THE ANALYTIC PROPERTY FOR $\zeta(s)$

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ABSTRACT. It's the analytic property of $\zeta(s)$ that is discussed in this article. The popular opinion is denied.

1. Introduction

$\zeta(s)$ [1]is defined (by Riemann) as:

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t - 1)dt \tag{1.1}$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people think this function is analytic except $s = 1$[1]. There is still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, R(s) > 0 \tag{1.2}$$

This expression is analytic except $s = 1$, If the Riemann's definition is also analytic the definitions 1.1 and 1.2 should be identical. In this article the analytic property in $R(s) = 0$ is discussed.

2. Discussion

Theorem 2.1. *The second definition of $\zeta(s)$ has divergent derivation near $s = 0$.*

Proof.

$$F_m(s) := \sum_{n=1}^{m} (-1)^{n+1} n^{-s}, R(s) > 0$$

Set $s \in (0, 1)$.

$$F'_m(s) = \sum_{n>0,2|n+1}^{\infty} \ln(n)(sn^{-s-1} - n^{-s-2}s(s+1)\theta/2), 0 < \theta < 1$$

$$F'_m(s) > \frac{1}{2}\int_{n=3}^{m-2} \ln(n)n^{-s-1}sdn - \sum_{n>0,2|n+1}^{\infty} \ln(n)n^{-s-2}s(s+1)\theta/2$$

$$\lim_{m\to\infty} F'_m(s) = \frac{1}{2}\int_3^{\infty} s\ln(x)x^{-s-1}dx - C_s, |C_s| < C$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} sxe^{-sx}dx - C_s$$

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$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} \frac{sxe^{-sx}d(sx)}{s} - C_s$$

It's easy to find when $s \to 0$ this term approaches to infinity. □

There is coming up sharp controversy, as is commonly known the $\zeta(s)$ hasn't infinity derivation in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition.

The problem happens on Γ function.

The derivations of each order of $\Gamma(s)$ near $R(s) = 0, R(s) > 0$ are

$$\int_0^{\infty} \ln^k(x) x^{\delta-1+ai} e^{-x} dx, \delta, a > 0, s = \delta + ai$$

$$\int_0^{\infty} x^k e^{-x\delta} e^{-aix} e^{-e^{-x}} dx,$$

$$\delta^{-k-1}\int_0^{\infty} x^k e^{-x} e^{-e^{-x/\delta}} e^{-aix/\delta} dx,$$

$$f(x) := x^k e^{-x} e^{-e^{-x/\delta}}$$

Because

$$\lim_{k\to\infty} \frac{\int_0^k f(x)dx}{\int_k^{\infty} f(x)dx} \to 0$$

and near $\delta = 0, \delta > 0$

$$\int_k^{\infty} f(x)\cos(-ax/\delta + \alpha)dx = Ck^k e^{-k}\delta + r\delta^2, C > 0$$

for some α. C is independent of k. And for sufficiently little δ

$$\frac{|r\delta|}{Ck^k e^{-k}} < 1/N, N > 4$$

Uniformly for k. Then

$$\lim_{\delta\to 0}\lim_{k\to\infty} \left|\frac{d^k\Gamma(s = \delta + ai)}{k!(ds)^k}\right|^{1/k} \to \infty$$

This explains the result:

Theorem 2.2. $\Gamma(s)$ *is not analytic at imaginary axis.*

The famous proof of Cauchy's [2] integration for the complex derivable function $f(x)$ in the considered domain

$$f(x) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z - x} dz$$

$$f^{(n)}(x) = \frac{n!}{2\pi i}\oint_{C'} \frac{f(z)}{(z - x)^{n+1}} dz$$

$$C' = re^{i\theta} - x, 0 \le \theta < 2\pi$$

is incorrect. The flaw of the proof is that the limit of integral contours $r \to 0$ is not valid, because the contours are not uniformly finite for the definition of a integration.

Counter-examples of the Cauchy's result are needed.

The *complex derivable continuation* $\zeta^*(s)$ of the second definition of $\zeta(s)$ in the whole complex plane is

$$(1 - e^{i2\pi(s-1)})\Gamma^*(s)\zeta^*(s) := \frac{1}{1-2^{1-s}} \int_C t^{s-1}/(e^t+1)dt$$

$$(1 - e^{i2\pi(s-1)})\Gamma^*(s) := \int_C x^{s-1}e^{-x}dx$$

$\Gamma^*(s)$ is the continuation of $\Gamma(s), R(s) > 0$, because

$$\int_C x^{s-1}e^{-x}dx = \lim_{r\to 0}(\int_{C_1+C_3} x^{s-1}e^{-x}dx + \int_{C_2} x^{s-1}e^{-x}dx)$$

and for $R(s) > 0$:

$$\lim_{r\to 0}\int_{C_2} x^{s-1}e^{-x}dx = \lim_{r\to 0}\int_{C_2} dx \sum_{j=0}^{\infty} \frac{(-1)^j x^{s-1+j}}{j!}$$

$$= \lim_{r\to 0}\sum_{j=0}^{\infty} \frac{(-1)^j x^{s+j}}{(s+j)j!}|_{re^{i2\pi}}^{r} = 0, R(s) > 0$$

As a fact we ought to comprehend this limit correctly instead of calculating on the meaningless infinitely little integral contour. It could be noticed that $\zeta^*(s)$ is very different from $\zeta(s)$ that defined by Riemann. In fact by L'hospital Law $\zeta^*(s)$ is finite at $s = 0$ but its derivation is infinite (at least as the theorem 2.1). Though for this, in the same logic of the cauchy's proof, the similar form of results can be obtained that violates the inequality deduced from cauchy's integral formula.

References

[1] E.C. Titchmarsh, The theory of the Riemann zeta function, Oxford University Press, 2nd ed, 1986.

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