THE ANALYTIC PROPERTY FOR $\zeta(s)$

WU SHENG-PING

Abstract. It's the analytic property of $\zeta(s)$ that is discussed in this article. The popular opinion is denied.

1. Introduction

$\zeta(s)$ [1]is defined (by Riemann) as:

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t - 1)dt \tag{1.1}$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people think this function is analytic except $s = 1$[1]. There is still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, R(s) > 0 \tag{1.2}$$

This expression is analytic except $s = 1$, If the Riemann's definition is also analytic the definitions 1.1 and 1.2 should be identical. In this article the analytic property in $R(s) = 0$ is discussed.

2. Discussion

Theorem 2.1. *The second definition of $\zeta(s)$ has divergent derivation near $s = 0$.*

Proof.

$$F_m(s) := \sum_{n=1}^{m} (-1)^{n+1} n^{-s}, R(s) > 0$$

Set $s \in (0, 1)$.

$$F'_m(s) = \sum_{n>0,2|n+1}^{\infty} \ln(n)(sn^{-s-1} - n^{-s-2}s(s+1)\theta/2), 0 < \theta < 1$$

$$F'_m(s) > \frac{1}{2} \int_{n=3}^{m-2} \ln(n)n^{-s-1}sdn - \sum_{n>0,2|n+1}^{\infty} \ln(n)n^{-s-2}s(s+1)\theta/2$$

$$\lim_{m\to\infty} F'_m(s) = \frac{1}{2} \int_3^{\infty} s\ln(x)x^{-s-1}dx - C_s, |C_s| < C$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2} \int_{\ln(3)}^{\infty} sxe^{-sx}dx - C_s$$

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$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} \frac{sxe^{-sx}d(sx)}{s} - C_s$$

It's easy to find when $s \to 0$ this term approaches to infinity. □

There is coming up sharp controversy, as is commonly known the $\zeta(s)$ hasn't infinity derivation in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition.

The problem happens on Γ function.

The derivations of each order of $\Gamma(s)$ near $R(s) = 0, R(s) > 0$ are

$$\int_0^{\infty} \ln^k(x) x^{\delta-1+ai} e^{-x} dx, \delta, a > 0, s = \delta + ai$$

$$\int_0^{\infty} x^k e^{-x\delta} e^{-aix} e^{-e^{-x}} dx,$$

$$\delta^{-k-1}\int_0^{\infty} x^k e^{-x} e^{-e^{-x/\delta}} e^{-aix/\delta} dx,$$

$$f(x) := x^k e^{-x} e^{-e^{-x/\delta}}$$

Because

$$\lim_{k\to\infty} \frac{\int_0^k f(x)dx}{\int_k^{\infty} f(x)dx} \to 0$$

and near $\delta = 0, \delta > 0$

$$\int_k^{\infty} f(x)\cos(-ax/\delta + \alpha)dx = Ck^k e^{-k}\delta + r\delta^2, C > 0$$

for some α. C is independent of k. And for sufficiently little δ

$$\frac{|r\delta|}{Ck^k e^{-k}} < 1/N, N > 4$$

Uniformly for k. Then

$$\lim_{\delta\to 0}\lim_{k\to\infty} \left|\frac{d^k\Gamma(s = \delta + ai)}{k!(ds)^k}\right|^{1/k} \to \infty$$

This explains the result:

Theorem 2.2. $\Gamma(s)$ *is not analytic at imaginary axis.*

The famous proof of Cauchy's [2] integration for the complex derivable function $f(x)$ in the considered domain

$$f(x) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z - x} dz$$

$$C' = re^{i\theta} - x, 0 \le \theta < 2\pi$$

is incorrect. The flaw of the proof is that the limit of integral contours $r \to 0$ is not valid, because the contours are not uniformly finite for the definition of a integration.

A counter-example of the Cauchy's result is needed. The *complex derivable continuation* of the second definition of $\zeta(s)$ in the whole complex plane is

$$(1 - e^{i2\pi(s-1)})\Gamma^*(s)\zeta(s) := \frac{1}{1 - 2^{1-s}}\int_C t^{s-1}/(e^t + 1)dt$$

$$(1 - e^{i2\pi(s-1)})\Gamma^*(s) := \int_C x^{s-1}e^{-x}dx$$

$\Gamma^*(s)$ is the continuation of $\Gamma(s), R(s) > 0$, because

$$\int_C x^{s-1}e^{-x}dx = \lim_{r\to 0}(\int_{C_1+C_3} x^{s-1}e^{-x}dx + \int_{C_2} x^{s-1}e^{-x}dx)$$

and for $R(s) > 0$:

$$\lim_{r\to 0}\int_{C_2} x^{s-1}e^{-x}dx = \lim_{r\to 0}\int_{C_2} dx\sum_{j=0}^{\infty}\frac{(-1)^j x^{s-1+j}}{j!}$$

$$= \lim_{r\to 0}\sum_{j=0}^{\infty}\frac{(-1)^j x^{s+j}}{(s+j)j!}|_{re^{i2\pi}}^{r} = 0, R(s) > 0$$

As a fact we ought to comprehend this limit correctly instead of calculating on the meaningless infinitely little integral contour.

From the theorem 2.1 a counter-example is obtained.

References

[1] E.C. Titchmarsh, The theory of the Riemann zeta function, Oxford University Press, 2nd ed, 1986.

[2] M. A. Lavrentieff, B. V. Shabat, Methods of functions of a complex variable, Russia, 2002

Wuhan university, Wuhan, China.

E-mail address: hiyaho@126.com