THE CONTINUOUS PROPERTY FOR $\zeta(s)$

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ABSTRACT. In this article it's discussed that the continuous property of $\zeta(s)$. The popular opinion is denied.

1. Introduction

$\zeta(s)$ [1]is defined (by Riemann) as:

$$(1-e^{i2pi(s-1)})\Gamma(s)\zeta(s)=\int_{C=C_1+C_2+C_3} t^{s-1}/(e^t-1)dt$$

$$C_1=(-\infty,r]e^{2i\pi}, C_2=re^{i\theta}, \theta=(2\pi,\pi], C_3=(r,\infty), 0<r<2\pi$$

Most of people thinks this function is analytic except $s=1$[1]. There still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta(s)=\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}(-1)^{n+1}n^{-s}, R(s)>0$$

This expression is analytic except $s=1$, If the Riemann's definition is also analytic they should be identical. In this article the analytic property in $R(s)=1$ is discussed.

2. Discussion

Theorem 2.1. *The second definition of $\zeta(s)$ has divergent derivation at the place near $s=0$.*

Proof.

$$F_m(s):=\sum_{n=1}^{m}(-1)^{n+1}n^{-s}, R(s)>0$$

Set $s\in(0,1)$.

$$F_m'(s)=\sum_{n>0,2|n+1}^{\infty}\ln(n)(sn^{-s-1}-n^{-s-2}s(s+1)\theta/2), 0<\theta<1$$

$$F_m'(s)>\frac{1}{2}\int_{n=3}^{m-2}\ln(n)n^{-s-1}sdn-\sum_{n>0,2|n+1}^{\infty}\ln(n)n^{-s-2}s(s+1)\theta/2$$

$$\lim_{m\to\infty}F_m'(s)=\frac{1}{2}\int_3^{\infty}s\ln(x)x^{-s-1}dx-C$$

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$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} sxe^{-sx}dx - C$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} \frac{sxe^{-sx}d(sx)}{s} - C$$

It's easy to find when $s \to 0$ this term approaches to infinity. □

There is coming up sharp controversy, as is commonly known the $\zeta(s)$ doesn't has infinity derivation in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition.

Definition 2.2.

$$g_t(s) = \int_{x\to 0}^{\infty} \frac{x^{s-1}}{e^x - 1}dx, R(s) \geq 1$$

$$G_t(s) = \int_C \frac{x^{s-1}}{e^x - 1}dx, R(s) \geq 1$$

C is defined in the section of Introduction.

Theorem 2.3. *$g_t(s)$ is continuous at $1 \leq R(s) < 2, s \neq 1$.*

Proof.

$$\int_{x\to 0}^{\delta'\to 0} \frac{x^{s-1}}{e^x - 1}dx$$

$$= \int_{x\to 0}^{\delta'\to 0} \sum_{n=0} B_n x^{s-2+n} dx/n!$$

$$= \sum_{n=0}^{\infty} \frac{B_n x^{s-1+n}}{(s-1+n)n!}\Big|_{x\to 0}^{\delta'\to 0}$$

B_n is Bernoulli number. Because[2]

$$B_{2n} = \frac{(-1)^{n-1}2(2n)!}{(2\pi)^{2n}}\zeta(2n), B_{2n+1} = 0, n > 0, |B_n/n!|^{1/n} < C$$

So that this integration is uniformly convergent at $1 \leq R(s) < 2, s \neq 1$. □

Lemma 2.4. *$\zeta(s), R(s) \leq 1$ is not the continuation of $\zeta(s), R(s) > 1$ at $R(s) = 1$ for Riemann's definition.*

Proof. This opposes the classical calculation. The classical calculation try to analyze:

$$\zeta(\delta + ai)$$

$$\leftrightarrow \lim_{r\to 0}\lim_{\delta'\to\delta}\int_C t^{\delta'+ai}/(e^t - 1)dt, \delta' > 0, a \in \mathbf{R}$$

$$\leftrightarrow \lim_{\delta'\to\delta}\lim_{r\to 0}\int_{C1+C_3} t^{\delta'+ai}/(e^t - 1)dt, \delta' > 0, a \in \mathbf{R}$$

If

$$\int_r^{\infty} t^{\delta'+ai}/(e^t - 1)dt$$

is uniformly continuous at $1 < R(s) < 2, 0 < r < 1$ (it is), the difference of the two terms is

$$\lim_{r\to 0}\lim_{\delta'\to\delta}\int_{C_2} t^{\delta'+ai}/(e^t - 1)dt \neq 0$$

□

But we are still convinced by the analytic property of $x^{s-1}/(e^x-1)$. In fact it's not the reason of the analytic of the function but possibly that the analytic of r^{ai} that the radium of C_2 involves. This function is calculated analytically but r is not analytic very much. As a fact if set $r=0$ then the controversy disappears.

References

[1] E.C. Titchmarsh, The theory of the Riemann zeta function, Oxford University Press, 2nd ed, 1986.

[2] James B Silva, Bernoulli Numbers and their Applications

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