THE ANALYTIC PROPERTY FOR RIEMANN ZETA FUNCTION

WU SHENG-PING

ABSTRACT. This article discusses the analytic property of Riemann zeta function. The popular opinion is denied.

1. Introduction

$\zeta(s)$ [1]is defined (by Riemann) as:

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t - 1)dt \tag{1.1}$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people think this function is analytic except $s = 1$[1]. There is still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, R(s) > 0 \tag{1.2}$$

Someone deduced applaudably

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \frac{1}{1 - 2^{1-s}} \int_{C=} t^{s-1}/(e^t + 1)dt \tag{1.3}$$

This expression is analytic except $s = 1$, If the Riemann's definition is also analytic the definitions 1.1 and 1.2 should be identical. In this article the analytic property in $R(s) = 0$ is discussed.

2. Discussion

Theorem 2.1. *The second definition of $\zeta(s)$ has divergent derivation near $s = 0$.*

Proof.

$$F_m(s) := \sum_{n=1}^{m} (-1)^{n+1} n^{-s}, R(s) > 0$$

Set $s \in (0, 1)$.

$$F'_m(s) = \sum_{n>0,2|n+1}^{\infty} \ln(n)(sn^{-s-1} - n^{-s-2}s(s+1)\theta/2), 0 < \theta < 1$$

$$F'_m(s) > \frac{1}{2} \int_{n=3}^{m-2} \ln(n)n^{-s-1}sdn - \sum_{n>0,2|n+1}^{\infty} \ln(n)n^{-s-2}s(s+1)\theta/2$$

Date: June 4, 2010.

2000 *Mathematics Subject Classification.* Primary 11M06.

Key words and phrases. Riemann zeta function, analytic continuation, Cauchy integral formula.

$$\lim_{m\to\infty} F'_m(s) = \frac{1}{2}\int_3^{\infty} s\ln(x)x^{-s-1}dx - C_s, |C_s| < C$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} sxe^{-sx}dx - C_s$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} \frac{sxe^{-sx}d(sx)}{s} - C_s$$

It's easy to find when $s \to 0$ this term approaches to infinity. □

There is coming up sharp controversy, as is commonly known the $\zeta(s)$ hasn't infinity derivation in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition.

The first probe is on that the proposition for analytic function $f(x)$, the similarity of the middle value theorem, like

$$\exists z'\forall z(f(z) - f(0) = f'(z')z), |z'| \leq |z|, |z| < r$$

is invalid. Here is a counterexample

$$f(x) = (1 + x)^a, a \in \mathbf{R}$$

If a is great it can be found a little $x_1, r : |x_1| < r$

$$\ln(f(x_1)) - \ln(f(\overline{x}_1)) = 2\pi i, f(x_1) = f(\overline{x}_1)$$

but the derivation zero is at

$$a(1 + x)^{a-1} = 0$$

It's impossible on $|x| \leq 3r$.

The second probe is on the definition of integration of real function $f(x, y)$ on piecewise smooth curve C with the length finite.

Theorem 2.2. *If define the integration real $f(x_1, x_2)$ on $C(x_1, x_2)$ is limit about Δ of the sum of $f(x_1, x_2)\Delta C$ or $f(x_1, x_2)||\Delta C||$, then a secure condition on the validity of this definition is that $f(x_1, x_2)$ is continuous uniformly.*

This is of no speciality.

The third probe is on Cauchy Integral Formula that said derivable function is analytic. Integrations about the complex derivable function $f(x)$ in the considered domain meet

$$f(x) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z - x}dz$$

$$f^{(n)}(x) = \frac{n!}{2\pi i}\oint_{C'} \frac{f(z)}{(z - x)^{n+1}}dz$$

$$C' = re^{i\theta} - x, 0 \leq \theta < 2\pi$$

Its famous proof of Cauchy's [2] is incorrect. The first reason is that the limit of integral contours $r \to 0$ causes the integrated is of no bound (hence not continuous) uniformly. The second is the similarity of middle value theorem is invalid any longer. In fact the reasonable calculation is like

$$\lim_{r\to 0} \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z - x}dz - f(x)$$

$$\lim_{r\to 0} \frac{1}{2\pi i}\oint_{C'} \frac{f(z) - f(x)}{z - x}dz$$

find the middle in real domain

$$\lim_{r\to 0}\frac{1}{2\pi i}\oint_{C'}\frac{F(x,z)(z-x)}{z-x}dz$$
$$=\lim_{r\to 0}\frac{1}{2\pi i}\oint_{C'}F(x,z)dz$$

If F is bounded uniformly at neighborhood

$$=0$$

A direct violation to the style of Cauchy's on the conclusion section.

Now it is critical to define the conception of Analytic as expandable to power series. French mathematician J.Dieudonne use this definition consistently in his famous book "Foundations of Modern Analysis". Following this definition of conception of analytic Cauchy integral formula is all right.

Maybe somebody had said

Theorem 2.3. *A smooth complex function is analytic if both its real part and imaginary part are analytic.*

It's obvious.

3. Conclusion

I have several reasons to denied the popular knowledge,
1)$\zeta(s)$ has infinite derivation at $s=0$. it can be verified in integral form on contours $C_1C_3, r=0$ by our noticing that

$$\int_C f(x)dx := \int_C \ln(x)x^{\delta-1}/(1+e^x)dx$$
$$\neq \int_{C_1,C_2,r=0}\ln(x)x^{\delta-1}/(1+e^x)dx = \frac{1}{\delta^2}\int_{C_1,C_2,r=0}\ln(x)/(1+e^{x^{1/\delta}})dx$$

The reason is explained by the probe second, there is another one

$$\int_S df(x)\wedge dx, df(x)$$

S is the measurable area enclosed by $(C, r=R)$ and $(C, r=0)$ (including this one), in which the integration is not well defined. In fact the integral formula of $\zeta(s)$ on contour C is all under critic.
3)The item 2) evinces that at $R(s)>0$ the zeta function should be not all analytic. (why? the supposed entire function, whose convergent radium calculated at near $s=0$ and others are not compact)

References

[1] E.C. Titchmarsh, The theory of the Riemann zeta function, Oxford University Press, 2nd ed, 1986.
[2] M. A. Lavrentieff, B. V. Shabat, Methods of functions of a complex variable, Russia, 2002

Wuhan university, Wuhan, China.
E-mail address: hiyaho@126.com