THE ANALYTIC PROPERTY FOR RIEMANN ZETA FUNCTION

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Abstract. This article discusses the analytic property of Riemann zeta function. The popular opinion is denied.

1. Introduction

$\zeta(s)$ [1]is defined (by Riemann) as:

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t - 1)dt \tag{1.1}$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people think this function is analytic except $s = 1$[1]. There is still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, R(s) > 0 \tag{1.2}$$

Someone deduced applaudably

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \frac{1}{1 - 2^{1-s}} \int_C t^{s-1}/(e^t + 1)dt \tag{1.3}$$

This expression is identical to Riemann's definition. In this article the analytic property is discussed.

2. Discussion

Theorem 2.1. *The second definition of $\zeta(s)$ has divergent derivation near $s = 0$.*

Proof.

$$F_m(s) := \sum_{n=1}^{m} (-1)^{n+1} n^{-s}, R(s) > 0$$

Set $s \in (0, 1)$.

$$F'_m(s) = \sum_{n>0,2|n+1}^{\infty} \ln(n)(sn^{-s-1} - n^{-s-2}s(s+1)\theta/2), 0 < \theta < 1$$

$$F'_m(s) > \frac{1}{2} \int_{n=3}^{m-2} \ln(n)n^{-s-1}sdn - \sum_{n>0,2|n+1}^{\infty} \ln(n)n^{-s-2}s(s+1)\theta/2$$

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$$\lim_{m\to\infty} F'_m(s) = \frac{1}{2}\int_3^\infty s\ln(x)x^{-s-1}dx - C_s, |C_s| < C$$
$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^\infty sxe^{-sx}dx - C_s$$
$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^\infty \frac{sxe^{-sx}d(sx)}{s} - C_s$$

It's easy to find when $s \to 0$ this term approaches to infinity. □

There is coming up sharp controversy, as is commonly known the $\zeta(s)$ hasn't infinity derivation in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition.

The first probe is on that the proposition for analytic function $f(x)$, the similarity of the middle value theorem, like

$$\exists z'\forall z(f(z) - f(0) = f'(z')z), |z'| \le |z|, |z| < r$$

is invalid. Here is a counterexample

$$f(x) = (1 + x)^a, a \in \mathbf{R}$$

If a is great it can be found a little $x_1, r : |x_1| < r$

$$\ln(f(x_1)) - \ln(f(\overline{x}_1)) = 2\pi i, f(x_1) = f(\overline{x}_1)$$

but the derivation zero is at

$$a(1 + x)^{a-1} = 0$$

It's impossible on $|x| \le 3r$.

The second probe is on the definition of integration of real function $f(x, y)$ on curve C.

Theorem 2.2. $l \in [0, c]$ *is the parametrization of the length of piecewise smooth curve* $C(l)$*. If define the integration of the real* $f(x, y)$ *on* $C(l)$ *is the limit of the sum of* $f(x, y)\Delta C$ *or* $f(x, y)||\Delta C||$ *when* $\Delta l \to 0$*. then this limit exists if* $f(x, y)$ *is continuous uniformly in a neighboring set of* C*.*

This is nothing special.

The third probe is on Cauchy Integral Formula that said derivable function is analytic. Integrations about the complex derivable function $f(x)$ in the considered domain meet

$$f(x) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z - x}dz$$
$$f^{(n)}(x) = \frac{n!}{2\pi i}\oint_{C'} \frac{f(z)}{(z - x)^{n+1}}dz$$
$$C' = re^{i\theta} - x, 0 \le \theta < 2\pi$$

Its famous proof of Cauchy's [2] is incorrect. The first reason is that the limit of integral contours $r \to 0$ causes the integrated is unbounded (hence not continuous) uniformly. The second is the similarity of middle value theorem is invalid any longer. In fact the reasonable calculation is like

$$\lim_{r\to 0} \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z - x}dz - f(x)$$
$$\lim_{r\to 0} \frac{1}{2\pi i}\oint_{C'} \frac{f(z) - f(x)}{z - x}dz$$

find the middle in real domain

$$= \lim_{r\to 0} \frac{1}{2\pi i} \oint_{C'} \frac{F(x,z)(z-x)}{z-x} dz$$

$$= \lim_{r\to 0} \frac{1}{2\pi i} \oint_{C'} F(x,z) dz$$

If F is bounded uniformly at neighborhood

$$= 0$$

A direct demonstration of the mistake of Cauchy's style is in the conclusion section.

Now it is critical to define the conception of Analytic as expandable to power series. French mathematician J.Dieudonne use this definition consistently in his famous book "Foundations of Modern Analysis". Following this definition of conception of analytic Cauchy integral formula is all right.

Maybe somebody had said

Theorem 2.3. *A smooth complex function is analytic if and only if both its real part and its imaginary part are analytic.*

It's obvious.

3. Conclusion

I have several reasons to denied the popular knowledge,

1)$\zeta(s)$ that defined by 1.2 has infinite derivation at $s = 0$. it can be verified in integral form on contours $C_1C_3, r = 0$ by our noticing that

$$\int_C f(x)dx := \int_C \ln(x)x^{\delta-1}/(1+e^x)dx$$

$$\neq \int_{C_1,C_2,r=0} \ln(x)x^{\delta-1}/(1+e^x)dx = \frac{1}{\delta^2}\int_{C_1^\delta,C_2^\delta,r=0} \ln(x)/(1+e^{x^{1/\delta}})dx$$

The reason is explained by the probe second, here is another explanation

$$\int_S df(x)\wedge dx, df(x)$$

S is the measurable area enclosed by $(C, r = R)$ and $(C, r = 0)$ (including this one), in which the integration is not well defined. In fact the integral formula of $\zeta(s)$ on the contour C is all under critics.

2)It's necessary to study the convergence of real 2-arguments power series $f(x, y)$, as a fact the convergent radius (r_1, r_2) respectively for arguments (x, y) meet

$$|\frac{\partial^{m+n} f(x,y)}{(\partial x)^m(\partial y)^n}| < m!n!r_1^{-m}r_2^{-n}$$

Study the real derivation of $\Gamma(s = x + yi), x > 0, y \in \mathbf{R}$ for sufficiently great m, n

$$|\Gamma^{(m,n)}(s)| > C(m+n)!|x_0 + y_0 i|^{-m-n}$$

Applying the precedent formula, one can find the convergent multi-cylinder is close to zero. This means the $\Gamma(s)$ is not analytic at all.

3)By calculating on the series expression of $\zeta(s)$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, R(s) > 1$$

one also can find that $\zeta(s)$ is not analytic on $R(s) > 1$.

References

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