THE ANALYTIC PROPERTY FOR RIEMANN ZETA FUNCTION

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ABSTRACT. This article discusses the analytic property of Riemann zeta function. The popular opinion is denied.

1. INTRODUCTION

$\zeta(s)$ [1]is defined (by Riemann) as:

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t - 1)dt \tag{1.1}$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people think this function is analytic except $s = 1$[1]. There is still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, R(s) > 0 \tag{1.2}$$

Someone deduced applaudably

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \frac{1}{1 - 2^{1-s}} \int_C t^{s-1}/(e^t + 1)dt \tag{1.3}$$

This expression is identical to Riemann's definition. In this article the analytic property is discussed.

2. DISCUSSION

Theorem 2.1. *The second definition of $\zeta(s)$ has divergent derivative near $s = 0$.*

Proof.

$$F_m(s) := \sum_{n=1}^{m} (-1)^{n+1} n^{-s}, R(s) > 0$$

Set $s \in (0, 1)$.

$$F_m'(s) = \sum_{n>0,2|n+1}^{\infty} \ln(n)(sn^{-s-1} - n^{-s-2}s(s+1)\theta/2), 0 < \theta < 1$$

$$F_m'(s) > \frac{1}{2} \int_{n=3}^{m-2} \ln(n) n^{-s-1} s dn - \sum_{n>0,2|n+1}^{\infty} \ln(n) n^{-s-2} s(s+1)\theta/2$$

Date: June 4, 2010.
2000 *Mathematics Subject Classification.* Primary 11M06.
Key words and phrases. Riemann zeta function, analytic continuation, Cauchy integral formula.

$$\lim_{m\to\infty} F'_m(s) = \frac{1}{2}\int_3^\infty s\ln(x)x^{-s-1}dx - C_s, |C_s| < C$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^\infty sxe^{-sx}dx - C_s$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^\infty \frac{sxe^{-sx}d(sx)}{s} - C_s$$

It's easy to find when $s \to 0$ this term approaches to infinity. □

There is coming up sharp controversy, as is commonly known the $\zeta(s)$ hasn't infinity derivative in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition.

The first probe is on that the proposition for analytic function $f(x)$, the similarity of the middle value theorem, like

$$\exists z'\forall z(f(z) - f(0) = f'(z')z), |z'| \leq |z|, |z| < r$$

is invalid. Here is a counterexample

$$f(x) = (1 + x)^a, a \in \mathbf{R}$$

If a is great it can be found a little $x_1, r : |x_1| < r$

$$\ln(f(x_1)) - \ln(f(\overline{x}_1)) = 2\pi i, f(x_1) = f(\overline{x}_1)$$

but the derivative zero is at

$$a(1 + x)^{a-1} = 0$$

It's impossible on $|x| \leq 3r$.

The second probe is on the definition of integration of real function $f(x, y)$ on curve C.

Theorem 2.2. *$l \in [0, c]$ is the parametrization of the length of piecewise smooth curve $C(l)$. l is divided into the collection of Δl. If the integration of the real $f(x, y)$ on $C(l)$ is defined as the limit of the sum of $f(x, y)\Delta C$ or $f(x, y)||\Delta C||$ when $\Delta l \to 0$. then this limit exists if $f(x, y)$ is continuous uniformly in a neighboring set of C.*

This is nothing special.

The third probe is on Cauchy Integral Formula that said derivable function is analytic. Integrations about the complex derivable function $f(x)$ in the considered domain meet

$$f(x) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z - x}dz$$

$$f^{(n)}(x) = \frac{n!}{2\pi i}\oint_{C'} \frac{f(z)}{(z - x)^{n+1}}dz$$

$$C' = re^{i\theta} - x, 0 \leq \theta < 2\pi$$

Its famous proof of Cauchy's [2] is incorrect. The first reason is that the limit of integral contours $r \to 0$ causes the integrated is unbounded (hence not continuous) uniformly. The second is the similarity of middle value theorem is invalid any longer. In fact the reasonable calculation is like

$$\lim_{r\to 0} \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z - x}dz - f(x)$$

$$\lim_{r\to 0}\frac{1}{2\pi i}\oint_{C'}\frac{f(z)-f(x)}{z-x}dz$$

find the middle in real domain

$$=\lim_{r\to 0}\frac{1}{2\pi i}\oint_{C'}\frac{F(x,z)(z-x)}{z-x}dz$$

$$=\lim_{r\to 0}\frac{1}{2\pi i}\oint_{C'}F(x,z)dz$$

If F is bounded uniformly at neighborhood

$$=0$$

Here is a direct illustration of the mistake of Cauchy's style

$$\int_C f(x)dx := \int_C \ln(x)x^{\delta-1}/(1+e^x)dx$$

$$\neq \int_{C_1,C_2,r=0} \ln(x)x^{\delta-1}/(1+e^x)dx$$

$$=\frac{1}{\delta^2}\int_{C_1^\delta,C_2^\delta,r=0} \ln(x)/(1+e^{x^{1/\delta}})dx$$

The reason is explained by the probe second. There is another explanation

$$\int_S df(x)\wedge dx, df(x)$$

S is the measurable area enclosed by $(C, r=R)$ and $(C, r=0)$ (including this one), in which the integration is not well defined.

Now it is critical to define the conception of Analytic as expandable to power series. French mathematician J.Dieudonne use this definition consistently in his famous book "Foundations of Modern Analysis".

Maybe somebody had said

Theorem 2.3. *A smooth complex function is analytic if and only if both its real part and its imaginary part are analytic.*

It's obvious.

3. Conclusion

I have several reasons to denied the popular knowledge.

1)It's necessary to study the convergence of real 2-arguments power series $f(x,y)$. As a fact the absolutely convergent radius (r_1,r_2) respectively for arguments (x,y) meet for sufficiently great m,n

$$|\frac{\partial^{m+n}f(x,y)}{(\partial x)^m(\partial y)^n}| < C'm!n!r_1^{-m}r_2^{-n}$$

C' is independent of m,n. The real derivatives of $\Gamma(s=x+yi), x>0, y\in\mathbf{R}$ are studied for sufficiently great m,n:

$$|\Gamma^{(m,n)}(s)|_{s=x_0+iy_0}| > C(m+n)!|x_0+y_0i|^{-m-n}, C>0$$

C is independent of m, n. The remainder term of a real function's Taylor expansion is considered

$$\lim_{k\to\infty}\prod_{1}^{k}\int_{x_0}^{x} g^{(k)}(x)d^k x = \lim_{k\to\infty} g^{(k)}(x_k)(x-x_0)^k/k! \to 0 \tag{3.1}$$

This condition is invalid for $R(\Gamma(x+ilx))$ in some case

$$x_0 > x > 0, x_k \in (x, x_0), l = 0$$

Thus the equation of 3.1 is invalid. Hence one can find the analytic property is invalid on $R(s) > 0$.
2)By calculating on $\zeta(s)$, one also can find that $\zeta(s)$ is not analytic on $R(s) > 0$.
3)Obviously the analytic continuation is unique, however, if the term "entirely analytic on set S" is means a power series can express the function for all members of S, one may ask whether a circle of local analytic function is also entirely analytic. This is a new problem after classical Cauchy integral formula is valid no longer, unsettled by me. For real function this proposition is not real:

$$1/(1+ix), x \in \mathbf{R}$$

References

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