# Classification of Elliptic/hyperelliptic Curves with Weak Coverings against GHS Attack without Isogeny Condition

Tsutomu Iijima * Fumiyuki Momose † Jinhui Chao *

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**Abstract**

The GHS attack is known as a method to map the discrete logarithm problem(DLP) in the Jacobian of a curve $C_0$ defined over the $d$ degree extension $k_d$ of a finite field $k$ to the DLP in the Jacobian of a new curve $C$ over $k$. Recently, classification and density analysis were shown for all elliptic curves and hyperelliptic curves $C_0/k_d$ of genus 2, 3 which possess $(2,\ldots,2)$ covering $C/k$ of $\mathbb{P}^1$, therefore subjected to GHS attack, under the isogeny condition (i.e. when $g(C) = d \cdot g(C_0)$). In this paper, we first show a general classification procedure for the odd characteristic case. Our main approach is to use representation of the extension of $Gal(k_d/k)$ acting on $cov(C/\mathbb{P}^1)$. Then a classification of small genus hyperelliptic curves $C_0/k_d$ which possesses $(2,..,2)$ covering $C$ over $k$ is presented without the isogeny condition. Explicit defining equations of such curves $C_0/k_d$ and existential conditions of a model of $C$ over $k$ are also presented.

Keywords : Weil descent attack, GHS attack, Elliptic curve cryptosystems, Hyperelliptic curve cryptosystems, Index calculus, Galois representation

## 1 Introduction

Let $k_d := \mathbb{F}_{q^d}, k := \mathbb{F}_q$ $(d > 1)$, $q$ be a power of a prime number.

Weil descent was firstly introduced by Frey [7] to elliptic curve cryptosystems. This idea is developed into the well-known GHS attack in [11]. This attack maps the discrete logarithm problem (DLP) in the Jacobian of

*Department of Information and System Engineering, Chuo University, 1-13-27 Kasuga, Bunkyo-ku, Tokyo, 112-8551 Japan

†Department of Mathematics, Chuo University, 1-13-27 Kasuga, Bunkyo-ku, Tokyo, 112-8551 Japan

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a curve $C_0$ defined over the $d$ degree extension field $k_d$ of the finite field $k$ to the DLP in the Jacobian of a curve $C$ over $k$ by a conorm-norm map. The GHS attack is further extended and analyzed by many researchers and is conceptually generalized to the cover attack [5]. The cover attack maps the DLP in the Jacobian of a curve $C_0/k_d$ to the DLP in the Jacobian of a covering curve $C/k$ of $C_0$ when a covering map or a non-constant morphism between $C_0$ and $C$ exists.

If the DLP in the Jacobian of $C_0$ can be solved more efficiently in the Jacobian of $C$, we call $C_0$ a weak curve or say that it has weak covering $C$ against GHS or cover attack. Thus, it is important and interesting to know what kind of curves $C_0$ have such coverings $C$, how many are they, etc..

It is known that the most efficient attack to DLP in the Jacobian of algebraic curve based systems is the index calculus algorithms. In [9], Gaudry first proposed his variant of the Adleman-DeMarrais-Huang algorithm [1] to attack hyperelliptic curve discrete logarithm problems, which is faster than Pollard's rho algorithm when the genus is larger than 4 but becomes impractical for large genera. Recently, a single-large-prime variation [25] and a double-large-prime variation [12][21] are proposed. These variations can be applied in the GHS attack if the curve $C/k$ is a hyperelliptic curve of $g(C) \geq 3$. The complexity of these double-large-prime algorithms are $\tilde{O}(q^{2-2/g})$. On the other hand, when $C/k$ is a non-hyperelliptic curve, Diem's recent proposal of a double-large-prime variation [4] can be applied with complexity of $\tilde{O}(q^{2-2/(g-1)})$. Besides, Gaudry showed a general algorithm solving discrete logarithms on Abelian varieties of dimension $n'$ in running time $\tilde{O}(q^{2-2/n'})$ [10]. In particular, for elliptic curves over cubic extension field $k_3$, the running time is $\tilde{O}(q^{4/3})$.

Recently, security analyses of elliptic and hyperelliptic curves $C_0/k_d$ with weak covering $C/k$ were shown under the following isogeny condition [2][18][19][20][22][23]. Assume that there exists a covering curve $C/k$ of $C_0/k_d$ and

$$^{\exists}\pi/k_d : C \twoheadrightarrow C_0 \tag{1}$$

such that for

$$\pi_* \quad : \quad J(C) \twoheadrightarrow J(C_0), \tag{2}$$
$$Res(\pi_*) \quad : \quad J(C) \longrightarrow Res_{k_d/k}J(C_0) \tag{3}$$

defines an isogeny over $k$, here $J(C)$ is the Jacobian variety of $C$ and $Res_{k_d/k}J(C_0)$ is its Weil restriction. Then $g(C) = d \cdot g(C_0)$. Under this condition, the curves $C_0/k_d$ which possess covering curves $C/k$ as $(2,\ldots,2)$ covering of $\mathbb{P}^1$ are already classified for hyperelliptic curves of genus 1,2,3 [18][19][20]. Here, classification means to give a complete list of all such weak curves $C_0$. In particular, defining equations are presented for these curves. Density of the weak curves are also obtained for certain cases.

In this paper, we first discuss the existence of a model of $C$ over $k$. Then we show a general classification procedure for the odd characteristic case of hyperelliptic curves $C_0/k_d : y^2 = c \cdot f(x)$ of genus 1,2,3 with $(2, \ldots, 2)$ covering $C/k$. By applying this procedure, we obtain a classification of these weak curves $C_0$ without isogeny condition. Specifically, we assume that $g(C) = d \cdot g(C_0) + e, e > 0$. Here, $e$ is the dimension of $\ker(Res(\pi_*))$. Our approach for the classification is a representation theoretical one, to investigate action of the extension of $Gal(k_d/k)$ on $cov(C/\mathbb{P}^1)$. As a result, we obtain a complete list of defining equations of these weak curves $C_0/k_d$ for small values of $e$ which is corresponding to cryptographically meaningful classes of $C_0$. Furthermore, representation theoretical approach gives the condition for a model of $C$ over $k$ explicitly.

## 2 GHS attack and $(2, \ldots, 2)$ covering

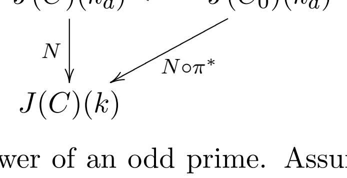
Firstly, we review briefly the GHS attack and the cover attack.

Let $k_d(C_0)$ be the function field of a curve $C_0/k_d$, $Cl^0(k_d(C_0))$ the class group of the degree 0 divisors of $k_d(C_0)$, $\sigma_{k_d/k}$ the Frobenius automorphism of $k_d$ over $k$, $x$ the transcendental element over $k_d$. Unless otherwise noted, we assume $\sigma_{k_d/k}$ is extended to an automorphism $\sigma$ of order $d$ in the separable closure of $k_d(x)$. It is showed by Diem [3] that $\sigma_{k_d/k}$ can extend an automorphism of the order $d$ when $d$ is odd for the odd characteristic case. We will extend the condition in the case of any $d > 1$ and the odd characteristic in the section 4. The Galois closure of $k_d(C_0)/k(x)$ is $\mathcal{F}' := k_d(C_0) \cdot \sigma(k_d(C_0)) \cdots \sigma^{d-1}(k_d(C_0))$ and the fixed field of $\mathcal{F}'$ by the automorphism $\sigma$ is $\mathcal{F} := \{\zeta \in \mathcal{F}' \mid \sigma(\zeta) = \zeta\}$. The DLP in $Cl^0(k_d(C_0)) \cong J(C_0)(k_d)$ is mapped to the DLP in $Cl^0(\mathcal{F}) \cong J(C)(k)$ using the following composition of conorm and norm maps:

$$N_{\mathcal{F}'/\mathcal{F}} \circ Con_{\mathcal{F}'/k_d(C_0)} : Cl^0(k_d(C_0)) \longrightarrow Cl^0(\mathcal{F}).$$

This map is called the conorm-norm homomorphism in the original GHS paper on the elliptic curve case [11].

This attack has been extended to wider classes of curves. The GHS attack is conceptually generalized to the cover attack by Frey and Diem [5]. When there exist an algebraic curve $C/k$ and a covering $\pi/k_d : C \longrightarrow C_0$, the DLP in $J(C_0)(k_d)$ can be mapped to the DLP in $J(C)(k)$ by a pullback-norm map.




Hereafter, let $q$ be a power of an odd prime. Assume $C_0$ is a hyperelliptic curve with $g(C_0) \in \{1,2,3\}$ given by

$$C_0/k_d : y^2 = c \cdot f(x) \tag{4}$$

where $c \in k_d^{\times}$, $f(x)$ is a monic polynomial in $k_d[x]$. Then assume that we have a tower of extensions of function fields such that $k_d(x, y, {}^{\sigma^1}y, \ldots, {}^{\sigma^{n-1}}y) \simeq k_d(C)\ /k_d(x)$ $(n \le d)$ is a $\overbrace{(2,\ldots,2)}^{n}$ type extension. Here, a $\overbrace{(2,\ldots,2)}^{n}$ covering is defined as a covering $\pi/k_d : C \longrightarrow \mathbb{P}^1$

$$\overbrace{C \longrightarrow \underbrace{C_0 \longrightarrow \mathbb{P}^1(x)}_{2}}^{\overbrace{(2,\ldots,2)}^{n}} \tag{5}$$

such that $cov(C/\mathbb{P}^1) \simeq \mathbb{F}_2^n$, here $cov(C/\mathbb{P}^1) := Gal(k_d(C)/k_d(x))$.

# 3 Representation of $Gal(k_d/k)$ on $cov(C/\mathbb{P}^1)$

Next, we consider the Galois group $Gal(k_d/k)$ acting on the covering group $cov(C/\mathbb{P}^1) \simeq \mathbb{F}_2^n$.

$$\begin{array}{ccc} Gal(k_d/k) \times cov(C/\mathbb{P}^1) & \longrightarrow & cov(C/\mathbb{P}^1) \\ (\sigma^i_{k_d/k}, \phi) & \longmapsto & {}^{\sigma^i}\phi := \sigma^i \phi \sigma^{-i} \end{array} \tag{6, 7}$$

Then one has a map onto $Aut(cov(C/\mathbb{P}^1))$.

$$\xi : Gal(k_d/k) \hookrightarrow Aut(cov(C/\mathbb{P}^1)) \simeq GL_n(\mathbb{F}_2) \tag{8}$$

Thus the representation of $\sigma$ for given $n, d$ is (we use the same notation for $\sigma$ and its representation):

$$\sigma = \begin{pmatrix} \boxed{\spadesuit_1} & O & \cdots & O \\ O & \boxed{\spadesuit_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & \boxed{\spadesuit_s} \end{pmatrix} \begin{matrix} \}n_1 \\ \\ \\ \}n_s \end{matrix} \tag{9}$$

is consisted of the diagonal blocks of matrices which are denoted by $\boxed{\spadesuit_i}$'s where $n = \sum_{i=1}^{s} n_i$ and the $O$ are zero matrices,

$$\boxed{\spadesuit_i} = \begin{pmatrix} \boxed{\star_i} & \boxed{\star_i} & \hat{O} & \cdots \\ \hat{O} & \boxed{\star_i} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \boxed{\star_i} \\ \hat{O} & \cdots & \hat{O} & \boxed{\star_i} \end{pmatrix} \begin{matrix} 1 \\ \vdots \\ \\ l_i \end{matrix} \tag{10}$$

is an $n_i \times n_i$ matrix which has a form of an $l_i \times l_i$ block matrix. The sub-blocks $\boxed{\star_i}$ are $n_i/l_i \times n_i/l_i$ matrices and $\hat{O}$'s are $n_i/l_i \times n_i/l_i$ zero matrices. Here, if $F_i(x)$ :=(the characteristic polynomial of $\boxed{\star_i}$)$^{l_i}$, then $F(x) := LCM\{F_i(x)\}$ is the minimal polynomial of $\sigma$. When $d_i$ :=ord($\boxed{\spadesuit_i}$), $d = LCM\{d_i\}$. The examples of the representation of $\sigma$ for given $n$ and $d$ are as follows:

**Example 3.1.** $n = 2, d = 2$

$$\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, F(x) = (x+1)^2 = x^2 + 1. \tag{11}$$

**Example 3.2.** $n = 2, d = 3$

$$\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, F(x) = x^2 + x + 1. \tag{12}$$

**Example 3.3.** $n = 3, d = 3$

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, F(x) = (x+1)(x^2+x+1) = x^3 + 1. \tag{13}$$

**Example 3.4.** $n = 3, d = 4$

$$\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, F(x) = (x+1)^3 = x^3 + x^2 + x + 1. \tag{14}$$

**Example 3.5.** $n = 4, d = 6$

$$\sigma = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \; or \; \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{15}$$

*Thus, $\sigma$ have the minimal polynomials as $F(x) = (x^2+x+1)^2$, or $F(x) = (x+1)^2(x^2+x+1)$.*

# 4 Existence of a model of $C$ over $k$

In this paper, we consider $C_0$ as a hyperelliptic curve over $k_d$ defined by $y^2 = c \cdot f(x)$ where $c \in k_d^\times$, $f(x)$ is a monic polynomial in $k_d[x]$. Denote by $F(x) \in \mathbb{F}_2[x]$ the minimal polynomial of $\sigma$. Here, we have the following

necessary and sufficient condition for given $n, d, \sigma$ :
$C$ has a model over $k_d \iff$

$$\begin{aligned} {}^{F(\sigma)}y^2 &= {}^{F(\sigma)}c \cdot {}^{F(\sigma)}f(x) = c^{F(q)} \cdot {}^{F(\sigma)}f(x) \equiv 1 \mod (k_d(x)^\times)^2, \\ {}^{G(\sigma)}y^2 &\not\equiv 1 \mod (k_d(x)^\times)^2 \text{ for}^\forall G(x) \mid F(x), G(x) \neq F(x). \end{aligned} \tag{16}$$

Firstly, we show conditions for existence of a model of $C$ over $k$. Now we know a model of $C$ over $k$ exists iff the extension $\sigma$ of the Frobenius automorphism $\sigma_{k_d/k}$ is an automorphism of $k_d(C)$ of order $d$ in the separable closure of $k_d(x)$. Diem showed in [3] that the Frobenius automorphism $\sigma_{k_d/k}$ on $k_d(x)$ is extended to an automorphism of $\mathcal{F}'/k_d(x)$ of order $d$ when $d$ is odd for the odd characteristic case. We extend the condition in the case of any $d > 1$. In the following lemma, we show explicitly the condition for $c$ in case of any $d > 1$. For the rest of this paper, define $\hat{F}(x) \in \mathbb{F}_2[x]$ as a polynomial such that $x^d + 1 = F(x)\hat{F}(x) \in \mathbb{F}_2[x]$.

**Lemma 4.1.** *In order that the curve $C$ has a model over $k$, when $\hat{F}(1) = 0$, $c$ needs to be a square $c \in (k_d^\times)^2$. When $\hat{F}(1) = 1$, there is a $\phi \in cov(C/\mathbb{P}^1)$ such that $\sigma\phi$ has order $d$ if $\sigma$ does not have order $d$, so we can adopt such $\sigma\phi$ instead of $\sigma$. Therefore $C$ always has a model over $k$ when $\hat{F}(1) = 1$.*

**Proof:** Let $M := \{\frac{b(x)}{a(x)} | k_d[x] \ni a(x), b(x) : \text{monic}\}$.
Since ${}^{F(\sigma)}f(x) \equiv 1 \mod (k_d(x)^\times)^2$, we have

$$ {}^{F(\sigma)}y^2 \equiv {}^{F(\sigma)}c = c^{F(q)} \mod (k_d(x)^\times)^2 \tag{17}$$

$$ {}^{F(\sigma)}y \equiv \epsilon c^{\frac{F(q)}{2}} \mod M, \qquad \text{here } \epsilon = \pm 1 \tag{18}$$

$$ {}^{\hat{F}(\sigma)F(\sigma)}y \equiv {}^{\hat{F}(\sigma)}\epsilon c^{\frac{\hat{F}(q)F(q)}{2}} \tag{19}$$

$$ {}^{\sigma^d+1}y \equiv \epsilon^{\hat{F}(1)} c^{\frac{q^d+1}{2}} \tag{20}$$

$$ {}^{\sigma^d}y \equiv \epsilon^{\hat{F}(1)} c^{\frac{q^d-1}{2}} y \tag{21}$$

We first consider two possibilities of $F(1) = 1$ and $F(1) = 0$ respectively.

- Case $F(1) = 1$ :
  We notice $\hat{F}(1) = 0$ in this case. Now, ${}^{\sigma^d}y \equiv c^{\frac{q^d-1}{2}} y$. In order that $\sigma$ has order $d$ (i.e. ${}^{\sigma^d}y \equiv y$), $c$ needs to be a square $c \in (k_d^\times)^2$.
- Case $F(1) = 0$ :
  Here, we consider further two possibilities of $\hat{F}(1) = 0$ and $\hat{F}(1) = 1$.
  (a) $\hat{F}(1) = 0$
  Similarly, ${}^{\sigma^d}y \equiv c^{\frac{q^d-1}{2}} y$. $c$ should be a square element in $k_d^\times$.
  (b) $\hat{F}(1) = 1$
  Then ${}^{\sigma^d}y \equiv \epsilon c^{\frac{q^d-1}{2}} y$.
  If $\epsilon = +1$ and $c \in (k_d^\times)^2$, then $\sigma$ has order $d$ (i.e. ${}^{\sigma^d}y = y$).

If $\epsilon = -1$ or $c \notin (k_d^{\times})^2$, then $\sigma$ has order $2d$.
However, we can show that $^{\exists}\phi \in cov(C/\mathbb{P}^1)$ such that $(\sigma\phi)^d = 1$.
Suppose $d = 2^r \cdot d'$ $(2 \nmid d')$. Since ${}^{\sigma}\phi := \sigma\phi\sigma^{-1}$, we have

$$\begin{aligned}(\sigma\phi)^d &= \sigma\phi\sigma^{-1}\cdot\sigma^2\phi\sigma^{-2}\cdots\sigma^d\phi\sigma^{-d}\cdot\sigma^d & (22)\\ &= {}^{\sigma}\phi\ {}^{\sigma^2}\phi\cdots{}^{\sigma^d}\phi\ \sigma^d & (23)\\ &= {}^{\sigma}\phi\ {}^{\sigma^2}\phi\cdots{}^{\sigma^{2^r d'}}\phi\ \sigma^d. & (24)\end{aligned}$$

Now, we choose $\phi := {}^t(\overbrace{0,0,\dots,1}^{m},0,\dots,0) \in cov(C/\mathbb{P}^1)$. Define

$$I \text{ as the identity matrix, } J := \left.\begin{pmatrix} 0 & 1 & & O \\ \vdots & \ddots & \ddots & \\ \vdots & O & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{pmatrix}\right\} m \le 2^r .$$

Then $J^m = O$. We notice that the representation of $\sigma$ is

$$\begin{pmatrix} \boxed{\spadesuit} & O \\ O & * \end{pmatrix} \text{ where } \boxed{\spadesuit} := I + J. \tag{25}$$

Here, ${}^{\sigma^i}\phi$ corresponds to $(I+J)^i \cdot {}^t(\overbrace{0,\dots 0,1}^{m})$. Now, since ${}^{\sigma^{2r}}\phi = \phi$, $(\sigma\phi)^d = (\phi\ {}^{\sigma}\phi\ {}^{\sigma^2}\phi\cdots{}^{\sigma^{2^r-1}}\phi)^{d'}\ \sigma^d$. Furthermore, since

$$I+(I+J)+\cdots+(I+J)^{2^r-1} = \begin{cases} O & \text{if } m < 2^r \\ \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix} & \text{if } m = 2^r, \end{cases} \tag{26}$$

where $O$ is the zero matrix, it follows that

$$\phi\ {}^{\sigma}\phi\ {}^{\sigma^2}\phi\cdots{}^{\sigma^{2^r-1}}\phi = \begin{cases} {}^t(0,0,\dots,0) & \text{if } m < 2^r \\ \psi := {}^t(1,0,\dots,0) & \text{if } m = 2^r. \end{cases} \tag{27}$$

On the other hand, $\sigma^d$ is an element in the center of $Gal(\mathcal{F}'/k(x))$, i.e., $\sigma^d \in Z(Gal(\mathcal{F}'/k(x))) = \{1,\psi\}$. When $\text{ord}(\sigma) = 2d$, $\sigma^d = \psi$. Furthermore, notice that $m = 2^r$ in the case of (b). Thus, in the multiplicative notation,

$$(\sigma\phi)^d = (\phi\ {}^{\sigma}\phi\ {}^{\sigma^2}\phi\cdots{}^{\sigma^{2^r-1}}\phi)^{d'}\ \sigma^d = \psi^{d'}\cdot\psi = 1 \tag{28}$$

As a result, we can adopt the above $\sigma\phi$ instead of $\sigma$.

□

**Example 4.1.** $n = 2, d = 2$
$x^2 + 1 = (x + 1)^2$, $F(x) = (x + 1)^2, \hat{F}(x) = 1$
*Since* $\hat{F}(x) = 1$, $\hat{F}(1) = 1$. *Therefore, c should be* 1 *or a non-square element in* $k_2$ *in order that the curve* $C$ *has a model over* $k$ *under the assumption* ${}^{F(\sigma)}f(x) \equiv 1 \mod (k_d(x)^\times)^2$.

**Example 4.2.** $n = 2, d = 3$
$x^3 + 1 = (x + 1)(x^2 + x + 1), F(x) = x^2 + x + 1, \hat{F}(x) = x + 1$
*Since* $\hat{F}(x) = x+1$, $\hat{F}(1) = 0$. *It follows that c is a square element* $c \in (k_3^\times)^2$ *(i.e.* $c = 1$*).*

**Example 4.3.** $n = 3, d = 3$
$x^3 + 1 = (x + 1)(x^2 + x + 1), F(x) = x^3 + 1, \hat{F}(x) = 1$
*Since* $\hat{F}(x) = 1$, $\hat{F}(1) = 1$. *Similarly, we obtain that c is* 1 *or a non-square element in* $k_3$.

**Example 4.4.** $n = 3, d = 4$
$x^4 + 1 = (x + 1)^4, F(x) = (x + 1)^3, \hat{F}(x) = x + 1$
*In this case,* $\hat{F}(1) = 0$. *Consequently,* $c \in (k_4^\times)^2$.

**Example 4.5.** $n = 4, d = 6$
$x^6 + 1 = (x + 1)^2(x^2 + x + 1)^2$

1. $F(x) = (x^2 + x + 1)^2, \hat{F}(x) = (x + 1)^2$
   *Now,* $\hat{F}(1) = 0$ *since* $\hat{F}(x) = x^2 + 1$. *Hence c is a square element* $c \in (k_6^\times)^2$.

2. $F(x) = (x + 1)^2(x^2 + x + 1), \hat{F}(x) = x^2 + x + 1$
   *Then,* $\hat{F}(1) = 1$. *As a result, c is* 1 *or a non-square element in* $k_6$.

# 5 Classification procedure of elliptic/hyperelliptic curves $C_0$ with weak coverings

From now, we show a procedure to classify all weak curves $C_0/k_d$ for given $n, d$. The procedure will output their defining equations and a complete list of such curves.

## 5.1 Ramification points of $C_0/\mathbb{P}^1$

Also for the rest of this paper, we assume the condition ${}^{F(\sigma)}f(x) \equiv 1 \mod (k_d(x)^\times)^2$. Recall that $\hat{F}(x) \in \mathbb{F}_2[x]$ is a polynomial such that $x^d + 1 = F(x)\hat{F}(x) \in \mathbb{F}_2[x]$. We will define the following notation as $b_i = 1$ when there exists a ramification point $(\alpha^{q^i}, 0)$ on $C_0$ and let $b_i = 0$ otherwise for $i = 0, \ldots, d - 1$. Here, $\alpha$ is either in $k_d$ ($\alpha \in k_d \setminus k_v, v \mid_{\neq} d$) or in certain extension of $k_d$ ($\alpha \in k_{d\tau} \backslash k_v, v \mid_{\neq} d\tau, {}^{\exists}\tau \in \mathbb{N}_{>1}$) if $f(x)$ contains all conjugate factors of $\alpha^{q^i}$ over $k_d$. Let $\Phi(x) := b_{d-1}x^{d-1} + \cdots + b_1 x + b_0$. Then $\Phi(x)$ defines a minimal Galois-invariant set of ramification points of $C_0/\mathbb{P}^1$ over $k_d$.

Since ${}^{F(\sigma)}f(x) \equiv 1 \mod (k_d(x)^{\times})^2$, $F(x)\Phi(x) \equiv 0 \mod (x^d+1)$. Then, $F(x)\Phi(x) \equiv 0 \mod (x^d + 1) \Leftrightarrow \Phi(x) \equiv 0 \mod \hat{F}(x)$. Therefore, it follows that $\Phi(x) \equiv a(x)\hat{F}(x) \mod (x^d + 1)$ for given $n, d$ ( ${}^{\exists}a(x) \in \mathbb{F}_2[x]$, $\deg a(x) < \deg F(x)$ ). Additionally, we can prove that $\hat{F}(x)\mathbb{F}_2[x]/(x^d+1) \cong \mathbb{F}_2[x]/(F(x))$. This suggests that we can know candidates of the ramification points of $C_0/\mathbb{P}^1$ if $a(x) \in \mathbb{F}_2[x]$ are determined for given $\hat{F}(x) \in \mathbb{F}_2[x]$. Hereafter, we assume that $\gcd(F(x), a(x)) = 1$ in order to treat $\Phi(x)$ corresponding to given $F(x)$. Next, we define the equivalence relation such that $(b_0, b_1, \ldots, b_{d-1}) \sim (b_j, \ldots, b_{d-1}, b_0, \ldots, b_{j-1})$ (i.e. the coefficients of $\Phi(x)$'s are cyclic permutation of each other), then corresponding $\Phi(x)$'s belong to the same class of $C_0$. Furthermore, $x^r a(x)\hat{F}(x) \equiv a(x)\hat{F}(x) \mod (x^d + 1)$ $\Leftrightarrow x^r + 1 \equiv 0 \mod F(x)$ for $1 \leq r \leq d$. Thus, we obtain that $r = d$. From these results, the number of the classes of $C_0$ is $N := \#\{(\mathbb{F}_2[x]/(F(x)))^{\times}\}/d$. This means that we obtain candidates of the ramification points of $C_0/\mathbb{P}^1$ if $N$ different $\Phi(x)$'s are found so that they are not cyclic permutation of each other for given $\hat{F}(x)$. From these facts, we show a procedure to derive candidates of the ramification points $\{(\alpha^{q^i}, 0) | b_i = 1\}$ on $C_0$ for given $n, d, \sigma$.

1. Choose a polynomial $a(x) = 1$, then $\Phi(x) := \hat{F}(x)$ gives ramification points $\{(\alpha^{q^i}, 0) | b_i = 1\}$ on $C_0$. If $N = 1$, then this procedure is completed. If $N \geq 2$, then repeat step 2 $\sim$ 4 until $N$ different $a(x)$'s are found so that the coefficients of $\Phi(x)$'s are not cyclic permutation of each other.

2. Choose another polynomial $a(x)$ such that $(a(x), F(x)) = 1$ and $\deg a(x) < \deg F(x)$ are satisfied. Next, define $\Phi(x) := a(x)\hat{F}(x)$.

3. Check whether all $\Phi(x)$'s are cyclic permutation of each other or not. If so, discard such $a(x)$. Go to step 2 again. If they are not cyclic permutation of each others, we add $\{(\alpha^{q^i}, 0) | b_i = 1\}$ defined by $\Phi(x)$ to the candidates.

4. Check whether $N$ different $a(x)$'s are found. If yes, then this procedure is completed. Otherwise, return to step 2.

**Example 5.1.** $n = 2, d = 2$
$x^2 + 1 = (x + 1)^2$, $F(x) = (x + 1)^2, \hat{F}(x) = 1$
*Now, $N = 1$. Choose $a(x) = 1$, then $\Phi(x) = a(x)\hat{F}(x) = 1$. Thus, there exists a ramification point $(\alpha, 0)$ on $C_0$ as a candidate.*

**Example 5.2.** $n = 2, d = 3$
$x^3 + 1 = (x + 1)(x^2 + x + 1), F(x) = x^2 + x + 1, \hat{F}(x) = x + 1$

*Similarly,* $N = 1$*. Choose* $a(x) = 1$*, then* $\Phi(x) = x + 1$*.* $C_0$ *has ramification points* $\{(\alpha, 0), (\alpha^q, 0)\}$ *on* $C_0$*.*

**Example 5.3.** $n = 3, d = 3$
$x^3 + 1 = (x + 1)(x^2 + x + 1), F(x) = x^3 + 1, \hat{F}(x) = 1, N = 1$
*Choose* $a(x) = 1$*, then* $\Phi(x) = 1$*. Consequently,* $C_0$ *has a ramification point* $(\alpha, 0)$ *on* $C_0$*.*

**Example 5.4.** $n = 3, d = 4$
$x^4 + 1 = (x + 1)^4, F(x) = (x + 1)^3, \hat{F}(x) = x + 1, N = 1$
*Choose* $a(x) = 1$*, then* $\Phi(x) = x+1$*.* $C_0$ *has ramification points* $\{(\alpha, 0), (\alpha^q, 0)\}$ *on* $C_0$*.*

**Example 5.5.** $n = 4, d = 6$
$x^6 + 1 = (x + 1)^2(x^2 + x + 1)^2$

1. $F(x) = (x^2 + x + 1)^2, \hat{F}(x) = (x + 1)^2, N = 2$
   *Now, choose* $a(x) = 1$ *and* $a(x) = x + 1$*, then* $\Phi(x) = x^2 + 1$ *and* $\Phi(x) = x^3 + x^2 + x + 1$*. In these cases,* $C_0$ *has ramification points* $\{(\alpha, 0), (\alpha^{q^2}, 0)\}$ *or* $\{(\alpha, 0), (\alpha^q, 0), (\alpha^{q^2}, 0), (\alpha^{q^3}, 0)\}$ *as candidates.*

2. $F(x) = (x + 1)^2(x^2 + x + 1), \hat{F}(x) = x^2 + x + 1, N = 1$
   *Now, choose* $a(x) = 1$*, then* $\Phi(x) = x^2 + x + 1$*. In the case,* $C_0$ *has ramification points* $\{(\gamma, 0), (\gamma^q, 0), (\gamma^{q^2}, 0)\}$*.*

### 5.2 Defining equations of $C_0$

Now, we are considering the case of a hyperelliptic curve $C_0/k_d$ for $g(C_0) \in \{1, 2, 3\}$ such that there is a covering $\pi/k_d : C \longrightarrow C_0$ and the covering curve $C/k$ has genus $g(C) = d \cdot g(C_0) + e$ (Notice that the procedure in the section 5 and Lemma 4.1 are applicable to any $e \geq 0$). Let $S$ be the number of fixed points of $C/\mathbb{P}^1$ covering. By the Riemann-Hurwitz theorem, $2g - 2 = 2^n(-2) + 2^{n-1}S$, then $S = 4 + \frac{dg_0+e-1}{2^{n-2}}$. Hereafter, we consider the following two types:

- Type (A) : ${}^{\exists}d_i$ s.t. $d_i = d$ $(= LCM\{d_i\})$
  then, $S = 4 + \frac{dg_0+e-1}{2^{n-2}} \geq \max\{d, 2g_0 + 3\}$
- Type (B) : $d_i \neq d$ for ${}^{\forall}d_i$
  then, $S = 4 + \frac{dg_0+e-1}{2^{n-2}} \geq \max\{q(d), 2g_0 + 4\}$

here $q(d) := \sum p_i^{e_i}$ for $d = \prod p_i^{e_i}$ ($p_i$'s are distinct prime numbers). Notice that a Type(B) matrix has Type(A) matrices as subrepresentations.

**Remark 5.1.** *See Example 3.5 again. We notice the left and right matrices are a Type (A) and a Type (B) respectively. Notice for the two cases:*

$$\begin{aligned} Type(A) : \sigma &= \begin{pmatrix} \boxed{\star_1} & \boxed{\star_1} \\ O & \boxed{\star_1} \end{pmatrix}, \boxed{\star_1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ Type(B) : \sigma &= \begin{pmatrix} \boxed{\spadesuit_1} & O \\ O & \boxed{\spadesuit_2} \end{pmatrix}, \boxed{\spadesuit_1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \boxed{\spadesuit_2} = \boxed{\star_2} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

The procedure in the section 5.1 showed us how to drive the candidates of the ramification points $\{(\alpha^{q^i}, 0)\}$ on $C_0$ ($\alpha \in k_d \setminus k_v, v \mid_{\neq} d$ or $\alpha \in k_{d\tau} \setminus k_v, v \mid_{\neq} d\tau, \tau \in \mathbb{N}_{>1}$ if $f(x)$ contains all conjugate factors of $\alpha^{q^i}$ over $k_d$). Below, we show main steps to find the defining equations for every weak curve $C_0$.

1. Calculate the number of fixed points of $C/\mathbb{P}^1$ covering $S = 4 + \frac{dg_0+e-1}{2^{n-2}}$ for given $n, d, g_0, e$ using Riemann-Hurwitz formula and test if $\sigma$ is Type (A) or (B).

2. If Type (A) case and $\sigma$ : irreducible, go to step 3 with ramification points $\{(\alpha^{q^i}, 0) | b_i = 1\}$ on $C_0$ obtained by using the procedure in the section 5.1.
   Otherwise, go to step 3 with ramification points obtained from all subrepresentations of $\sigma$ except the trivial representation (1) by the procedure in the section 5.1. Notice that the Type (B) matrix has Type (A) matrices as sub-blocks. Therefore, we can reuse the results for Type (A) in the section 5.1.

3. Find $f(x)$ to try all combinations of polynomials which contain all conjugate factors of $x - \alpha^{q^i}$ for each ramification point and have the right degree of genus $g_0$.

The above operations are explained in further details in the following examples.

**Example 5.6.** $n = 2, d = 2, g_0 = 2, e = 1, g = 5$ *(Type A)*
*In this case, we can know that* $f(x)$ *has a factor* $x - \alpha_i$ *as in Example 5.1* ($\alpha_i \in k_2 \setminus k$ *or* $\alpha_i \in k_{2\tau} \setminus k_v, v \mid_{\neq} 2\tau, \tau \in \mathbb{N}_{>1}$ *if* $f(x)$ *contains all conjugate factors of* $\alpha_i$ *over* $k_2$). *Since* $S = 4 + (d \cdot g_0 + e - 1)/2^{n-2} = 8$, *we have the following two forms as candidates of* $C_0/k_2$:
*(a)* $S = \#\{\alpha_1, \alpha_1^q\} + \#\{\alpha_2, \alpha_2^q\} + 4 = 2 + 2 + 4$
  $C_0/k_2 : \ y^2 = (x - \alpha_1)(x - \alpha_2)h_1(x)$.
*(b)* $S = \#\{\alpha_1, \alpha_1^q\} + \#\{\alpha_2, \alpha_2^q\} + \#\{\alpha_3, \alpha_3^q\} + \#\{\alpha_4, \alpha_4^q\} = 2 + 2 + 2 + 2$
  $C_0/k_2 : y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)$.
*Here,* $h_1(x) \in k[x]$, $\deg h_1(x) \in \{4, 3\}$, $\prod(x - \alpha_i) \in k_2[x] \setminus k[x]$. *As* $g_0 = 2$ *in this case, (a) should be chosen from two forms. We remark the ramification*

*points are* $\alpha_1, \alpha_2 \in k_2 \setminus k$ *or* $\alpha_1 \in k_4 \setminus k_2$, $\alpha_2 := \alpha_1^{q^2}$ *in consideration of conjugate factors of* $\alpha_1$ *over* $k_2$. *Recall Example 4.1.* $\hat{F}(1) = 1$ *since* $\hat{F}(x) = 1$. *Let* $\eta$ *be* 1 *or a non-square element in* $k_2$. *As a result, we obtain* $C_0/k_2 : y^2 = \eta(x - \alpha_1)(x - \alpha_2)h_1(x)$. *Now,* $g = d \cdot g_0 + e = 5$. *Roughly, the attacking costs on* $J(C/k)$ *is lower than on* $J(C/k_d)$ *as follows:*

| $C_0/k_d$ : | $C/k$ : *hyper* | $C/k$ : *non-hyper* |
|---|---|---|
| $\tilde{O}(q^{\frac{d \cdot g_0}{2}}) = \tilde{O}(q^2)$ | $\tilde{O}(q^{2-\frac{2}{d \cdot g_0 + e}}) = \tilde{O}(q^{8/5})$ | $\tilde{O}(q^{2-\frac{2}{d \cdot g_0 + e - 1}}) = \tilde{O}(q^{3/2})$ |

**Example 5.7.** $n = 3, d = 3, g_0 = 2, e = 3, g = 9$ *(Type A)*
*Now,* $f(x)$ *has a factor* $x - \alpha$. *Additionally, consider also* $(n, d) = (2, 3)$ *(i.e.* $(x - \alpha_i)(x - \alpha_i^q)|f(x)$*). Since* $S = 4 + (d \cdot g_0 + e - 1)/2^{n-2} = 8$, *there exist two cases as follows:*
*(1)* $S = 3 + 5$
$C_0/k_3 : y^2 = \eta(x - \alpha)h_1(x)$
*Here,* $\alpha \in k_3 \setminus k, h_1(x) \in k[x]$, $\deg h_1(x) \in \{5, 4\}$.
*(2)* $S = 3 + 3 + 2$
$C_0/k_3 : y^2 = \eta(x - \alpha_1)(x - \alpha_1^q)(x - \alpha_2)(x - \alpha_2^q)h_1(x)$
*Here,* $(x-\alpha_1)(x-\alpha_1^q)(x-\alpha_2)(x-\alpha_2^q) \in k_3[x] \setminus k[x], h_1(x) \in k[x]$, $\deg h_1(x) \in \{2, 1\}$. *The ramification points are* $\alpha_1, \alpha_2 \in k_3 \setminus k$ *or* $\alpha_1 \in k_6 \setminus (k_2 \cup k_3)$, $\alpha_2 := \alpha_1^{q^3}$. *Remark that* $\eta := 1$ *or a non-square element in* $k_3$. *The rough estimation of the attacking costs between* $J(C_0/k_d)$ *and* $J(C/k)$ *is as follows:*

| $C_0/k_d$ : | $C/k$ : *hyper* | $C/k$ : *non-hyper* |
|---|---|---|
| $\tilde{O}(q^{\frac{d \cdot g_0}{2}}) = \tilde{O}(q^3)$ | $\tilde{O}(q^{2-\frac{2}{d \cdot g_0 + e}}) = \tilde{O}(q^{16/9})$ | $\tilde{O}(q^{2-\frac{2}{d \cdot g_0 + e - 1}}) = \tilde{O}(q^{7/4})$ |

**Example 5.8.** $n = 3, d = 3, g_0 = 2, e = 1, g = 7$ *(Type A)*
*Similarly, we consider factors* $x-\alpha$ *and* $(x-\alpha)(x-\alpha^q)$. *Now* $S = 4+(d \cdot g_0 + e-1)/2^{n-2} = 7$. *Consequently, we obtain* $C_0/k_3 : y^2 = \eta(x-\alpha)(x-\alpha^q)h_1(x)$ *when* $S = 3 + 4$. *Here,* $\alpha \in k_3 \setminus k, h_1(x) \in k[x]$, $\deg h_1(x) \in \{4, 3\}$, $\eta := 1$ *or a non-square element in* $k_3$. *The rough estimation between the attacking costs is as follows:*

| $C_0/k_d$ : | $C/k$ : *hyper* | $C/k$ : *non-hyper* |
|---|---|---|
| $\tilde{O}(q^{\frac{d \cdot g_0}{2}}) = \tilde{O}(q^3)$ | $\tilde{O}(q^{2-\frac{2}{d \cdot g_0 + e}}) = \tilde{O}(q^{12/7})$ | $\tilde{O}(q^{2-\frac{2}{d \cdot g_0 + e - 1}}) = \tilde{O}(q^{5/3})$ |

**Example 5.9.** $n = 3, d = 4, g_0 = 2, e = 1, g = 9$ *(Type A)*
*Recall that* $(x - \alpha)(x - \alpha^q)|f(x)$ *(*$\alpha \in k_4 \setminus k_2$ *or* $\alpha \in k_{4\tau} \setminus k_v, v \not| 4\tau, \tau \in \mathbb{N}_{>1}$ *if* $f(x)$ *contains all conjugate factors of* $\alpha^{q^i}$ *over* $k_4$*) when* $(n, d) = (3, 4)$, *and* $(x - \beta)|f(x)$ *(*$\beta \in k_2 \setminus k$ *or* $\beta \in k_{2\tau} \setminus k_v, v \not| 2\tau, \tau \in \mathbb{N}_{>1}$ *if* $f(x)$ *contains all conjugate factors of* $\beta$ *over* $k_2$*) when* $(n, d) = (2, 2)$. *Then,* $S = 4 + (d \cdot g_0 + e - 1)/2^{n-2} = 8$. *Since* $g_0 = 2$, *we obtain* $C_0/k_4 : y^2 = (x - \alpha)(x - \alpha^q)h_1(x)$ *when* $S = 4 + 4$. *Here,* $\alpha \in k_4 \setminus k_2, h_1(x) \in k[x]$, $\deg h_1(x) \in \{4, 3\}$. *The comparison similar to the above examples is as follows:*

| $C_0/k_d$ : | $C/k$ : *hyper* | $C/k$ : *non-hyper* |
|---|---|---|
| $\tilde{O}(q^{\frac{d\cdot g_0}{2}}) = \tilde{O}(q^4)$ | $\tilde{O}(q^{2-\frac{2}{d\cdot g_0+e}}) = \tilde{O}(q^{16/9})$ | $\tilde{O}(q^{2-\frac{2}{d\cdot g_0+e-1}}) = \tilde{O}(q^{7/4})$ |

**Example 5.10.** $n = 4, d = 6, g_0 = 1, e = 3, g = 9$ *(Type A)*
*In this case, consider the combination of* $(x-\alpha)(x-\alpha^{q^2})|f(x)$ *and* $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})(x-\alpha^{q^3})|f(x)$. *Now,* $S = 4 + (d \cdot g_0 + e - 1)/2^{n-2} = 6$. *Since* $g_0 = 1$, *we obtain* $C_0/k_6 : y^2 = (x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})(x-\alpha^{q^3})$ $(\alpha \in k_6 \setminus (k_3 \cup k_2))$ *when* $S = 6+0$. *The comparison between attacking costs is :*

| $C_0/k_d$ : | $C/k$ : *hyper* | $C/k$ : *non-hyper* |
|---|---|---|
| $\tilde{O}(q^{\frac{d\cdot g_0}{2}}) = \tilde{O}(q^3)$ | $\tilde{O}(q^{2-\frac{2}{d\cdot g_0+e}}) = \tilde{O}(q^{16/9})$ | $\tilde{O}(q^{2-\frac{2}{d\cdot g_0+e-1}}) = \tilde{O}(q^{7/4})$ |

**Example 5.11.** $n = 4, d = 6, g_0 = 1, e = 3, g = 9$ *(Type B)*
*We know* $(x-\gamma)(x-\gamma^q)(x-\gamma^{q^2})|f(x)$ *as in Example 5.5. Next, consider all proper subrepresentations of* $\sigma$ *except the trivial representation* $(1)$. *Derive candidates of the ramification points for* $(n, d) = (3, 3), (2, 3), (2, 2)$. *From the results of Example 5.3, 5.2 and 5.1, they have been already obtained :* $(x-\alpha)|f(x)$, $(x-\alpha)(x-\alpha^q)|f(x)$ *and* $(x-\beta)|f(x)$ *(Here,* $\alpha \in k_3 \setminus k$ *or* $\alpha \in k_{3\tau} \setminus k_v, v \mid_{\neq} 3\tau, \tau \in \mathbb{N}_{>1}$, *and* $\beta \in k_2 \setminus k$ *or* $\beta \in k_{2\tau} \setminus k_v, v \mid_{\neq} 2\tau, \tau \in \mathbb{N}_{>1}$*respectively). Finally, find* $f(x)$ *to try all combinations of polynomials which contain all conjugate factors of the aboves to consider that* $C_0/k_6$ *have* $g_0 = 1$ *and* $S = 6$. *In this case, it follows that* $C_0/k_6$ *has the form* $y^2 = \eta(x-\alpha)(x-\alpha^q)(x-\beta)h_1(x)$ *when* $S = 3+2+1$. *Here,* $\alpha \in k_3 \setminus k$, $\beta \in k_2 \setminus k$, $h_1(x) \in k[x]$, $\deg h_1(x) \in \{1, 0\}$, $\eta := 1$ *or a non-square element in* $k_6$. *The comparison between attacking costs is the same as Example 5.10.*

See the lists in Appendices for other defining equations $C_0/k_d$.

# 6 Classification of elliptic/hyperelliptic curves $C_0$ without isogeny condition

Here, we apply the procedure in section 5 to classify $C_0/k_d$. In particular, we consider cases of a hyperelliptic curve $C_0/k_d$ for $g(C_0) \in \{1, 2, 3\}$ such that there is a covering $\pi/k_d : C \longrightarrow C_0$ and the covering curve $C/k$ has genus $g(C) = d \cdot g(C_0) + e$ $(e > 0)$.

## 6.1 Upper bound of $e$ in $g(C) = dg(C_0) + e$ $(e > 0)$

Firstly, since $C_0$ are used in the cryptographic applications, we need to restrict $C_0$ to a practically meaningful class. Thus we will tentatively estimate an upper bound of $e$ for $g(C_0) \in \{1, 2, 3\}$. In algebraic curve based cryptosystems, the standard key length is above 160 bits at present. This means the size of the Jacobian of $C_0/k_d$ is

$$q^{g(C_0)d} \geq 2^{160}. \tag{29}$$

Next, we assume that the size of Jacobian of $C/k$ is $q^{dg_0+e} \leq 2^a$.

**Remark 6.1.** *Hereafter, we discuss within* $a \leq 320$*. Meanwhile, Lemma 4.1 and the procedures in the previous section can apply to any* $e \geq 0$ *and* $q^{dg_0+e} > 2^{320}$*. The treatment of these cases will be reported in the near future.*

### 6.1.1 Case $g(C_0) = 1$

Then, we have the following situation for $g_0 = 1$

$$\begin{cases} q^{d+e} \leq 2^a \\ 2^{160} \leq q^d. \end{cases} \tag{30}$$

Now, since $\frac{q^{d+e}}{q^d} \leq \frac{2^a}{2^{160}}$, $q^e \leq 2^{a-160}$. Consequently,

$$\log q^e \leq \log 2^{a-160}.$$

It follows that an upper bound of $e$ is

$$e \leq \frac{(a-160)d}{160}. \tag{31}$$

When we assume $a \leq 320$, $e \leq d$ is obtained.

### 6.1.2 Case $g(C_0) = 2, 3$

Similarly, when $g(C_0) = 2$, assume that

$$\begin{cases} q^{2d+e} \leq 2^a \\ 2^{160} \leq q^{2d}. \end{cases} \tag{32}$$

Then $e \leq 2d$ if $a \leq 320$. When $g(C_0) = 3$, the double-large-prime algorithms have the cost of $\tilde{O}(q^{\frac{4}{3}d})$. Accordingly, the condition $q^{3d} \geq 2^{180}$ (i.e. $q^{\frac{4}{3}d} \geq 2^{80}$) should be adopted instead of $q^{3d} \geq 2^{160}$ ($q^{\frac{4}{3}d} \geq 2^{71.11\cdots}$) to keep the same security level with $g_0 = 1, 2$ hyperelliptic curves (the costs of attack to each DLP are $q^{\frac{d}{2}} \geq 2^{80}$ for $g_0 = 1$, $q^d \geq 2^{80}$ for $g_0 = 2$ as a key length of more than $2^{160}$ respectively). Thus, one can assume

$$\begin{cases} q^{3d+e} \leq 2^a \\ 2^{180} \leq q^{3d}. \end{cases} \tag{33}$$

Consequently, $e \leq \frac{7}{3}d$ if $a \leq 320$. In the next subsection, we enumerate the candidates of $n, d, e, S$ within these bounds of $e$ for $g(C_0) = 1, 2, 3$.

### 6.2 The candidates of $(n, d, e, S)$

#### 6.2.1 Type (A)

• Case $g_0 = 1$:
From the above, $d + e - 1 \geq 2^{n-2}d - 2^n$ when $g_0 = 1$. Since we assume $0 < e \leq d$, $2d-1 \geq d+e-1 \geq 2^{n-2}d-2^n$. Then $2^n-1 \geq (2^{n-2}-2)d$ $(n \geq 3)$. Now, if $n > 3$,

$$(n \leq)\ d \leq 4 + \frac{7}{2^{n-2} - 2}. \tag{34}$$

Consequently, it follows that $n \geq 6$ is not within the candidates. From this result and the property of $\sigma$, the candidates of 4-triple $(n, d, e, S)$ are: $(5,5,4,5), (4,4,1,5), (4,5,4,6), (4,6,3,6), (4,7,6,7), (3,3,2,6), (3,4,1,6), (3,4,3,7), (3,7,2,8), (3,7,4,9), (3,7,6,10), (2,2,1,6), (2,2,2,7), (2,3,1,7), (2,3,2,8), (2,3,3,9)$.

• Case $g_0 = 2$:
Similarly, when $g_0 = 2$, since we assume $0 < e \leq 2d$, $4d - 1 \geq 2d + e - 1 \geq 2^{n-2}d - 2^n$. Then, if $n > 4$,

$$(n \leq)\ d \leq 4 + \frac{15}{2^{n-2} - 4}. \tag{35}$$

Thus the candidates of $(n, d, e, S)$ are: $(4,4,5,7), (4,5,3,7), (4,5,7,8), (4,6,1,7), (4,6,5,8), (4,6,9,9), (4,7,3,8), (4,7,7,9), (4,7,11,10), (4,15,15,15), (4,15,19,16), (4,15,23,17), (4,15,27,18), (3,3,1,7), (3,3,3,8), (3,3,5,9), (3,4,1,8), (3,4,3,9), (3,4,5,10), (3,4,7,11), (3,7,1,11), (3,7,3,12), (3,7,5,13), (3,7,7,14), (3,7,9,15), (3,7,11,16), (3,7,13,17), (2,2,1,8), (2,2,2,9), (2,2,3,10), (2,2,4,11), (2,3,1,10), (2,3,2,11), (2,3,3,12), (2,3,4,13), (2,3,5,14), (2,3,6,15)$.

• Case $g_0 = 3$:
Next, if $g_0 = 3$ $(0 < e \leq \frac{7}{3}d)$, then

$$(5 \leq n \leq)\ d \leq 4 + \frac{61}{3(2^{n-2} - \frac{16}{3})}. \tag{36}$$

Hence possible $(n, d, e, S)$ are: $(5,8,17,9), (4,4,9,9), (4,5,6,9), (4,5,10,10), (4,6,3,9), (4,6,7,10), (4,6,11,11), (4,7,4,10), (4,7,8,11), (4,7,12,12), (4,7,16,13), (4,15,4,16), (4,15,8,17), (4,15,12,18), (4,15,16,19), (4,15,20,20), (4,15,24,21), (4,15,28,22), (4,15,32,23), (3,3,2,9), (3,3,4,10), (3,3,6,11), (3,4,1,10), (3,4,3,11), (3,4,5,12), (3,4,7,13), (3,4,9,14), (3,7,2,15), (3,7,4,16), (3,7,6,17), (3,7,8,18), (3,7,10,19), (3,7,12,20), (3,7,14,21), (3,7,16,22), (2,2,1,10), (2,2,2,11), (2,2,3,12), (2,2,4,13), (2,3,1,13), (2,3,2,14), (2,3,3,15), (2,3,4,16), (2,3,5,17), (2,3,6,18), (2,3,7,19)$.

#### 6.2.2 Type (B)

• Case $2 \nmid d$:
Now, $d = LCM\{d_i\} \leq \prod d_i \leq \prod(2^{n_i} - 1) < 2^n$. ($d_i$ is the order of $\boxed{\spadesuit_i}$ in

(9)). Here, if $g_0 = 1$ $(0 < e \leq d)$, then

$$d + e - 1 \leq 2d - 1 < 2^{n+1}. \tag{37}$$

On the other hand, it follows that

$$d + e - 1 \geq 2^{n-2}(q(d) - 4) \tag{38}$$

since $S = 4 + \frac{d+e-1}{2^{n-2}} \geq q(d)$. From (37)(38), one obtains

$$2^{n+1} > 2^{n-2}(q(d) - 4). \tag{39}$$

Consequently, $12 > q(d)$. Besides, we have $20 > q(d)$ for $g_0 = 2$ $(0 < e \leq 2d)$ since $2^{n-2}(q(d) - 4) \leq 2d + e - 1 < 2^{n+2}$. By the similar manner, $26 > q(d)$ when $g_0 = 3$ $(0 < e \leq \frac{7}{3}d)$.

• Case $2 \mid d$:

In this case, $n_i = l_i m_i, d_i = 2^{r_i} d_i'$ $(2 \nmid d_i')$, then $d_i' \mid 2^{m_i} - 1$. Let $r := \max\{r_i\}$. Here, we obtain $2^{r_i - 1} + 1 \leq l_i \leq 2^{r_i}$ for $r_i \geq 1$. Accordingly, $2^{r-1} + 1 \leq l_1 \leq 2^r$ when we assume $l_1$ with $r_1 \geq 1$. Now, notice that

$$\boxed{\spadesuit_i} = \begin{pmatrix} \boxed{\star_i} & \boxed{\star_i} & \hat{O} & \cdots \\ \hat{O} & \boxed{\star_i} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \boxed{\star_i} \\ \hat{O} & \cdots & \hat{O} & \boxed{\star_i} \end{pmatrix} \begin{matrix} 1 \\ \\ \\ l_i \end{matrix} \qquad (\boxed{\star_i})\ \}m_i. \tag{40}$$

Then

$$\begin{aligned} d = LCM\{2^{r_i} d_i'\} = 2^r \cdot LCM\{d_i'\} &\leq 2^r \cdot \prod d_i' && (41) \\ &\leq 2^r \cdot \prod (2^{m_i} - 1) && (42) \\ &< \begin{cases} 2^{r + \sum_{i \geq 1} m_i} & (m_1 \geq 2) \\ 2^{r + \sum_{i \geq 2} m_i} & (m_1 = 1). \end{cases} && (43) \end{aligned}$$

On the other hand, we know

$$dg_0 + e - 1 \geq 2^{n-2}(q(d) - 4). \tag{44}$$

Hence, if $g_0 = 1$ $(0 < e \leq d)$, then

$$2d - 1 \geq 2^{n-2}(q(d) - 4). \tag{45}$$

From (43) (45), we obtain

$$\begin{aligned} 2^{r + (\sum_{i \geq 1} m_i) + 1} &> 2^{n-2}(q(d) - 4) && (46) \\ 2^{3 + r + (\sum_{i \geq 1} m_i) - n} &> q(d) - 4 && (47) \\ 2^{3 + r - 2^{r-1} m_1} &> q(d) - 4 && (48) \end{aligned}$$

for $m_1 \geq 2$. Similarly, $2^{3+r-2^{r-1}-1} > q(d) - 4$ for $m_1 = 1$. Therefore, we obtain $8 > q(d)$. In the same way, we have $12 > q(d)$ and $15 > q(d)$ for $g_0 = 2$ and $g_0 = 3$ respectively.

From these upper bounds and the property of $\sigma$, we obtain a list of possible $(g_0, n, d, e, S)$ :
$(1,4,6,3,6), (2,5,12,9,8), (2,5,12,17,9), (2,5,14,13,9), (2,5,14,21,10),$
$(2,5,21,7,10), (2,5,21,15,11), (2,5,21,23,12), (2,5,21,31,13), (2,5,21,39,14),$
$(2,4,6,5,8), (2,4,6,9,9), (3,6,21,34,10), (3,6,28,29,11), (3,6,28,45,12),$
$(3,6,28,61,13), (3,5,21,2,12), (3,5,21,10,13), (3,5,21,18,14), (3,5,21,26,15),$
$(3,5,21,34,16), (3,5,21,42,17), (3,5,14,7,10), (3,5,14,15,11), (3,5,14,23,12),$
$(3,5,14,31,13), (3,5,12,13,10), (3,5,12,21,11), (3,4,6,7,10), (3,4,6,11,11).$

Within the above lists, we constructed explicitly all classes of hyperelliptic curves $C_0/k_d$ for $g(C_0) \in \{1,2,3\}$ such that there is a covering $\pi/k_d : C \longrightarrow C_0$ and the covering curve $C/k$ has genus $g(C) = d \cdot g(C_0) + e$ $(e > 0)$. Lists for all defining equation $C_0/k_d$ are shown in the Appendices.

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# Appendices

## A Classification for Type (A) : ${}^{\exists}d_i = d$

Let $h_1(x) \in k[x]$, $h_d(x) \in k_d[x] \setminus k_u[x]$ $(u \mid_{\neq} d)$, $\eta := 1$ or a non-square element in $k_d$, $\alpha, \gamma \in k_d \setminus k_v$ $(v \mid_{\neq} d)$, $\alpha_i \in k_{\tau_i} \setminus k_{w_i}$ $(w_i \mid_{\neq} \tau_i)$. Here, choose $\alpha_i$ and $\tau_i \in \{d, 2d, \cdots, max\{i\}d\}$ such that $h_d(x) \in k_d[x] \setminus k_u[x]$ $(u \mid_{\neq} d)$. Refer to Example 5.6 $(n, d, g_0, e, S) = (2, 2, 2, 1, 8)$ as an example of how to choose $\alpha_i$ and $\tau_i$. Let $C_0/k_d : y^2 = c \cdot h_d(x)h_1(x)$.

| $(n,d,g_0,e,S)$ | $c$ | $h_d(x)$ | $\deg h_1(x)$ |
|---|---|---|---|
| $(4,4,1,1,5)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})$ | $1,0$ |
| $(4,4,2,5,7)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})$ | $3,2$ |
| $(4,4,3,9,9)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})$ | $5,4$ |
| | $\eta$ | $(x-\alpha)(x-\gamma^q)(x-\gamma^{q^2})$ | $5,4$ |
| $(4,5,3,10,10)$ | $1$ | $\prod_{i=1}^{2}(x-\alpha_i)(x-\alpha_i^q)(x-\alpha_i^{q^2})(x-\alpha_i^{q^3})$ | $0$ |
| $(4,6,1,3,6)$ | $1$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})(x-\alpha^{q^3})$ | $0$ |
| $(4,7,2,7,9)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})(x-\alpha^{q^4})$ | $2,1$ |
| | $\eta$ | $(x-\alpha)(x-\alpha^{q^2})(x-\alpha^{q^3})(x-\alpha^{q^4})$ | $2,1$ |
| $(4,7,2,11,10)$ | $\eta$ | $(x-\alpha)(x-\alpha^{q^2})(x-\alpha^{q^3})$ | $3,2$ |
| | $\eta$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^3})$ | $3,2$ |
| $(4,7,3,8,11)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})(x-\alpha^{q^4})$ | $4,3$ |
| | $\eta$ | $(x-\alpha)(x-\alpha^{q^2})(x-\alpha^{q^3})(x-\alpha^{q^4})$ | $4,3$ |
| $(4,7,3,12,12)$ | $\eta$ | $(x-\alpha)(x-\alpha^{q^2})(x-\alpha^{q^3})$ | $5,4$ |
| | $\eta$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^3})$ | $5,4$ |
| $(3,3,1,2,6)$ | $\eta$ | $x-\alpha$ | $3,2$ |
| $(3,3,2,1,7)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)$ | $4,3$ |
| $(3,3,2,3,8)$ | $\eta$ | $x-\alpha$ | $5,4$ |
| | $\eta$ | $\prod_{i=1}^{2}(x-\alpha_i)(x-\alpha_i^q)$ | $2,1$ |
| $(3,3,2,5,9)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)(x-\gamma)$ | $3,2$ |
| $(3,3,3,2,9)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)$ | $6,5$ |
| $(3,3,3,4,10)$ | $\eta$ | $x-\alpha$ | $7,6$ |
| | $\eta$ | $\prod_{i=1}^{2}(x-\alpha_i)(x-\alpha_i^q)$ | $4,3$ |
| $(3,3,3,6,11)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)(x-\gamma)$ | $5,4$ |
| | $\eta$ | $\prod_{i=1}^{3}(x-\alpha_i)(x-\alpha_i^q)$ | $2,1$ |
| $(2,2,1,1,6)$ | $\eta$ | $\prod_{i=1}^{2}(x-\alpha_i)$ | $2,1$ |
| $(2,2,1,2,7)$ | $\eta$ | $\prod_{i=1}^{3}(x-\alpha_i)$ | $1,0$ |
| $(2,2,2,1,8)$ | $\eta$ | $\prod_{i=1}^{2}(x-\alpha_i)$ | $4,3$ |
| $(2,2,2,2,9)$ | $\eta$ | $\prod_{i=1}^{3}(x-\alpha_i)$ | $3,2$ |
| $(2,2,2,3,10)$ | $\eta$ | $\prod_{i=1}^{4}(x-\alpha_i)$ | $2,1$ |
| $(2,2,2,4,11)$ | $\eta$ | $\prod_{i=1}^{5}(x-\alpha_i)$ | $1,0$ |
| $(2,2,3,1,10)$ | $\eta$ | $\prod_{i=1}^{2}(x-\alpha_i)$ | $6,5$ |
| $(2,2,3,2,11)$ | $\eta$ | $\prod_{i=1}^{3}(x-\alpha_i)$ | $5,4$ |
| $(2,2,3,3,12)$ | $\eta$ | $\prod_{i=1}^{4}(x-\alpha_i)$ | $4,3$ |
| $(2,2,3,4,13)$ | $\eta$ | $\prod_{i=1}^{5}(x-\alpha_i)$ | $3,2$ |

Let $\beta \in k_2 \setminus k$, $\beta_j \in k_{\omega_j} \setminus k_{\rho_j}$ $(\rho_j \mid_{\neq} \omega_j)$, $h_2(x) \in k_2[x] \setminus k[x]$.
Here, choose $\beta_j$ and $\omega_j \in \{d, 2d, \cdots, max\{j\}d\}$ such that $h_2(x) \in k_2[x] \setminus k[x]$.
Let $C_0/k_d : y^2 = c \cdot h_d(x)h_2(x)h_1(x)$.

| $(n, d, g_0, e, S)$ | $c$ | $h_d(x)$ | $h_2(x)$ | $\deg h_1(x)$ |
|---|---|---|---|---|
| $(3, 4, 1, 1, 6)$ | 1 | $(x-\alpha)(x-\alpha^q)$ | 1 | $2, 1$ |
| $(3, 4, 1, 3, 7)$ | 1 | $(x-\alpha)(x-\alpha^q)$ | $x-\beta$ | $1, 0$ |
| $(3, 4, 2, 1, 8)$ | 1 | $(x-\alpha)(x-\alpha^q)$ | 1 | $4, 3$ |
| $(3, 4, 2, 3, 9)$ | 1 | $(x-\alpha)(x-\alpha^q)$ | $x-\beta$ | $3, 2$ |
| $(3, 4, 2, 5, 10)$ | 1 | $\prod_{i=1}^{2}(x-\alpha_i)(x-\alpha_i^q)$ | 1 | $2, 1$ |
| | 1 | $(x-\alpha)(x-\alpha^q)$ | $\prod_{j=1}^{2}(x-\beta_i)$ | $2, 1$ |
| $(3, 4, 2, 7, 11)$ | 1 | $(x-\alpha)(x-\alpha^q)$ | $\prod_{j=1}^{3}(x-\beta_i)$ | $1, 0$ |
| | 1 | $\prod_{i=1}^{2}(x-\alpha_i)(x-\alpha_i^q)$ | $x-\beta$ | $1, 0$ |
| $(3, 4, 3, 1, 10)$ | 1 | $(x-\alpha)(x-\alpha^q)$ | 1 | $6, 5$ |
| $(3, 4, 3, 3, 11)$ | 1 | $(x-\alpha)(x-\alpha^q)$ | $x-\beta$ | $5, 4$ |
| $(3, 4, 3, 5, 12)$ | 1 | $\prod_{i=1}^{2}(x-\alpha_i)(x-\alpha_i^q)$ | 1 | $4, 3$ |
| | 1 | $(x-\alpha)(x-\alpha^q)$ | $\prod_{j=1}^{2}(x-\beta_i)$ | $4, 3$ |
| $(3, 4, 3, 7, 13)$ | 1 | $(x-\alpha)(x-\alpha^q)$ | $\prod_{j=1}^{3}(x-\beta_j)$ | $3, 2$ |
| | 1 | $\prod_{i=1}^{2}(x-\alpha_i)(x-\alpha_i^q)$ | $x-\beta$ | $3, 2$ |
| $(3, 4, 3, 9, 14)$ | 1 | $(x-\alpha)(x-\alpha^q)$ | $\prod_{j=1}^{4}(x-\beta_j)$ | $2, 1$ |
| | 1 | $\prod_{i=1}^{2}(x-\alpha_i)(x-\alpha_i^q)$ | $\prod_{j=1}^{2}(x-\beta_j)$ | $2, 1$ |
| | 1 | $\prod_{i=1}^{3}(x-\alpha_i)(x-\alpha_i^q)$ | 1 | $2, 1$ |

# B Classification for Type (B): $^\forall d_i \neq d$

Here, $h_v(x) \in k_v[x] \setminus k_w[x]$ $(w \mid_{\neq} v)$, $\eta := 1$ or a non-square element in $k_d$.
(1) $n = 6, d = 28$, $\alpha \in k_7 \setminus k, \beta \in k_4 \setminus k_2$
$C_0/k_d : y^2 = c \cdot h_7(x)h_4(x)h_1(x)$

| $(n, d, g_0, e, S)$ | $c$ | $h_7(x)$ | $h_4(x)$ | $\deg h_1(x)$ |
|---|---|---|---|---|
| $(6, 28, 3, 61, 13)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})(x-\alpha^{q^4})$ | $(x-\beta)(x-\beta^q)$ | $2, 1$ |
| | $\eta$ | $(x-\alpha)(x-\alpha^{q^2})(x-\alpha^{q^3})(x-\alpha^{q^4})$ | $(x-\beta)(x-\beta^q)$ | $2, 1$ |

(2) $n = 5, d = 12$, $\alpha \in k_4 \setminus k_2, \beta \in k_3 \setminus k$
$C_0/k_d : y^2 = c \cdot h_4(x)h_3(x)h_1(x)$

| $(n, d, g_0, e, S)$ | $c$ | $h_4(x)$ | $h_3(x)$ | $\deg h_1(x)$ |
|---|---|---|---|---|
| $(5, 12, 2, 17, 9)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)$ | $(x-\beta)(x-\beta^q)$ | $2, 1$ |
| $(5, 12, 3, 21, 11)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)$ | $(x-\beta)(x-\beta^q)$ | $4, 3$ |

(3) $n=5, d=14,\ \alpha\in k_7\setminus k, \beta\in k_2\setminus k,\ \ \beta_1,\beta_2\in k_2\setminus k$ or $\beta_1\in k_4\setminus k_2$, $\beta_2:=\beta_1^{q^2},\ \ C_0/k_d: y^2=c\cdot h_7(x)h_2(x)h_1(x)$

| $(n,d,g_0,e,S)$ | $c$ | $h_7(x)$ | $h_2(x)$ | $\deg h_1(x)$ |
|---|---|---|---|---|
| $(5,14,2,21,10)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})(x-\alpha^{q^4})$ | $x-\beta$ | $1,0$ |
| | $\eta$ | $(x-\alpha)(x-\alpha^{q^2})(x-\alpha^{q^3})(x-\alpha^{q^4})$ | $x-\beta$ | $1,0$ |
| $(5,14,3,23,12)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})(x-\alpha^{q^4})$ | $x-\beta$ | $3,2$ |
| | $\eta$ | $(x-\alpha)(x-\alpha^{q^2})(x-\alpha^{q^3})(x-\alpha^{q^4})$ | $x-\beta$ | $3,2$ |
| $(5,14,3,31,13)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})(x-\alpha^{q^4})$ | $\prod_{i=1}^2(x-\beta_i)$ | $2,1$ |
| | $\eta$ | $(x-\alpha)(x-\alpha^{q^2})(x-\alpha^{q^3})(x-\alpha^{q^4})$ | $\prod_{i=1}^2(x-\beta_i)$ | $2,1$ |
| | $\eta$ | $(x-\alpha)(x-\alpha^{q^2})(x-\alpha^{q^3})$ | $x-\beta$ | $4,3$ |
| | $\eta$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^3})$ | $x-\beta$ | $4,3$ |

(4) $n=5, d=21,\ \alpha\in k_7\setminus k, \beta\in k_3\setminus k,\ \ \beta_1,\beta_2\in k_3\setminus k$ or $\beta_1\in k_6\setminus(k_2\cup k_3)$, $\beta_2:=\beta_1^{q^3},\ \ C_0/k_d: y^2=c\cdot h_7(x)h_3(x)h_1(x)$

| $(n,d,g_0,e,S)$ | $c$ | $h_7(x)$ | $h_3(x)$ | $\deg h_1(x)$ |
|---|---|---|---|---|
| $(5,21,2,7,10)$ | $1$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})(x-\alpha^{q^4})$ | $(x-\beta)(x-\beta^q)$ | $0$ |
| | $1$ | $(x-\alpha)(x-\alpha^{q^2})(x-\alpha^{q^3})(x-\alpha^{q^4})$ | $(x-\beta)(x-\beta^q)$ | $0$ |
| $(5,21,3,2,12)$ | $1$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})(x-\alpha^{q^4})$ | $(x-\beta)(x-\beta^q)$ | $2,1$ |
| | $1$ | $(x-\alpha)(x-\alpha^{q^2})(x-\alpha^{q^3})(x-\alpha^{q^4})$ | $(x-\beta)(x-\beta^q)$ | $2,1$ |
| $(5,21,3,10,13)$ | $1$ | $(x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})(x-\alpha^{q^4})$ | $\prod_{i=1}^2(x-\beta_i)(x-\beta_i^q)$ | $0$ |
| | $1$ | $(x-\alpha)(x-\alpha^{q^2})(x-\alpha^{q^3})(x-\alpha^{q^4})$ | $\prod_{i=1}^2(x-\beta_i)(x-\beta_i^q)$ | $0$ |

(5) $n=4, d=6,\ \alpha\in k_3\setminus k,\ \beta\in k_2\setminus k,\ \gamma\in k_6\setminus(k_2\cup k_3),\quad \alpha_1,\alpha_2\in k_3\setminus k$ or $\alpha_1\in k_6\setminus(k_2\cup k_3)$ , $\alpha_2:=\alpha_1^{q^3},\ \ \beta_1,\beta_2\in k_2\setminus k$ or $\beta_1\in k_4\setminus k_2,\ \beta_2:=\beta_1^{q^2}$, $C_0/k_d: y^2=c\cdot h_3(x)h_2(x)h_1(x)$

| $(n,d,g_0,e,S)$ | $c$ | $h_3(x)$ | $h_2(x)$ | $\deg h_1(x)$ |
|---|---|---|---|---|
| $(4,6,1,3,6)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)$ | $x-\beta$ | $1,0$ |
| $(4,6,2,5,8)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)$ | $x-\beta$ | $3,2$ |
| $(4,6,2,9,9)$ | $\eta$ | $x-\alpha$ | $x-\beta$ | $4,3$ |
| | $\eta$ | $\prod_{i=1}^2(x-\alpha_i)(x-\alpha_i^q)$ | $x-\beta$ | $1,0$ |
| | $\eta$ | $(x-\alpha)(x-\alpha^q)$ | $\prod_{j=1}^2(x-\beta_j)$ | $2,1$ |
| $(4,6,3,7,10)$ | $\eta$ | $(x-\alpha)(x-\alpha^q)$ | $x-\beta$ | $5,4$ |
| $(4,6,3,11,11)$ | $\eta$ | $x-\alpha$ | $x-\beta$ | $6,5$ |
| | $\eta$ | $\prod_{i=1}^2(x-\alpha_i)(x-\alpha_i^q)$ | $x-\beta$ | $3,2$ |
| | $\eta$ | $(x-\alpha)(x-\alpha^q)$ | $\prod_{j=1}^2(x-\beta_j)$ | $4,3$ |

$C_0/k_d: y^2=c\cdot h_6(x)h_1(x)$

| $(n,d,g_0,e,S)$ | $c$ | $h_6(x)$ | $\deg h_1(x)$ |
|---|---|---|---|
| $(4,6,2,9,9)$ | $\eta$ | $(x-\gamma)(x-\gamma^q)(x-\gamma^{q^2})$ | $3,2$ |