THE ANALYTICAL PROPERTY FOR $\zeta(s)$

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Abstract. In this article it's discussed the analytic property of $\zeta(s)$. The popular opinion is denied.

1. Introduction

$\zeta(s)$ [1]is defined (by Riemann) as:

$$\Gamma(s)\zeta(s) = \frac{1}{1-e^{i2\pi s}} \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t-1)dt$$

$$C_1 = (-\infty e^{2i\pi}, re^{2i\pi}], C_2 = re^{i\theta}, \theta = (2\pi, 0], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people thinks this function is analytic except $s = 1$[1]. There still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, R(s) > 0$$

This expression is analytic except $s = 1$, If the Riemann's definition is also analytic they should be identical. In this article the analytic property in $R(s) = 1$ is discussed.

2. Discussion

Theorem 2.1. *The second definition of $\zeta(s)$ has divergent derivation at the place near $s = 0$.*

Proof.

$$F_m(s) := \sum_{n=1}^{m} (-1)^{n+1} n^{-s}, R(s) > 0$$

Set $s \in (0, 1)$.

$$F'_m(s) = \sum_{n>0,2|n+1}^{\infty} \ln(n)(sn^{-s-1} - n^{-s-2}s(s+1)\theta/2), 0 < \theta < 1$$

$$F'_m(s) > \frac{1}{2} \int_{n=3}^{m-2} \ln(n) n^{-s-1} s dn - \sum_{n>0,2|n+1}^{\infty} \ln(n) n^{-s-2} s(s+1)\theta/2$$

$$\lim_{m\to\infty} F'_m(s) = \frac{1}{2} \int_3^{\infty} s \ln(x) x^{-s-1} dx - C$$

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$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} sxe^{-sx}dx - C$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} \frac{sxe^{-sx}d(sx)}{s} - C$$

It's easy to find when $s \to 0$ this term is close to infinity. □

There is coming up sharp controversy, as we commonly know the $\zeta(s)$ doesn't has infinity derivation in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition. Though that, the calculation of Riemann's definition is still sound. The question is that though the every derivation exists, the function still possible not analytic, the Taylor's remainder term are possibly not close to zero. That's to say some function seemed the algebraic of the complex argument but is not Taylor expandable.

Theorem 2.2.

$$\int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx > C(\int_a^b |f(x)g(x)|dx)^2$$

$$C = 1 + \frac{(\int_a^c f(x)dx \int_c^b g(x)dx - \int_a^c g(x)dx \int_c^b f(x)dx)^2}{(c-a)(b-c)(\int_a^b |f(x)g(x)|dx)^2}$$

$f(x), g(x)$ is non-negative real functions, $a < c < b$.

Theorem 2.3.

$$\lim_{k\to\infty} \int_{k^2}^{\infty} \ln^k(x)/(e^x - 1)dx \to 0$$

Proof.

$$\int_{k^2}^{\infty} \ln^k(x)e^{-x}dx$$

$$= \int_k^{\infty} \ln^k(x^2)e^{-x^2}dx^2$$

$$> \int_k^{\infty} \ln(x^2)e^{-x}dx^2$$

□

Theorem 2.4.

$$H_n(s) := \int_1^{\infty} \frac{\ln^n(x)x^{s-1}}{e^x - 1}$$

$$\lim_{n\to\infty} |\frac{H_n(1)}{H_{n-1}(1)n}| \to \infty$$

Proof. Consider

$$h(s) := \int_1^{\infty} \frac{x^{s-1}}{e^x - 1} = \int_1^{\infty} x^{s-1}e^{-x}dx \sum_{n=1}^{\infty} n^{-s}, 1 < s < 1.1$$

and note that

$$\lim_{s\to 1} h(s) \to \infty$$

Because

$$h(s+\delta) = \sum_{n=0}^{\infty} \frac{H_n(s)\delta^n}{n!}, 1 < s, s+\delta < 1.1$$

then

$$\exists n \exists \delta (|\frac{H_n(s)\delta}{H_{n-1}(s)n}| > 0.9), 1 < s, s+\delta < 1.1$$

$$\forall N \in R \exists n (\lim_{s\to 1} |\frac{H_n(s)}{H_{n-1}(s)n}| > N)$$

In another way use the theorem 2.2 and choose $c = e^2, b = (n+1)^2$, exists for great enough n:

$$|\frac{H_{n+1}(s)}{H_n(s)(n+1)}| > |\frac{H_n(s)}{H_{n-1}(s)n}|$$

hence

$$\lim_{n\to\infty} |\frac{H_n(1)}{H_{n-1}(1)n}| \to \infty$$

□

Theorem 2.5.

$$C_0 \int_a^b \frac{\ln^k(x)}{e^x} dx < \int_a^b \frac{\ln^k(x)}{e^x - 1} dx < C_1 \int_a^b \frac{\ln^k(x)}{e^x} dx, b > a > 1$$

C_0, C_1 is positives depending on a, b.

Definition 2.6.

$$g_k(x) := \frac{x^k(\cos(ax) + i\sin(ax))e^x}{e^{e^x} - 1} =: g_{k1}(x) + ig_{k2}(x)$$

$$G_k(x) := \ln^k(x)/(e^x - 1)$$

Theorem 2.7.

$$\int_0^\infty |g_k(x)|dx < O(\int_1^\infty G_k(x)dx)$$

Theorem 2.8.

$$|\int_0^\infty g_k(x)dx| = O(\int_0^\infty |g_k(x)|dx)$$

Proof. Because

$$|\int \frac{x^k(cos(a(x+\beta)) + isin(a(x+\beta)))e^x}{e^{e^x} - 1} dx| = |\int g_k(x)dx|$$

make shift of β to set the point with maximal value $x_m : k = x_m \ln(x_m)$ meeting $a(x+\beta) = k'\pi - \pi/2, k' \in \mathbf{N}$. On this case to calculating the integration.

It can be found that

$$\lim_{k\to\infty} \frac{\int_{x_m}^\infty G_k(x)dx}{\int_1^{x_m} G_k(x)dx} = 0$$

For any $0 < \alpha < 1$ existing N for $k > N$ it's valid that $x_m > k^\alpha$, hence the integrations is calculated like

$$\lim_{k\to\infty} \frac{\int_{x_m}^\infty G_k(x)dx}{\int_1^{x_m} G_k(x)dx}$$

$$= C \lim_{k\to\infty} \frac{\int_1^2 G'_k(\alpha)d\alpha}{\int_0^1 G'_k(\alpha)d\alpha}$$

$$G'_k(\alpha)d\alpha = \frac{\ln^k(k^\alpha)}{e^{k^\alpha}} dk^\alpha$$

Calculate to find for great enough k:

$$\frac{G_k(e^x)e^x}{G_k(e^{x-\pi/a})e^{x-\pi/a}}|_{\ln(x)<x_m} > C > 3$$

□

Considering the $\Gamma(s)\zeta(s)$ at $s = ai + 1, a \in R, a \neq 0$, the k-th order derivation on s of the its part generated from integration in line C_2 is of magnitude less than: $C''C'^k k!$. From those previous theorems it can be found that

$$\frac{d^k}{(ds)^k}\Gamma(s)\zeta(s)|_{s=ia+1}, a \in R$$

can't be the coefficients of any convergent Taylor's series at some area.

References

[1] E.C. Titchmarsh, The theory of the Riemann zeta function, Oxford University Press, 2nd ed, 1986.

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