THE ANALYTIC PROPERTY FOR RIEMANN ZETA FUNCTION

WU SHENG-PING

ABSTRACT. This article discusses the analytic property of Riemann zeta function. The popular opinion is denied.

1. Introduction

$\zeta(s)$ [1]is defined (by Riemann) as:

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t - 1)dt \tag{1.1}$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people think this function is analytic except $s = 1$[1]. There is still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, R(s) > 0 \tag{1.2}$$

This expression is analytic except $s = 1$, If the Riemann's definition is also analytic the definitions 1.1 and 1.2 should be identical. In this article the analytic property in $R(s) = 0$ is discussed.

2. Discussion

Theorem 2.1. *The second definition of $\zeta(s)$ has divergent derivation near $s = 0$.*

Proof.

$$F_m(s) := \sum_{n=1}^{m} (-1)^{n+1} n^{-s}, R(s) > 0$$

Set $s \in (0, 1)$.

$$F'_m(s) = \sum_{n>0,2|n+1}^{\infty} \ln(n)(sn^{-s-1} - n^{-s-2}s(s+1)\theta/2), 0 < \theta < 1$$

$$F'_m(s) > \frac{1}{2}\int_{n=3}^{m-2} \ln(n)n^{-s-1}sdn - \sum_{n>0,2|n+1}^{\infty} \ln(n)n^{-s-2}s(s+1)\theta/2$$

$$\lim_{m\to\infty} F'_m(s) = \frac{1}{2}\int_3^{\infty} s\ln(x)x^{-s-1}dx - C_s, |C_s| < C$$

Date: June 4, 2010.

2000 *Mathematics Subject Classification.* Primary 11M06.

Key words and phrases. Riemann zeta function, analytic continuation, Cauchy integral formula.

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} sxe^{-sx}dx - C_s$$
$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} \frac{sxe^{-sx}d(sx)}{s} - C_s$$

It's easy to find when $s \to 0$ this term approaches to infinity. □

There is coming up sharp controversy, as is commonly known the $\zeta(s)$ hasn't infinity derivation in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition.

The probe is on Cauchy Integral Formula that said derivable function is analytic. Integrations about the complex derivable function $f(x)$ in the considered domain meet

$$f(x) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z-x}dz$$
$$f^{(n)}(x) = \frac{n!}{2\pi i}\oint_{C'} \frac{f(z)}{(z-x)^{n+1}}dz$$
$$C' = re^{i\theta} - x, 0 \le \theta < 2\pi$$

Its famous proof of Cauchy's [2] is incorrect. The flaw of the proof is that the limit of integral contours $r \to 0$ is not valid, because the contours are not uniformly finite for the definition of a integration.

Counter-examples of the Cauchy's result are needed.

The *complex derivable continuation* $\zeta^*(s)$ of the second definition of $\zeta(s)$ in the whole complex plane is

$$(1 - e^{i2\pi(s-1)})\Gamma^*(s)\zeta^*(s) := \frac{1}{1-2^{1-s}}\int_C t^{s-1}/(e^t+1)dt$$
$$(1 - e^{i2\pi(s-1)})\Gamma^*(s) := \int_C x^{s-1}e^{-x}dx$$

$\Gamma^*(s)$ is the classical continuation of $\Gamma(s), R(s) > 0$, because

$$\int_C x^{s-1}e^{-x}dx = \lim_{r\to 0}(\int_{C_1+C_3} x^{s-1}e^{-x}dx + \int_{C_2} x^{s-1}e^{-x}dx)$$

and for $R(s) > 0$:

$$\lim_{r\to 0}\int_{C_2} x^{s-1}e^{-x}dx = \lim_{r\to 0}\int_{C_2} dx \sum_{j=0}^{\infty} \frac{(-1)^j x^{s-1+j}}{j!}$$
$$= \lim_{r\to 0}\sum_{j=0}^{\infty} \frac{(-1)^j x^{s+j}}{(s+j)j!}|_{re^{i2\pi}}^{r} = 0, R(s) > 0$$

As a fact we ought to comprehend this limit correctly instead of calculating on the meaningless infinitely little integral contour. It could be noticed that $\zeta^*(s)$ is quite different from $\zeta(s)$ that defined by Riemann at $R(s) < 1$. In fact by L'hospital Law (it's proved by integration not by analytic form) $\zeta^*(s)$ is finite and continuous at $s = 0$ so its residue is zero, but its derivation is infinite (at least as the theorem 2.1). Though this, in the same logic of the cauchy's proof, the similar form of results can be obtained that violates the inequality deduced from cauchy's integral formula.

Now it is critical to define the conception of Analytic as expandable to power series. French mathematician J.Dieudonne use this definition consistently in his famous book "Foundations of Modern Analysis". Following this definition of conception of analytic Cauchy integral formula is all right.

Maybe somebody have said

Theorem 2.2. *A complex function is analytic if both its real part and imaginary part are analytic.*

It's obvious. By this theorem $\zeta^*(s)$ is analytic on $R(s) > 0$, and $\Gamma(s)$ is analytic on $R(s) > 0$. $\zeta^*(s)$ is even different from $\zeta(s)$ at near $R(s) = 1$,

$$((s-1)(1-e^{i2\pi(s-1)})\Gamma(s)\zeta(s))' = \frac{d}{ds}(s-1)\int_C t^{s-1}/(e^t-1)dt$$

$$= \int_C t^{s-1}/(e^t-1)dt + (s-1)^2\int_C t^{s-2}/(e^t-1)dt$$

near $s = 1$ the integrations is continuous

$$= -2\pi i + P_1(s-1)(s-1) + (s-1)^2(\pi i + P_2(s-1)(s-1))$$

at $s = 1$

$$((s-1)(1-e^{i2\pi(s-1)})\Gamma(s)\zeta(s))'|_1 = -2\pi i$$

but the other

$$((s-1)(1-e^{i2\pi(s-1)})\Gamma(s)\zeta^*(s))' = \frac{d}{ds}(s-1)\frac{1}{1-2^{1-s}}\int_C t^{s-1}/(e^t+1)$$

$$= \int_C t^{s-1}/(e^t+1)dt \cdot \frac{d}{ds}(s-1)\frac{1}{1-2^{1-s}} + (s-1)^2\frac{1}{1-2^{1-s}} \cdot \int_C t^{s-2}/(e^t+1)dt$$

$$((s-1)(1-e^{i2\pi(s-1)})\Gamma(s)\zeta^*(s))'|_1 = 0$$

The fact is that for the definition of Riemann: $(s-1)\zeta(s)$, though it's continuous and derivable of any order at $R(s) = 1$, it's not analytic i.e. can't be expanded as power series.

References

[1] E.C. Titchmarsh, The theory of the Riemann zeta function, Oxford University Press, 2nd ed, 1986.

[2] M. A. Lavrentieff, B. V. Shabat, Methods of functions of a complex variable, Russia, 2002

Wuhan university, Wuhan, China.

E-mail address: hiyaho@126.com