x_i - y_i = x_i > 0$, and hence $\hat{u}_i > 0$. Therefore, $u - \mathbf{v}^i \in H$ and $x' = y + u - \mathbf{v}^i \in \Gamma$. □

Lemma 10.3. *If there exist* $j \in \{2, \ldots, r\}$ *and* $i \in J_m$ *such that* $m_{j-1} + 1 < i \le m_j$ *and* $x_i^k = |P_i|$ *for all* $k = i, \ldots, r$, *then* Γ *is not strictly* m*-partite.*

Proof. We claim that, in this situation, the participants in P_{i-1} and those in P_i are hierarchically equivalent. Consider $x \in \min \Gamma$ such that $x' = x - \mathbf{e}^{i-1} + \mathbf{e}^i = x - \mathbf{v}^{i-1} \in \mathbf{P}$. This implies that $x_i < |P_i|$ Consider an H-minimal point y with $u = x - y \in G$. Observe that $m(y) \ge i$ because $m_{j-1} < i - 1$. Then $\hat{u}_i - \hat{u}_{i-1} = u_i = x_i - y_i = x_i - |P_i| < 0$, and hence $\hat{u}_{i-1} > 0$. Therefore, $u - \mathbf{v}^{i-1} \in H$ and $x' = y + u - \mathbf{v}^{i-1} \in \Gamma$. □

Lemma 10.4. *If* $r \ge 2$ *and there exist* $j \in \{1, \ldots, r-1\}$ *and* $i \in [1, m_j]$ *such that* $x^j([1, m_j]) = x^k([1, m_j]) + \mathbf{e}^i$ *for all* $k = j+1, \ldots, r$, *then* Γ *is decomposable.*

Proof. Suppose first that $i = m_j$. Consider $p \notin P$ and define $P_i' = P_i \cup \{p\}$. Consider as well the points $y^j = x^j([1, m_j]) + \mathbf{e}^i$, and $y^k = x^k([1, m_j])$ for $1 \le k < j$, and $z^k = x^k([m_j + 1, m])$ for $j < k \le r$. Let Γ_1 be the $(P_1, \ldots, P_{i-1}, P_i')$-hierarchical access structure with $\min_H \Gamma_1 = \{y^1, \ldots, y^j\}$, and let Γ_2 be the $(P_{i+1}, \ldots, P_m)$-hierarchical access structure with $\min_H \Gamma_2 = \{z^{j+1}, \ldots, z^r\}$. It is easy to check that $\Gamma = \Gamma_1[\Gamma_2; p]$.

Suppose now that $i < m_j$. In this case, $x_k^j = |P_k|$ for all $k = i+1, \ldots, m_j$ by Theorem 9.2. Consider the point $y^j = x^j([1, m_j]) + \mathbf{e}^{m_j} - \mathbf{e}^i$, and the points $y^k = x^k([1, m_j])$ for all $1 \le k < j$ and $z^k = x^k([m_j + 1, m])$ for all $j < k \le r$. Consider Γ_1 and Γ_2 defined as in the previous case and observe that $\Gamma = \Gamma_1[\Gamma_2; p]$. □

Lemma 10.5. *If* $m \ge 2$ *and* $x_1^j = |P_1|$ *for all* $j = 1, \ldots, r$, *then* Γ *is decomposable.*

Proof. Consider $p \notin P$ and $P_1' = P_1 \cup \{p\}$, and the points $z^j = x^j([2, m])$ for all $j = 1, \ldots, r$. Let Γ_1 be the $(x_1^1 + 1, x_1^1 + 1)$-threshold access structure on P_1' and let Γ_2 be the $(P_2, \ldots, P_m)$-hierarchical access structure with $\min_H \Gamma_2 = \{z^1, \ldots, z^r\}$. Then $\Gamma = \Gamma_1[\Gamma_2; p]$. □

Lemma 10.6. *If $m \geq 2$ and Γ is indecomposable, then $x^r_m > 1$*

Proof. Suppose that $m_{r-1} < m - 1$ and $x^r_m = 1$. Consider $p \notin P$ and define $P'_{m-1} = P_{m-1} \cup \{p\}$, and the points $y^j = x^j([1, m-1])$ for $1 \leq j \leq m$. Let Γ_1 be the $(P_1, \ldots, P_{m-2}, P'_{m-1})$-hierarchical access structure with $\min_H \Gamma_1 = \{y^1, \ldots, y^r\}$ and let Γ_2 the $(1, |P_m|)$-threshold access structure on P_m. One can check that $\Gamma = \Gamma_1[\Gamma_2; p]$. If $m_{r-1} = m - 1$, then $x^r_m > 1$ by Theorem 9.2. □

We can now proceed to prove Theorem 10.1 by considering several cases depending on the number m of levels in the structure. Recall that a weighted threshold access structure with weight vector $w = (w_1, \ldots, w_m) \in \mathbb{R}^m_+$, where $w_1 > \cdots > w_m > 0$, is W-stable for $W = W(w) = \{v \in \mathbb{Z}^m : v \cdot w \geq 0\}$. The fact that $W \cup (-W) = \mathbb{Z}^m$ will be very useful in our discussion.

The case $m = 1$ clearly corresponds to the threshold access structures. We discuss in Proposition 10.7 the case $m = 2$, that is, the characterization of ideal weighted threshold access structures with two weights. Actually, this was previously solved in [28, 29], but we are only interested in the indecomposable ones. The case $m \geq 3$ is analyzed in Propositions 10.8, 10.10 and 10.12.

Proposition 10.7. *Every ideal indecomposable weighted threshold access structure that is strictly bipartite is in one of the families $\mathbf{B}_1$, $\mathbf{B}_2$ or $\mathbf{B}_3$.*

Proof. Let Γ be an ideal indecomposable weighted threshold access structure with weight vector $w = (w_1, w_2) \in \mathbb{R}^2$. Suppose that $\min_H \Gamma = \{(x_1, x_2)\}$. Taking into account Lemmas 10.2, 10.3 and 10.5, it is clear that $0 < x_1 < |P_1|$ and $1 < x_2 < |P_2|$. If $|P_2| \geq x_2 + 2$, then $(x_1, x_2) + (-1, 2) \in \mathbf{P} \smallsetminus \Gamma$, which implies that $(-1, 2) \notin W$, and hence $(1, -2) \in W$. But $(x_1, x_2) + (1, -2) \in \mathbf{P} \smallsetminus \Gamma$, a contradiction implying that $|P_2| = x_2 + 1$. Then $\Gamma \in \mathbf{B}_1$ in this case. Suppose now that $\min_H \Gamma = \{(x_1, 0), (y_1, y_2)\}$. Since $y_2 \geq 2$ by Lemma 10.6 and $x_1 - y_1 \geq 2$ by Lemma 10.4, $(y_1, y_2) + (1, -2) \in \mathbf{P} \smallsetminus \Gamma$, so $(1, -2) \notin W$ and $w_1 < 2w_2$. In addition, $w_1 > (y_2 + y_1 - x_1)w_2$ because $(x_1, x_2) + (-1, y_2 + y_1 - x_1) \in \mathbf{P} \smallsetminus \Gamma$. This implies that $x_1 = y_2 + y_1 - 1$. If $y_1 = 0$ then $y_2 = x_1 + 1$, and hence $\Gamma \in \mathbf{B}_2$. Suppose that $y_1 > 0$. If $|P_2| \geq y_2 + 2$, then $(y_1, y_2) + (-1, 2) \in \mathbf{P} \smallsetminus \Gamma$, which implies that $(-1, 2) \notin W$, and hence $(1, -2) \in W$. But $(y_1, y_2) + (1, -2) \in \mathbf{P} \smallsetminus \Gamma$, a contradiction implying that $|P_2| \leq y_2 + 1$. Then $\Gamma \in \mathbf{B}_3$. This concludes the proof because, by Theorem 9.2, all possible cases for ideal hierarchical bipartite access structures have been analyzed. □

Proposition 10.8. *Let Γ be an ideal indecomposable weighted threshold access structure. If Γ is strictly m-partite with $m \geq 3$, then $r = |\min_H(\Gamma)| = 2$.*

Proof. Let Γ be an ideal indecomposable weighted threshold access structure with weight vector $w \in \mathbb{R}^m_+$. Suppose that $r = 1$. From Lemmas 10.2, 10.3 and 10.5, $0 < x^1_i < |P_i|$ for all $i = 1, \ldots, m$. This implies that the points $x^1 + (\mathbf{e}^1 - \mathbf{e}^2 - \mathbf{e}^m)$ and $x^1 - (\mathbf{e}^1 - \mathbf{e}^2 - \mathbf{e}^m)$ are in $\mathbf{P} \smallsetminus \Gamma$, and hence the vector $\mathbf{e}^1 - \mathbf{e}^2 - \mathbf{e}^m$ is not in W nor in $-W$, a contradiction.

Suppose that $r \geq 3$. Define $x = x^{r-2}$, $y = x^{r-1}$, $z = x^r$, $i = m_{r-2}$, and $j = m_{r-1}$. By Theorem 9.2, $x' = x - \mathbf{e}^i + \mathbf{e}^j + \mathbf{e}^m \in \mathbf{P} \smallsetminus \Gamma$ because $|x'([1, m_k])| < |x^k|$ for all $k = 1, \ldots, r$. Thus $-\mathbf{e}^i + \mathbf{e}^j + \mathbf{e}^m \notin W$ and so $w_i > w_j + w_m$.

Suppose that $z_i < |P_i|$ and define the point $z' = z + \mathbf{e}^i - 2\mathbf{e}^m$, which is in $\mathbf{P}$ by Lemma 10.5. We claim that $z' \notin \Gamma$. Observe that $z' - x^k \notin H$ for all $k = 1, \ldots, r-2$ because $z'([1, m_k]) = z([1, m_k]) < x^k([1, m_k])$. Moreover, $z' - z \notin H$ because $|z'| < |z|$. Suppose now that $z' - x \in H$. In this case it is clear that $|z'([1, i])| \geq |x([1, i])|$. By Theorem 9.2, $|x([1, i])| = |z([1, i])| + 1 = |z'([1, i])|$. Since $z_i < |P_i|$, by applying Theorem 9.2 again it follows $x([1, i]) = z([1, i]) + \mathbf{e}^i$. Observe that

$y([1,i]) = z([1,i])$ because $x([1,i]) > y([1,i])$. By Lemma 10.4, this is a contradiction. Therefore, $z' - x \notin H$. We prove now that $z' - y \notin H$. On the contrary, $|z'([1,j])| = |y([1,j])|$. If $y_j > z_j$, then $y([1,j]) = z([1,j]) + \mathbf{e}^j$, a contradiction by Lemma 10.4. If $y_j = z_j$ then there exists by Theorem 9.2 a value $k \in \{1, \dots, j-1\}$ for which $y_k = z_k + 1$, $y_\ell = z_\ell = |P_\ell|$ for all $\ell = k+1, \dots, j$, and $y_\ell = z_\ell$ for all $\ell = 1, \dots, k-1$, a contradiction by Lemma 10.4. Therefore $z' \in \mathbf{P} \smallsetminus \Gamma$, and hence $w_i < 2w_m$, a contradiction.

Suppose that $z_i = |P_i|$. By Theorem 9.2 there exists $k \in \{1, \dots, i-1\}$ for which $x_k > z_k$ and $|P_\ell| = z_\ell = x_\ell$ for all $\ell = k+1, \dots, i$. Define $z' = z + \mathbf{e}^k - 2\mathbf{e}^m$. Analogously to the previous case, $z' \in \mathbf{P} \smallsetminus \Gamma$. Therefore, $w_k < 2w_m$, a contradiction. □

Lemma 10.9. *Let Γ be an ideal weighted threshold access structure that is strictly m-partite and indecomposable. If $r = 2$ and $m_1 > 1$, then $x_1^2 > 0$.*

Proof. Suppose that $x_1^2 = 0$. By Lemma 10.2, $x_1^1 > 0$ and, as consequence of Theorem 9.2, $x_\ell^1 = |P_\ell|$ for all $\ell = 2, \dots, m_1$. Then observe that participants in P_1 and P_2 are hierarchically equivalent, and hence Γ is $(P_1 \cup P_2, P_3, \dots, P_m)$-partite with H-minimal points $(x_1^i + x_2^i, x_3^i, \dots, x_m^i)$ for $i = 1, 2$. □

Proposition 10.10. *Every ideal indecomposable weighted threshold access structure that is strictly tripartite is in one of the families $\mathbf{T}_1$, $\mathbf{T}_2$ or $\mathbf{T}_3$.*

Proof. Let Γ be an ideal indecomposable weighted threshold access structure with vector of weights $w \in \mathbb{R}_+^3$. Assume that Γ is strictly tripartite. By Proposition 10.8, Γ has exactly two minimal points.

Suppose that $\min_H \Gamma = \{x, y\} = \{(x_1, 0, 0), (y_1, y_2, y_3)\}$. Taking into account Lemmas 10.2, 10.3, and 10.5, it is clear that $0 < y_2$ and $1 < y_3 < |P_3|$. By Lemma 10.4, $x_1 > y_1 + 1$, which implies that $y + (1, -1, -1) \in \mathbf{P} \smallsetminus \Gamma$. Hence $(1, -1, -1) \notin W$ and so $w_1 < w_2 + w_3$. Suppose that $y_2 = |P_2|$. If $|P_3| > y_3 + 1$, then $y + (0, -1, 2) \in \mathbf{P} \smallsetminus \Gamma$ and so $w_2 > 2w_3$. But $w_1 < 2w_3$ because $y + (1, 0, -2) \in \mathbf{P} \smallsetminus \Gamma$, a contradiction. Therefore, $|P_3| = y_3 + 1$ and Γ is in $\mathbf{T}_2$. Now suppose that $y_2 < |P_2|$. In this case $y + (0, 1, -2) \in \mathbf{P} \smallsetminus \Gamma$. If $|P_3| > y_3 + 1$, then $y + (0, -1, 2) \in \mathbf{P} \smallsetminus \Gamma$, a contradiction implying that $|P_3| = y_3 + 1$. If $y_1 > 0$, then $y + (-1, 1, 1) \in \mathbf{P} \smallsetminus \Gamma$, a contradiction. Consequently, $y_1 = 0$ and Γ is in $\mathbf{T}_1$.

Suppose that $\min_H \Gamma = \{x, y\} = \{(x_1, x_2, 0), (y_1, y_2, y_3)\}$ with $x_2 > 0$. Observe that $y_3 \geq 2$ by Lemma 10.6. Suppose, for the sake of contradiction, that $x_1 = y_1$. Taking into account Lemmas 10.4 and 10.5, it is clear that $x_2 \geq y_2 + 2$ and $x_1 < |P_1|$. In this case, both $y + (1, 0, -2)$ and $y + (-1, 2, 0)$ are in $\mathbf{P} \smallsetminus \Gamma$, a contradiction. Hence $x_1 > y_1$. As a consequence of Theorem 9.2, $x_2 = |P_2|$ and so $x_2 > y_2$ by Lemma 10.3. Note that $y_1 > 0$ by Lemma 10.9. Since $y + (1, 0, -2) \in \mathbf{P} \smallsetminus \Gamma$, $w_1 < 2w_3$. If $|x| < |y| - 1$, then $x + (-1, 0, 2) \in \mathbf{P} \smallsetminus \Gamma$ and so $w_1 > 2w_3$, a contradiction. Hence $|x| = |y| - 1$. If $y_3 < |P_3|$ then $y + (-1, 1, 1) \in \mathbf{P} \smallsetminus \Gamma$ and so $w_1 > w_2 + w_3$, a contradiction implying $y_3 = |P_3|$. Observe that $x_2 = y_2 + 1$, because if $x_2 > y_2 + 1$ then $y + (-1, 2, 0) \in \mathbf{P} \smallsetminus \Gamma$ and hence $w_1 > 2w_2$, a contradiction. Therefore, Γ is in $\mathbf{T}_3$.

This concludes the proof because, by Theorem 9.2, all possible tripartite hierarchical ideal access structures have been analyzed. □

Lemma 10.11. *Let Γ be an ideal weighted threshold access structure that is strictly m-partite and indecomposable. If $r = 2$, then $|x^1([1, m_1])| > |x^2([1, m_1])| + 1$.*

Proof. From Theorem 9.2, $x^1([1, m_1]) > x^2([1, m_1])$, and if $|x^1([1, m_1])| = |x^2([1, m_1])| + 1$, then there exists $1 \leq i \leq m_1$ for which $x^1([1, m_1]) = x^2([1, m_1]) + \mathbf{e}^i$, which contradicts Lemma 10.4. □

Proposition 10.12. *If an ideal weighted threshold access structure is strictly m-partite with $m > 3$, then it is decomposable.*

Proof. Let Γ be an ideal weighted threshold access structure that is strictly m-partite with $m > 3$. By Proposition 10.8, it has exactly two H-minimal points, that we call x and y, with $m(x) < m(y)$. Define $i = m(x)$ and observe that $m(y) = m$.

Suppose that $m - i = 1$ and $x_1 = y_1$. Since $i \geq 3$, by Lemmas 10.2, 10.3, and 10.5 we obtain that $x_j > 0$ and $y_j < |P_j|$ for all $j = 1, 2, 3$. Thus both $x + \mathbf{e}^1 - \mathbf{e}^2 - \mathbf{e}^3$ and $y - \mathbf{e}^1 + \mathbf{e}^2 + \mathbf{e}^3$ are in $\mathbf{P} \smallsetminus \Gamma$, a contradiction. Now suppose that $m - i = 1$ and $x_1 > y_1$. By Theorem 9.2, $x_j = |P_j|$ for all $j = 2, \ldots, i$, and by Lemma 10.3, $y_2 < |P_2|$ and $y_3 < |P_3|$. As a consequence of Lemma 10.9 we obtain that $y_1 > 0$, and by following an analogous reasoning, we obtain that $y_2 > 0$. Hence both $y - \mathbf{e}^1 + \mathbf{e}^2 + \mathbf{e}^3$ and $y + \mathbf{e}^1 - \mathbf{e}^2 - \mathbf{e}^m$ are in $\mathbf{P} \smallsetminus \Gamma$, which implies that $w_3 < w_m$, a contradiction. Therefore $m - i > 1$.

Now suppose that $m - i \geq 2$ and $i > 1$. By Lemmas 10.3 and 10.5, $y_2 < |P_2|$ and $1 < y_m < |P_m|$. Suppose that $x_1 = y_1$. It is clear that $y_1 > 0$ by Lemma 10.2, so taking into account Lemmas 10.11 and 10.5 we obtain that both $y + \mathbf{e}^1 - 2\mathbf{e}^m$ and $y - \mathbf{e}^1 + \mathbf{e}^2 + \mathbf{e}^m$ are in $\mathbf{P} \smallsetminus \Gamma$, a contradiction. Now suppose that $x_1 > y_1$. In this case, $y_1 > 0$ by Lemma 10.9, and $x_j = |P_j|$ for all $j = 2, \ldots, i$ by Theorem 9.2. As a consequence of Lemma 10.11, both $y + \mathbf{e}^1 - \mathbf{e}^{m-1} - \mathbf{e}^m$ and $y - \mathbf{e}^1 + \mathbf{e}^{m-1} + \mathbf{e}^m$ are in $\mathbf{P} \smallsetminus \Gamma$, a contradiction.

Finally, suppose that $i = 1$. By Lemmas 10.2, 10.3, and 10.4 we obtain that $x_1 - y_1 \geq 2$, $y_{m-j} > 0$ for $j = 0, 1, 2$, and $y_{m-j} < |P_{m-j}|$ for $j = 0, 1$. Both $y + \mathbf{e}^1 - \mathbf{e}^{m-1} - \mathbf{e}^m$ and $y - \mathbf{e}^{m-2} + \mathbf{e}^{m-1} + \mathbf{e}^m$ are in $\mathbf{P} \smallsetminus \Gamma$, a contradiction. □

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