THE ANALYTICAL PROPERTY FOR $\zeta(s)$

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ABSTRACT. In this article it's discussed that the analytic property of $\zeta(s)$. The popular opinion is denied.

1. Introduction

$\zeta(s)$ [1]is defined (by Riemann) as:

$$2\sin(is\pi)\Gamma(s)\zeta(s) = i\int_{C=C_1+C_2+C_3}(-t)^{s-1}/(e^t-1)dt$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people thinks this function is analytic except $s = 1$[1]. There still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta(s) = \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}(-1)^{n+1}n^{-s}, R(s) > 0$$

This expression is analytic except $s = 1$, If the Riemann's definition is also analytic they should be identical. In this article the analytic property in $R(s) = 1$ is discussed.

2. Discussion

Theorem 2.1. *The second definition of $\zeta(s)$ has divergent derivation at the place near $s = 0$.*

Proof.

$$F_m(s) := \sum_{n=1}^{m}(-1)^{n+1}n^{-s}, R(s) > 0$$

Set $s \in (0, 1)$.

$$F_m'(s) = \sum_{n>0,2|n+1}^{\infty}\ln(n)(sn^{-s-1} - n^{-s-2}s(s+1)\theta/2), 0 < \theta < 1$$

$$F_m'(s) > \frac{1}{2}\int_{n=3}^{m-2}\ln(n)n^{-s-1}sdn - \sum_{n>0,2|n+1}^{\infty}\ln(n)n^{-s-2}s(s+1)\theta/2$$

$$\lim_{m\to\infty}F_m'(s) = \frac{1}{2}\int_3^{\infty}s\ln(x)x^{-s-1}dx - C$$

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$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} sxe^{-sx}dx - C$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} \frac{sxe^{-sx}d(sx)}{s} - C$$

It's easy to find when $s \to 0$ this term approaches to infinity. □

There is coming up sharp controversy, as is commonly known the $\zeta(s)$ doesn't has infinity derivation in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition.

Definition 2.2.

$$g_t(s) = \int_0^{\infty} \frac{(-x)^{s-1}}{te^x - 1}dx, t \geq 1$$

$$G_t(s) = \int_C \frac{(-x)^{s-1}}{te^x - 1}dx, t \geq 1$$

C is defined in the section of Introduction.

Theorem 2.3.

$$g_t(s) = (-1)^{s-1}\Gamma(s)\sum_{n=1}^{\infty} t^{-n}n^{-s}$$

$$1 < R(s) < 2, s \neq 1, t \geq 1$$

$$g_t(s : R(s) = 1) := \lim_{s'\to s} g_t(s')$$

$g_t(s)$ is continuous at $1 \leq R(s) < 2, t \geq 1, s \neq 1$ if $g_1(s)$ continuous at $1 \leq R(s) < 2, s \neq 1$.

Proof. If not, the exist infinite point (s_i, t_i) in any neighboring set of $P = (s, t) = (1 + ai, 1), a \in \mathbf{R}$ meeting

$$\exists C > 0 \forall i(|g_{t_i}(s_i) - g_1(1 + ai)| > C)$$

but in some order of i

$$\lim_{i\to\infty} ||(s_i, t_i) - (s_i, 1)|| + ||(s_i, 1) - P|| \to 0$$

and $(s_i, t_i), (s_i, 1), P$ in the continuous domain. □

Lemma 2.4. *$\zeta(s), R(s) \leq 1$ is not the continuation of $\zeta(s), R(s) > 1$ at $R(s) = 1$ for Riemann's definition.*

Proof.

$$1 < R(s) < 2, s \neq 1, t \in [1, 2)$$

$$s = 1 + \delta + ai, a \in \mathbf{R}$$

$$\lim_{t=e^\delta\to 1} (G_t(s) - 2\sin(is\pi)g_t(s)/i)$$

$$= \lim_{t=e^\delta\to 1} \int_{C', r'\to\ln(t)} \frac{(-x)^{s-1}}{te^x - 1}dx$$

C' is circle with center in $-\ln(t)$ and radium r' except a gap between the intersections with $\arg(x) = 0, 2\pi$.

$$\lim_{t=e^\delta\to 1} |(G_t(s) - 2\sin(is\pi)g_t(s)/i)| = 2|\sinh(a\pi)/a|$$

If the Riemann's continuation is analytic at $R(s) = 1$, because $g_t(s)$ continuous at $1 \leq R(s) < 2, t \geq 1, s \neq 1$ if $g_1(s)$ continuous at $1 \leq R(s) < 2, s \neq 1$. And $G_t(s)$ is continuous at $0 < s < 2, t \geq 1$. It can be found easily:

$$\lim_{t=e^{\delta}\to 1}(G_t(s) - 2\sin(is\pi)g_t(s)/i) = 0$$

That's not real obviously. □

This opposes the classical calculation. The classical calculation try to analyse:

$$\zeta(\delta + ai)$$
$$\leftrightarrow \lim_{r\to 0}\lim_{\delta'\to\delta}\int_C (-t)^{\delta'+ai}/(e^t - 1)dt, \delta' > 0, a \in \mathbf{R}$$
$$\leftrightarrow \lim_{\delta'\to\delta}\lim_{r\to 0}\int_{C1+C_3} (-t)^{\delta'+ai}/(e^t - 1)dt, \delta' > 0, a \in \mathbf{R}$$

If

$$\int_r^{\infty} (-t)^{\delta'+ai}/(e^t - 1)dt$$

uniformly continuous at $1 < R(s) < 2, 0 < r < 1$, the difference of the two terms is

$$\lim_{r\to 0}\lim_{\delta'\to\delta}\int_{C_2} (-t)^{\delta'+ai}/(e^t - 1)dt$$

As a fact, from the calculation of the second definition of $\zeta(s)$ the function

$$\int_0^{\infty} (-x)^{s-1}/(e^x - 1)dx$$

is uniformly continuous at $1 < R(s) < 2$.

References

[1] E.C. Titchmarsh, The theory of the Riemann zeta function, Oxford University Press, 2nd ed, 1986.

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