A New Improved Distinguisher for HC-128

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Abstract. In this paper, we present a new distinguisher for HC-128 which is the best known so far. The distinguisher requires approximately 2^{138} keystream words with success probability 0.9772.

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1 Introduction

The eSTREAM [2] Portfolio (revision 1 in September 2008) contains the stream cipher HC-128 [6] in Profile 1 (SW). Apart from the analysis by the author (Wu) himself to conjecture the security of this cipher, the only other observation is by Dunkelman [3] in the eSTREAM discussion forum to show that the keystream words of HC-128 leak information regarding secret states. Recently, generalization of these results has been studied in [4]. In this paper, we identify a new and improved distinguisher for HC-128. To the best of our knowledge, this is currently the strongest distinguisher available.

Each keystream word of HC-128 is 32 bit long (the 0th bit is the least significant bit and the 31st bit is the most significant bit). In [6], bitwise XOR of least significant bits of 10 (possibly) different keystream words (rotated by certain amounts) are considered to propose a distinguisher. In [4], the distinguisher is extended for other bits too. The distinguishers presented in [6, 4] require approximately 2^{156} words of keystream with success probability of 0.9772. Our distinguisher requires approximately 2^{138} keystream words with the same success probability. For every block of 512 many keystream words of HC-128, corresponding to either the P or the Q array, we show that XOR of the least significant bits (LSBs) of four keystream words (two taken from the initial sub-block of 256 keystream

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words and two taken from the latter sub-block of 256 keystream words) is biased towards 0. The distinguisher can also be extended for bits other than the LSBs.

The complete study of the new distinguisher is presented in Section 3. Let us start with the description of HC-128 in the following section.

2 Description of HC-128

This is adapted from [6, Section 2].

2.1 Notations and Data Structures

The following operations are used in HC-128:

$+$: $x + y$ means $x + y \bmod 2^{32}$, where $0 \le x < 2^{32}$ and $0 \le y < 2^{32}$.
$\boxminus$: $x \boxminus y$ means $x - y \bmod 512$.
$\oplus$: bit-wise exclusive OR.
$\|$: concatenation.
$\gg$: right shift operator. $x \gg n$ means x being right shifted n bits.
$\ll$: left shift operator. $x \ll n$ means x being left shifted n bits.
$\ggg$: right rotation operator. $x \ggg n$ means $((x \gg n) \oplus (x \ll (32 - n))$, where $0 \le n < 32$, $0 \le x < 2^{32}$.
$\lll$: left rotation operator. $x \lll n$ means $((x \ll n) \oplus (x \gg (32 - n))$, where $0 \le n < 32$, $0 \le x < 2^{32}$.

Two tables P and Q, each with 512 many 32-bit elements are used as internal states of HC-128. A 128-bit key array $K[0, \ldots, 3]$ and a 128-bit initialization vector $IV[0, \ldots, 3]$ are used, where each entry of the array is a 32-bit element. Let s_t denote the keystream word generated at the t-th step, $t = 0, 1, 2, \ldots$.

The following six functions are used in HC-128:

$f_1(x) = (x \ggg 7) \oplus (x \ggg 18) \oplus (x \gg 3)$,
$f_2(x) = (x \ggg 17) \oplus (x \ggg 19) \oplus (x \gg 10)$,
$g_1(x, y, z) = \big((x \ggg 10) \oplus (z \ggg 23)\big) + (y \ggg 8)$,
$g_2(x, y, z) = \big((x \lll 10) \oplus (z \lll 23)\big) + (y \lll 8)$,
$h_1(x) = Q[x^{(0)}] + Q[256 + x^{(2)}]$,
$h_2(x) = P[x^{(0)}] + P[256 + x^{(2)}]$,

where $x^{(0)}$ (least significant byte) $, x^{(1)}, x^{(2)}$ and $x^{(3)}$ (most significant byte) are the four bytes of a 32-bit word $x = x^{(3)} \| x^{(2)} \| x^{(1)} \| x^{(0)}$.

2.2 Key and IV Setup

1. Let $K[0, \ldots, 3]$ be the secret key and $IV[0, \ldots, 3]$ be the initialization vector. Let $K[i + 4] = K[i]$ and $IV[i + 4] = IV[i]$ for $0 \le i \le 3$.

2. The key and IV are expanded into an array $W[0, \ldots, 1279]$ as follows.

$$W[i] = \begin{cases} K[i], & 0 \le i \le 7; \\ IV[i-8], & 8 \le i \le 15; \\ f_2(W[i-2]) + W[i-7] & \\ \quad + f_1(W[i-15]) + W[i-16] + i, & 16 \le i \le 1279. \end{cases}$$

3. Update the tables P and Q with the array W as follows.

$$P[i] = W[i+256], \text{ for } 0 \le i \le 511$$
$$Q[i] = W[i+768], \text{ for } 0 \le i \le 511$$

4. Run the cipher 1024 steps and use the outputs to replace the table elements as follows.
for $i = 0$ to 511, do
$\quad P[i] = \big(P[i] + g_1(P[i \boxminus 3], P[i \boxminus 10], P[i \boxminus 511])\big) \oplus h_1(P[i \boxminus 12])$;
for $i = 0$ to 511, do
$\quad Q[i] = \big(Q[i] + g_2(Q[i \boxminus 3], Q[i \boxminus 10], Q[i \boxminus 511])\big) \oplus h_2(Q[i \boxminus 12])$;

2.3 The Keystream Generation Algorithm

$i = 0$;
repeat until enough keystream bits are generated
{
  $j = i \bmod 512$;
  if $(i \bmod 1024) < 512$
  {
    $P[j] = P[j] + g_1(P[j \boxminus 3], P[j \boxminus 10], P[j \boxminus 511])$;
    $s_i = h_1(P[j \boxminus 12]) \oplus P[j]$;
  }
  else
  {
    $Q[j] = Q[j] + g_2(Q[j \boxminus 3], Q[j \boxminus 10], Q[j \boxminus 511])$;
    $s_i = h_2(Q[j \boxminus 12]) \oplus Q[j]$;
  }
  end-if
  $i = i + 1$;
}
end-repeat

3 Our New Distinguisher

In this section we present our new distinguisher. Before getting into the technical details, let us explain one more notation. For any n-bit integer I, $[I]^b$ denotes the b-th least significant bit, $0 \le b \le n-1$. Thus $[I]^0$ denotes the LSB of I. Also in the following discussion, we abuse the notation of s_i described in Section 2.3. Here,

by s_t, we mean a keystream word generated in the t-th step in a block of 512 words corresponding to either the P array (completely) or the Q array (completely). As we will be concentrating on the relationship between four keystream words in a block of 512, (to keep the notation simple) we do not use any index to identify different blocks.

3.1 Least Significant Bit Based Distinguisher

We first consider the term $h_2(Q[\alpha])$, $0 \leq \alpha \leq 511$. We know,

$$h_2(Q[\alpha]) = P[\beta^{(0)}] + P[256 + \beta^{(2)}],$$

where $\beta = Q[\alpha]$. When we consider the LSBs, the "+" operator can be replaced by "$\oplus$". So, we have

$$[h_2(\beta)]^0 = \left[P[\beta^{(0)}] \oplus P[256 + \beta^{(2)}]\right]^0.$$

Following the idea of [3] and [4, Lemma 3, Corollary 3, Theorem 5], we have the following result.

Proposition 1. *In HC-128, consider a block of 512 many keystream words corresponding to the array P. For $0 \leq u \neq l \leq 511$,*

$$Prob\big([s_u \oplus s_l]^0 = [P[u] \oplus P[l]]^0\big) \geq \frac{1}{2} + \frac{1}{2^{17}}.$$

In the following discussion, we will consider the lower bound of this probability, i.e., we will replace $Prob\big([s_u \oplus s_l]^0 = [P[u] \oplus P[l]]^0\big)$ by $\frac{1}{2} + \frac{1}{2^{17}}$.

We note that the equality

$$[s_u \oplus s_l]^0 = [P[u] \oplus P[l]]^0$$

holds if and only if the equality $[h_1(P[u \boxminus 12]) \oplus h_1(P[l \boxminus 12])]^0 = 0$ holds, i.e., $[F_1(\beta)]^0 = 0$ where $F_1(\beta) = h_1(P[u \boxminus 12]) \oplus h_1(P[l \boxminus 12])$, and $\beta = Q[\alpha], u = \beta^{(0)}, l = 256 + \beta^{(2)}$ for some α such that $0 \leq \alpha \leq 511$. The equality

$$[(s_u \oplus s_l)]^0 = [(P[u] \oplus P[l])]^0 \oplus 1$$

holds if and only if $[F_1(\beta)]^0 = 1$.

We also need to consider the next technical result that follows similar to some subcases in the proof of [4, Lemma 3]. For clarity we write the complete proof here.

Proposition 2. *Let $h'(x) = U'[x^{(0)}] + U'[x^{(2)} + 2^m]$ be an n-bit to n-bit mapping, where each entry of the array U' is an n-bit number, U' contains 2^{m+1} many elements and $x^{(0)}$ and $x^{(2)}$ are two disjoint m-bit segments from the n-bit input x. Suppose x and x' are two n-bit random inputs to h'. Assuming that the entries of U' are distributed uniformly at random, we have $Prob\big(h'(x) = h'(x') | x^{(0)} \neq x'^{(0)}$ and $x^{(2)} \neq x'^{(2)}\big) = 2^{-2n} + 2^{-n}(1 - 2^{-n})^2$.*

Proof. Given $x^{(0)} \neq x'^{(0)}$ and $x^{(2)} \neq x'^{(2)}$, the value of $h'(x)$ equals the value $h'(x')$ in the following two ways.

- The event "$U'[x^{(0)}] = U'[x'^{(0)}]$ and $U'[x^{(2)}] = U'[x'^{(2)}]$" (this gives $h'(x) = h'(x')$) happens by random association with probability $2^{-n} \cdot 2^{-n}$.
- We consider that $U'[x^{(0)}] \neq U'[x'^{(0)}]$ and $U'[x^{(2)}] \neq U'[x'^{(2)}]$, but still we have $h'(x) = h'(x')$ due to random association. This happens with probability $(1 - 2^{-n}) \cdot (1 - 2^{-n}) \cdot 2^{-n}$.

Adding the above two cases, we get the result[1]. □

Remark 1. Following [4, Lemma 3], $Prob\big(h(x) = h(x')\big) = 2^{-2m} + 2^{1-m-n}(1 - 2^{-m}) + 2^{-2n}(1 - 2^{-m})^2 + 2^{-n}(1 - 2^{-m})^2(1 - 2^{-n})^2$, which gives the value approximately 2^{-16} for $m = 8$ and $n = 32$. This association between two 32-bit integers is quite high as the inputs are actually 16 bits long. On the other hand, for $m = 8$ and $n = 32$, the probability from Proposition 2 comes to be $2^{-32} - 2^{-64} + 2^{-96} \approx 2^{-32}$, which is the random association probability between two 32-bit numbers. This is due to the fact that we cannot use the equality of $x^{(0)}, x'^{(0)}$ or $x^{(2)}, x'^{(2)}$.

Lemma 1. *Consider the consecutive P and Q arrays in the execution of HC-128 and let s_t, $0 \leq t \leq 511$, be the keystream words generated corresponding to the P array. For $0 \leq \alpha \neq \alpha' \leq 511$, we have*

$$Prob\left([s_u \oplus s_l]^0 = [s_{u'} \oplus s_{l'}]^0\right) = \frac{1}{2} + \frac{1}{2^{65}},$$

where $u = Q[\alpha]^{(0)}$, $l = 256 + Q[\alpha]^{(2)}$, $u' = Q[\alpha']^{(0)}$ and $l' = 256 + Q[\alpha']^{(2)}$ and $u \neq u'$, $l \neq l'$.

Proof. First we denote the event $\left([s_u \oplus s_l]^0 = [s_{u'} \oplus s_{l'}]^0\right)$ by C.

We define $F_2(\beta, \beta') = F_1(\beta) \oplus F_1(\beta')$.

$$Prob(C) = Prob(C, [F_2(\beta, \beta')]^0 = 0) + Prob(C, [F_2(\beta, \beta')]^0 = 1). \quad (1)$$

Now, $[F_2(\beta, \beta')]^0$ can be equal to zero in the following two mutually exclusive paths.

1. $[F_1(\beta)]^0 = [F_1(\beta')]^0 = 0$. This implies that $[s_u \oplus s_l]^0 = [h_2(\beta)]^0$ and $[s_{u'} \oplus s_{l'}]^0 = [h_2(\beta')]^0$.
2. $[F_1(\beta)]^0 = [F_1(\beta')]^0 = 1$. This occurs when $[s_u \oplus s_l]^0 = [h_2(\beta)]^0 \oplus 1$ and $[s_{u'} \oplus s_{l'}]^0 = [h_2(\beta')]^0 \oplus 1$.

[1] Note that, in the proof of [4, Lemma 3], the term $(1 - 2^{-m})^2$ has been multiplied in each of the two cases for taking care of the condition $x^{(0)} \neq x'^{(0)}$ and $x^{(2)} \neq x'^{(2)}$. However, the term $(1 - 2^{-m})^2$ is not multiplied here as Proposition 2 considers that $x^{(0)} \neq x'^{(0)}$ and $x^{(2)} \neq x'^{(2)}$ is given.

Given that $[F_2(\beta, \beta')]^0 = 0$ the event C occurs if only if $[h_2(\beta)]^0 = [h_2(\beta')]^0$ with the constraint that $\beta^{(0)} \neq \beta'^{(0)}$ and $\beta^{(2)} \neq \beta'^{(2)}$. The condition $\beta^{(0)} \neq \beta'^{(0)}$ (respectively $\beta^{(2)} \neq \beta'^{(2)}$) is required to guarantee $u \neq u'$ (respectively $l \neq l'$).

With the assumption $\beta^{(0)} \neq \beta'^{(0)}$ and $\beta^{(2)} \neq \beta'^{(2)}$, we get $Prob([h_2(\beta)]^0 = [h_2(\beta')]^0) = 1 \cdot 2^{-32} + (1 - 2^{-32}) \cdot \frac{1}{2} = \frac{1}{2} + 2^{-33}$, where the first part comes following Proposition 2 and the second part comes when the LSBs of $h_2(\beta)$ and $h_2(\beta')$ are equal by random association. Thus,

$$Prob(C|\ [F_2(\beta, \beta')]^0 = 0) = Prob([h_2(\beta)]^0 = [h_2(\beta')]^0) = \frac{1}{2} + \frac{1}{2^{33}}.$$

Similarly, $[F_2(\beta, \beta')]^0$ can be equal to 1 in the following two mutually exclusive ways.

1. $[F_1(\beta)]^0 = 0$ and $[F_1(\beta')]^0 = 1$. This implies that $[s_u \oplus s_l]^0 = [h_2(\beta)]^0$ and $[s_{u'} \oplus s_{l'}]^0 = [h_2(\beta')]^0 \oplus 1$.
2. $[F_1(\beta)]^0 = 1$ and $[F_1(\beta')]^0 = 0$. This occurs when $[s_u \oplus s_l]^0 = [h_2(\beta)]^0 \oplus 1$ and $[s_{u'} \oplus s_{l'}]^0 = [h_2(\beta')]^0$.

Given that $[F_2(\beta, \beta')]^0 = 1$, the event C occurs if only if $[h_2(\beta)]^0 = [h_2(\beta')]^0 \oplus 1$. Therefore,

$$Prob(C|\ [F_2(\beta, \beta')]^0 = 1) = Prob([h_2(\beta)]^0 = [h_2(\beta')]^0 \oplus 1) = \frac{1}{2} - \frac{1}{2^{33}}.$$

From Proposition 1, the event $([F_1(\beta)]^0 = 0)$ occurs with a probability approximately $\frac{1}{2} + \frac{1}{2^{17}}$ and the event $([F_1(\beta)]^0 = 1)$ occurs with a probability approximately $\frac{1}{2} - \frac{1}{2^{17}}$. Thus,

$$Prob([F_2(\beta, \beta')]^0 = 0) = (\frac{1}{2} + \frac{1}{2^{17}})^2 + (\frac{1}{2} - \frac{1}{2^{17}})^2, \text{ and}$$

$$\begin{aligned} Prob([F_2(\beta, \beta')]^0 = 1) &= (\frac{1}{2} + \frac{1}{2^{17}})(\frac{1}{2} - \frac{1}{2^{17}}) + (\frac{1}{2} + \frac{1}{2^{17}})(\frac{1}{2} - \frac{1}{2^{17}}) \\ &= 2(\frac{1}{2} + \frac{1}{2^{17}})(\frac{1}{2} - \frac{1}{2^{17}}). \end{aligned}$$

From Equation 1, we have,

$$\begin{aligned} Prob(C) = &Prob([F_2(\beta, \beta')]^0 = 0) Prob(C|[F_2(\beta, \beta')]^0 = 0) \\ &+ Prob([F_2(\beta, \beta')]^0 = 1) Prob(C|[F_2(\beta, \beta')]^0 = 1) \\ = &\frac{1}{2} + \frac{1}{2^{65}}. \end{aligned}$$

□

Now we provide an alternative approach to the proof of Lemma 1 for better understanding. We have considered the term $h_2(Q[\alpha])$, for $0 \leq \alpha \leq 511$. Note that

$$h_2(Q[\alpha]) = P[\beta^{(0)}] + P[256 + \beta^{(2)}],$$

where $\beta = Q[\alpha]$ and then we get

$$h_2(Q[\alpha]) = \left(s_{\beta^{(0)}} \oplus h_1(P[\beta^{(0)} \boxminus 12])\right) + \left(s_{256+\beta^{(2)}} \oplus h_1(P[256 + \beta^{(2)} \boxminus 12])\right). \quad (2)$$

As noted in [6], for the least significant bit, '+' can be replaced by '$\oplus$' and hence,

$$[s_{\beta^{(0)}} \oplus s_{256+\beta^{(2)}}]^0 = [h_1(P[\beta^{(0)} \boxminus 12]) \oplus h_1(P[256 + \beta^{(2)} \boxminus 12]) \oplus h_2(\beta)]^0. \quad (3)$$

Denoting $u = \beta^{(0)}, l = 256 + \beta^{(2)}, \mu = P[\beta^{(0)} \boxminus 12], \nu = P[256 + \beta^{(2)} \boxminus 12]$, we have

$$[s_u \oplus s_l]^0 = [h_1(\mu) \oplus h_1(\nu) \oplus h_2(\beta)]^0.$$

Thus for $u \neq u'$ and $l \neq l'$, we get,

$$[s_u \oplus s_l]^0 = [s_{u'} \oplus s_{l'}]^0,$$

if and only if

$$[h_1(\mu) \oplus h_1(\nu) \oplus h_2(\beta)]^0 = [h_1(\mu') \oplus h_1(\nu') \oplus h_2(\beta')]^0,$$

where μ', ν' and β' correspond to the indices u' and l' in the same manner as μ, ν and β correspond to the indices u and l.

The function h_1 uses actually 16 bits only from the 32-bit input. The situation is similar for h_2. Thus, one may be tempted to approximate $[h_1(.) \oplus h_1(.) \oplus h_2(.)]^0$ as a random 48-bit-to-1-bit S-box. Thus, $Prob([h_1(\mu) \oplus h_1(\nu) \oplus h_2(\beta)]^0 = [h_1(\mu') \oplus h_1(\nu') \oplus h_2(\beta')]^0)$ is equal to the collision probability of a random 48-bit-to-1-bit S-box. According to [6, Theorem 1], this is equal to $2^{-48} + 2^{-1} - 2^{-48-1}$.

However, considering the random 48-bit-to-1-bit S-box is not correct. In our analysis, we cannot have "$\beta^{(0)} = \beta'^{(0)}$ or $\beta^{(2)} = \beta'^{(2)}$" as that will violate the condition "$u \neq u'$ and $l \neq l'$". Thus for the $h_2(.)$ function, the equality of the 16-bit inputs from β and β' cannot happen. Thus the event $h_2(\beta) = h_2(\beta')$ occurs with a probability approximately 2^{-32} when "$\beta^{(0)} \neq \beta'^{(0)}$ and $\beta^{(2)} \neq \beta'^{(2)}$" (similar to the discussion presented in Remark 1). This is the reason that while we consider the inputs of $h_1(.), h_2(.)$ as 16 bits in general with respect to HC-128, in this case we have to consider the input of $h_2(.)$ as 32 bits while modelling the S-box. Thus, one can approximate $[h_1(.) \oplus h_1(.) \oplus h_2(.)]^0$ as a random 64-bit-to-1-bit S-box. Hence, $Prob([h_1(\mu) \oplus h_1(\nu) \oplus h_2(\beta)]^0 = [h_1(\mu') \oplus h_1(\nu') \oplus h_2(\beta')]^0)$ is equal to the collision probability of a random 64-bit-to-1-bit S-box. According to [6, Theorem 1], this is equal to $2^{-64} + 2^{-1} - 2^{-64-1} = 2^{-1} + 2^{-65}$, which supports the result of Lemma 1.

We started concentrating on $Q[\alpha]$ by noting the expression of $h_2(Q[\alpha])$. This leads to the relationship among the keystream words generated corresponding to the previous P array. In a similar direction, one can start with $P[\alpha]$ by noting the expression of $h_1(P[\alpha])$. This will lead to similar relationship among the keystream words generated corresponding to the previous Q array.

Lemma 2. *Consider the consecutive Q and P arrays in the execution of HC-128 and let s_t, $0 \leq t \leq 511$, be the keystream words generated corresponding to the Q array. For $0 \leq \alpha \neq \alpha' \leq 511$, we have*

$$Prob\left([s_u \oplus s_l]^0 = [s_{u'} \oplus s_{l'}]^0\right) = \frac{1}{2} + \frac{1}{2^{65}},$$

where $u = P[\alpha]^{(0)}$, $l = 256 + P[\alpha]^{(2)}$, $u' = P[\alpha']^{(0)}$ and $l' = 256 + P[\alpha']^{(2)}$, and $u \neq u'$, $l \neq l'$.

Given α, α', there is no way to observe the values of $Q[\alpha], Q[\alpha']$ (or $P[\alpha], P[\alpha']$) and hence we cannot identify the indices u, l, u', l'.

We overcome this problem in the following manner. We know that any element $Q[\alpha]$ (or $P[\alpha]$), $0 \leq \alpha \leq 511$ provides one pair of keystream words of the form (s_u, s_l). So there are $\binom{512}{2}$ many quadruples of the form (u, l, u', l') for which Lemma 1 (or Lemma 2) holds. We refer to these quadruples as *favourable quadruples*. The following result uses Lemma 1 and Lemma 2 to compute the expression of the probability for "any" quadruple (u, l, u', l') where the pair (u, u') corresponds to the initial half ($0 \leq u \neq u' \leq 255$) and the pair (l, l') corresponds to the latter half ($256 \leq l \neq l' \leq 511$) of the array Q (or P). This is our main result that will be used to find the new distinguisher.

Theorem 1. *Let s_t, $0 \leq t \leq 511$, be the keystream words generated corresponding to either P or Q array. For $0 \leq u \neq u' \leq 255$, $256 \leq l \neq l' \leq 511$,*

$$Prob\left([s_u \oplus s_l]^0 = [s_{u'} \oplus s_{l'}]^0\right) \approx \frac{1}{2} + \frac{1}{2^{78}}.$$

Proof. From the ranges of u, u', l, l', it is clear that there are $\binom{256}{2}^2$ many quadruples of the form (u, l, u', l'). Let F be the event that an arbitrary quadruple (u, u', l, l') is favourable and further let E be the event $\left([s_u \oplus s_l]^0 = [s_{u'} \oplus s_{l'}]^0\right)$.

We can choose any α, α' for $0 \leq \alpha \neq \alpha' \leq 511$ to build one equation of the form $[s_u \oplus s_l]^0 = [s_{u'} \oplus s_{l'}]^0$ with the following constraint. If at least one of the equalities $(Q[\alpha]^{(0)} = Q[\alpha']^{(0)})$ or $(Q[\alpha]^{(2)} = Q[\alpha']^{(2)})$ holds, then we cannot form the above combination of keystream bits to generate the equation. The situation is similar for the case of P. However, the expected number of such cases is very small (around 4 out of $\binom{512}{2} \approx 2^{17}$) if we consider that Q (or P) contains 512 many 32-bit integers chosen uniformly at random from the set of all 32-bit integers. Thus, we can approximate $Prob(F) = \frac{\binom{512}{2}}{\binom{256}{2}^2}$.

Further, from Lemma 1 and Lemma 2, $Prob(E|F) = \frac{1}{2} + \frac{1}{2^{65}}$. We can assume that for a non-favourable quadruple, the event E occurs due to random association only, i.e., $Prob(E|F^C) = \frac{1}{2}$, where F^C is the complement of the event F. Thus,
$Prob(E) = Prob(F) \cdot Prob(E|F) + Prob(F^C) \cdot Prob(E|F^C)$
$= \frac{\binom{512}{2}}{\binom{256}{2}^2} \cdot (\frac{1}{2} + \frac{1}{2^{65}}) + \left(1 - \frac{\binom{512}{2}}{\binom{256}{2}^2}\right) \cdot \frac{1}{2} \approx \frac{1}{2} + \frac{1}{2^{78}}.$ □

Hence, Theorem 1 gives us a distinguisher. The number of keystream words required to mount the above distinguisher is computed in Theorem 2 below.

Theorem 2. *HC-128 can be distinguished from an ideal random word generator by observing* 2^{138} *keystream words with a success probability of 0.9772.*

Proof. According to Theorem 1, the event $\left([s_u \oplus s_l]^0 = [s_{u'} \oplus s_{l'}]^0\right)$ based on which the distinguisher is constructed occurs with a probability $p(1+q)$, where $p = \frac{1}{2}$ and $q \approx \frac{1}{2^{77}}$. According to [1, Section 4.1], to get a success probability of 0.9772, one would require $\frac{4^2}{pq^2} = 2^{159}$ many samples. In our case, each sample consists of a set of 4 keystream words of the form $(s_u, s_{u'}, s_l, s_{l'})$. Since each block of $512 = 2^9$ many keystream words (corresponding to either the array P or the array Q) gives $\binom{256}{2}^2 \approx 2^{30}$ many samples, one needs $2^{159+9-30} = 2^{138}$ many keystream words to mount the above distinguisher. □

In terms of data complexity, we have a significant improvement over [6, 4], where the number of keystream words required is around 2^{156} for the same success probability.

3.2 Distinguisher Based on Any Bit of the Keystream Words

So far we have concentrated on the LSBs and now we extend this to other bits also. We use the following result from [5].

Proposition 3. *Suppose* X *and* Y *are two* n*-bit integers. Let* $S = X + Y$ *and* $T = X \oplus Y$*. Then* $Prob([S]^b = [T]^b) = \frac{1}{2}(1 + \frac{1}{2^b})$*,* $0 \le b \le n-1$*.*

Now we present the main result.

Lemma 3. *Consider the consecutive* P *and* Q *arrays in the execution of HC-128 and let* s_t*,* $0 \le t \le 511$*, be the keystream words generated corresponding to the* P *array. For* $0 \le \alpha \ne \alpha' \le 511$ *and* $0 \le b \le 31$*, we have*

$$Prob\left([s_u \oplus s_l]^b = [s_{u'} \oplus s_{l'}]^b\right) = \frac{1}{2} + \frac{1}{2^{65+2b}},$$

where $u = Q[\alpha]^{(0)}$*,* $l = 256 + Q[\alpha]^{(2)}$*,* $u' = Q[\alpha']^{(0)}$ *and* $l' = 256 + Q[\alpha']^{(2)}$*, and* $u \ne u'$*,* $l \ne l'$*.*

Proof. Let us denote

$$F_{1,b}(\beta) = \left[h_2(\beta) \oplus P[u] \oplus P[l]\right]^b.$$

Further, we define

$$F_{2,b}(\beta, \beta') = F_{1,b}(\beta) \oplus F_{1,b}(\beta').$$

The event $([s_u \oplus s_l]^b = [s_{u'} \oplus s_{l'}]^b)$ is denoted by K. Therefore,

$$Prob(K) = Prob(K, F_{2,b}(\beta, \beta') = 0) + Prob(K, F_{2,b}(\beta, \beta') = 1). \tag{4}$$

The event that $F_{2,b}(\beta, \beta') = 0$ occurs following two mutually exclusive paths.

1. $F_{1,b}(\beta) = F_{1,b}(\beta') = 0$,

2. $F_{1,b}(\beta) = F_{1,b}(\beta') = 1$.

Given $F_{2,b}(\beta, \beta') = 0$, proceeding along the same lines as the proof of Lemma 2, one can check that the event K occurs with the probability $\frac{1}{2} + \frac{1}{2^{65}}$, i.e.,

$$Prob(K|F_{2,b}(\beta, \beta') = 0) = \frac{1}{2} + \frac{1}{2^{65}}.$$

Again, the event that $F_{2,b}(\beta, \beta') = 1$ occurs following two mutually exclusive ways.

1. $F_{1,b}(\beta) = 0, F_{1,b}(\beta') = 1$,
2. $F_{1,b}(\beta) = 1, F_{1,b}(\beta') = 0$.

Given $F_{2,b}(\beta, \beta') = 1$, in a similar manner to Lemma 2, one can check that the event K occurs with the probability $\frac{1}{2} - \frac{1}{2^{65}}$, i.e.,

$$Prob(K|F_{2,b}(\beta, \beta') = 1) = \frac{1}{2} - \frac{1}{2^{65}}.$$

Using Proposition 3,

$$\begin{aligned} Prob(F_{2,b}(\beta, \beta') = 0) &= (\frac{1}{2} + \frac{1}{2^{b+1}})^2 + (\frac{1}{2} - \frac{1}{2^{b+1}})^2 \text{ and} \\ Prob(F_{2,b}(\beta, \beta') = 1) &= (\frac{1}{2} + \frac{1}{2^{b+1}})(\frac{1}{2} - \frac{1}{2^{b+1}}) + (\frac{1}{2} + \frac{1}{2^{b+1}})(\frac{1}{2} - \frac{1}{2^{b+1}}) \\ &= 2(\frac{1}{2} + \frac{1}{2^{b+1}})(\frac{1}{2} - \frac{1}{2^{b+1}}). \end{aligned}$$

From Equation 4,

$$\begin{aligned} Prob(K) = &Prob(F_{2,b}(\beta, \beta') = 0) Prob(K|F_{2,b}(\beta, \beta') = 0) \\ &+ Prob(F_{2,b}(\beta, \beta') = 1) Prob(K|F_{2,b}(\beta, \beta') = 1) \\ = &\frac{1}{2} + \frac{1}{2^{65+2b}}. \end{aligned}$$

This completes the proof. □

The above result can be used to construct 32 many distinguishers, each based on a particular bit position of the keystream words. These distinguishers are characterized by Theorem 3 below, which is a generalized version of Theorems 1 and 2.

Theorem 3. *Let s_t, $0 \leq t \leq 511$, be the keystream words generated corresponding to either P or Q array. For $0 \leq u \neq u' \leq 255$, $256 \leq l \neq l' \leq 511$,*

$$Prob\left([s_u \oplus s_l]^b = [s_{u'} \oplus s_{l'}]^b\right) \approx \frac{1}{2} + \frac{1}{2^{78+2b}},$$

for $0 \leq b \leq 31$. Thus, if one concentrates on the b-th bit of each keystream word, HC-128 can be distinguished from an ideal random word generator by observing 2^{138+4b} keystream words with a success probability of 0.9772.

Proof. We present the proof for the keystream words corresponding to the P array. The situation is similar for the keystream words corresponding to the Q array.

Suppose, samples of quadruples (u, l, u', l') are randomly selected satisfying $0 \leq u \neq u' \leq 255$, $256 \leq l \neq l' \leq 511$. Let F be the event that $u = Q[\alpha]^{(0)}$, $l = 256 + Q[\alpha]^{(2)}$, $u' = Q[\alpha']^{(0)}$ and $l' = 256 + Q[\alpha']^{(2)}$ for some $\alpha \neq \alpha'$ in $[0, 511]$. Further, for $0 \leq b \leq 31$, let E_b be the event that the equality $\left([s_u \oplus s_l]^b = [s_{u'} \oplus s_{l'}]^b\right)$ holds for arbitrary u, u', l, l' satisfying $0 \leq u \neq u' \leq 255$, $256 \leq l \neq l' \leq 511$. As argued in the proof of Theorem 1, we have $Prob(F) = \frac{\binom{512}{2}}{\binom{256}{2}^2} \approx \frac{1}{2^{13}}$ and according to Lemma 3, we have $Prob(E_b|F) = \frac{1}{2} + \frac{1}{2^{65+2b}}$. As before, we may assume that $Prob(E_b|F^C) = \frac{1}{2}$. Thus, $Prob(E_b) \approx \frac{1}{2^{13}} \cdot (\frac{1}{2} + \frac{1}{2^{65+2b}}) + (1 - \frac{1}{2^{13}}) \cdot \frac{1}{2} = \frac{1}{2} + \frac{1}{2^{78+2b}} = p(1 + q_b)$, where $p = \frac{1}{2}$ and $q_b = \frac{1}{2^{77+2b}}$.

Similar to the analysis presented in the proof of Theorem 2, for a success probability of 0.9772 for the distinguisher based on the b-th bits of the keystream words, one would require $\frac{4^2}{pq_b^2} = 2^{159+4b}$ many samples, i.e., $2^{159+4b+9-30} = 2^{138+4b}$ many keystream words. □

Based on Theorem 3, we can find the following result.

Theorem 4. *Let s_t, $0 \leq t \leq 511$, be the keystream words generated corresponding to either P or Q array. Then the expected number of 0's in the bit pattern of $s_u \oplus s_l \oplus s_{u'} \oplus s_{l'}$ is $16 + \frac{1}{3}(\frac{1}{2^{76}} - \frac{1}{2^{140}})$, where $0 \leq u \neq u' \leq 255$, $256 \leq l \neq l' \leq 511$.*

Proof. Let $\psi = s_u \oplus s_l \oplus s_{u'} \oplus s_{l'}$. Let $m_b = 1$, if $[\psi]^b = 0$; otherwise, let $m_b = 0$, $0 \leq b \leq 31$. Hence, the total number of zeros in the bit pattern of ψ is given by $M = \sum_{b=0}^{31} m_b$. From Theorem 3, we have $Prob(m_b = 1) = \frac{1}{2} + \frac{1}{2^{78+2b}}$. Hence, $E(m_b) = \frac{1}{2} + \frac{1}{2^{78+2b}}$ and by linearity of expectation, $E(M) = \sum_{b=0}^{31} E(m_b)$ $= 16 + \frac{1}{3}(\frac{1}{2^{76}} - \frac{1}{2^{140}})$. □

Our result is sharper than [4, Theorem 4], where the expected number of 0's in the bit pattern of the XOR of 10 properly chosen different keystream words was $16 + \frac{13}{12} \cdot 2^{-79}$.

4 Conclusion

In this paper, we present the currently best known distinguisher for HC-128 that requires 2^{138} keystream words for a success probability of 0.9772. This distinguisher involves LSBs of four properly chosen keystream words. Further, it can be extended to any bit of the keystream words.

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