THE ANALYTIC PROPERTY FOR $\zeta(s)$

WU SHENG-PING

ABSTRACT. In this article it's discussed that the analytic property of $\zeta(s)$. The popular opinion is denied.

1. Introduction

$\zeta(s)$ [1]is defined (by Riemann) as:

$$(1-e^{i2\pi(s-1)})\Gamma(s)\zeta(s)=\int_{C=C_1+C_2+C_3} t^{s-1}/(e^t-1)dt$$

$$C_1=(-\infty,r]e^{2i\pi},C_2=re^{i\theta},\theta=(2\pi,\pi],C_3=(r,\infty),0<r<2\pi$$

Most of people thinks this function is analytic except $s=1$[1]. There still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta(s)=\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}(-1)^{n+1}n^{-s},R(s)>0$$

This expression is analytic except $s=1$, If the Riemann's definition is also analytic they should be identical. In this article the analytic property in $R(s)=0$ is discussed.

2. Discussion

Theorem 2.1. *The second definition of $\zeta(s)$ has divergent derivation at the place near $s=0$.*

Proof.

$$F_m(s):=\sum_{n=1}^{m}(-1)^{n+1}n^{-s},R(s)>0$$

Set $s\in(0,1)$.

$$F'_m(s)=\sum_{n>0,2|n+1}^{\infty}\ln(n)(sn^{-s-1}-n^{-s-2}s(s+1)\theta/2),0<\theta<1$$

$$F'_m(s)>\frac{1}{2}\int_{n=3}^{m-2}\ln(n)n^{-s-1}sdn-\sum_{n>0,2|n+1}^{\infty}\ln(n)n^{-s-2}s(s+1)\theta/2$$

$$\lim_{m\to\infty}F'_m(s)=\frac{1}{2}\int_{3}^{\infty}s\ln(x)x^{-s-1}dx-C$$

Date: June 4, 2010.
2000 *Mathematics Subject Classification.* Primary 11M06.
Key words and phrases. Riemann zeta function, analytic continuation.

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} sxe^{-sx}dx - C$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} \frac{sxe^{-sx}d(sx)}{s} - C$$

It's easy to find when $s \to 0$ this term approaches to infinity. □

There is coming up sharp controversy, as is commonly known the $\zeta(s)$ doesn't has infinity derivation in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition.

The problem happens on Γ function.

The derivations of each order of $\Gamma(s)$ near $R(s) = 0, R(s) > 0$ are

$$\int_0^\infty \ln^k(x)x^{\delta-1+ai}e^{-x}dx, \delta, a > 0, s = \delta + ai$$

$$\int_0^\infty x^k e^{-x\delta}e^{-aix}e^{-e^{-x}}dx,$$

$$\delta^{-k-1}\int_0^\infty x^k e^{-x}e^{-e^{-x/\delta}}e^{-aix/\delta}dx,$$

$$f(x) := x^k e^{-x}e^{-e^{-x/\delta}}$$

Because

$$\lim_{k\to\infty} \frac{\int_0^k f(x)dx}{\int_k^\infty f(x)dx} \to 0$$

and near $\delta = 0, \delta > 0$

$$\int_k^\infty f(x)\cos(-ax/\delta + \alpha)dx = Ck^k e^{-k}\delta + r\delta^2, C > 0$$

for some α. C is independent of k. And for little enough δ

$$\frac{|r\delta|}{Ck^k e^{-k}} < 1/N, N > 4$$

Uniformly for k. Then

$$\lim_{\delta\to 0}\lim_{k\to\infty} |\frac{d^k\Gamma(s = \delta + ai)}{k!(ds)^k}|^{1/k} \to \infty$$

This explains the result:

Theorem 2.2. $\Gamma(s)$ *is not analytic at imaginary axis.*

The famous proof of Cauchy [2] on the result

$$|f'(x_0)| \le \sup_A |f(x)|/r$$

for analytic function $f(x)$ at area of round A of radium r and center x_0, is not correct because exists counter-example

$$f(x) = e^{((x+1/2)^k+3)^k}, r = 0.1, x_0 = 0$$

for a great k. The flow of its proof is that the limit of integral contour

$$C = re^{i\theta}, 0 \le \theta < 2\pi, r \to 0$$

is not valid, because the contour is not uniform finite for the definition of a integration.

References

[1] E.C. Titchmarsh, The theory of the Riemann zeta function, Oxford University Press, 2nd ed, 1986.

[2] M. A. Lavrentieff, B. V. Shabat, Methods of functions of a complex variable, Russia, 2002

State Key Laboratory of Software Engineering, Wuhan university, China. Post-code: 430074

E-mail address: hiyaho@126.com