THE ANALYTIC PROPERTY FOR RIEMANN ZETA FUNCTION

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Abstract. This article discusses the analytic property of Riemann zeta function. The popular opinion is denied.

1. Introduction

$\zeta(s)$ [1]is defined (by Riemann) as:

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t - 1)dt \tag{1.1}$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people think this function is analytic except $s = 1$[1]. There is still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta^*(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, R(s) > 0 \tag{1.2}$$

Someone deduced applaudably

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \frac{1}{1-2^{1-s}} \int_C t^{s-1}/(e^t + 1)dt \tag{1.3}$$

This expression is identical to Riemann's definition. In this article the analytic property is discussed.

2. Discussion

Theorem 2.1. *The second definition of $\zeta(s)$ has divergent derivative near $s = 0$.*

Proof.

$$F_m(s) := \sum_{n=1}^{m} (-1)^{n+1} n^{-s}, R(s) > 0$$

Set $s \in (0, 1)$.

$$F'_m(s) = \sum_{n>0,2|n+1}^{\infty} \ln(n)(sn^{-s-1} - n^{-s-2}s(s+1)\theta/2), 0 < \theta < 1$$

$$F'_m(s) > \frac{1}{2}\int_{n=3}^{m-2} \ln(n)n^{-s-1}sdn - \sum_{n>0,2|n+1}^{\infty} \ln(n)n^{-s-2}s(s+1)\theta/2$$

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$$\lim_{m\to\infty} F'_m(s) = \frac{1}{2}\int_3^\infty s\ln(x)x^{-s-1}dx - C_s, |C_s| < C$$
$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^\infty sxe^{-sx}dx - C_s$$
$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^\infty \frac{sxe^{-sx}d(sx)}{s} - C_s$$

It's easy to find when $s \to 0$ this term approaches to infinity. □

There is coming up sharp controversy, as is commonly known the $\zeta(s)$ hasn't infinity derivative in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition.

The first probe is on that the proposition for analytic function $f(x)$, the similarity of the middle value theorem, like

$$\exists z'\forall z(f(z) - f(0) = f'(z')z), |z'| \le |z|, |z| < r$$

is invalid. Here is a counterexample

$$f(x) = (1 + x)^a, a \in \mathbf{R}$$

If a is great it can be found a little $x_1, r : |x_1| < r$

$$\ln(f(x_1)) - \ln(f(\overline{x}_1)) = 2\pi i, f(x_1) = f(\overline{x}_1)$$

but the derivative zero is at

$$a(1 + x)^{a-1} = 0$$

It's impossible on $|x| \le 3r$.

The second probe is on the definition of integration of real function $f(x, y)$ on curve C.

Theorem 2.2. *$l \in [0, c]$ is the parametrization of the length of piecewise smooth curve $C(l)$. l is divided into the collection of Δl. If the integration of the real $f(x, y)$ on $C(l)$ is defined as the limit of the sum of $f(x, y)\Delta C$ or $f(x, y)||\Delta C||$ when $\Delta l \to 0$. then this limit exists if $f(x, y)$ is continuous uniformly in a neighboring set of C.*

This is nothing special.

The third probe is on Cauchy Integral Formula that said derivable function is analytic. Integrations about the complex derivable function $f(x)$ in the considered domain meet

$$f(x) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z - x}dz$$
$$f^{(n)}(x) = \frac{n!}{2\pi i}\oint_{C'} \frac{f(z)}{(z - x)^{n+1}}dz$$
$$C' = re^{i\theta} - x, 0 \le \theta < 2\pi$$

Its famous proof of Cauchy's [2] is incorrect. The first reason is that the limit of integral contours $r \to 0$ causes the integrated is unbounded (hence not continuous) uniformly. The second is the similarity of middle value theorem is invalid any longer. In fact the reasonable calculation is like

$$\lim_{r\to 0} \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z - x}dz - f(x)$$

$$\lim_{r\to 0}\frac{1}{2\pi i}\oint_{C'}\frac{f(z)-f(x)}{z-x}dz$$

find the middle in real domain

$$=\lim_{r\to 0}\frac{1}{2\pi i}\oint_{C'}\frac{F(x,z)(z-x)}{z-x}dz$$

$$=\lim_{r\to 0}\frac{1}{2\pi i}\oint_{C'}F(x,z)dz$$

If F is bounded uniformly at neighborhood

$$=0$$

Here is a direct illustration of the mistake of Cauchy's style

$$\int_C f(x)dx := \int_C \ln(x)x^{\delta-1}/(1+e^x)dx$$

$$\neq \int_{C_1,C_2,r=0}\ln(x)x^{\delta-1}/(1+e^x)dx$$

$$=\frac{1}{\delta^2}\int_{C_1^\delta,C_2^\delta,r=0}\ln(x)/(1+e^{x^{1/\delta}})dx$$

The reason is explained by the probe second. There is another explanation

$$\int_S df(x)\wedge dx, df(x)$$

S is the measurable area enclosed by $(C, r=R)$ and $(C, r=0)$ (including this one), in which the integration is not well defined.

Now it is critical to define the conception of Analytic as expandable to power series. French mathematician J.Dieudonne use this definition consistently in his famous book "Foundations of Modern Analysis".

Maybe somebody had said

Theorem 2.3. *A smooth complex function is analytic if and only if both its real part and its imaginary part are analytic.*

It's obvious.

3. Conclusion

I have several reasons to denied the popular knowledge.

Obviously the analytic continuation is unique, however, if the term "entirely analytic on circle S" is means a power *2-dimensional* series expansion on the center of S can express the function for all members of S, one may ask whether a circle of locally analytic function is also entirely analytic. This is a new problem after classical Cauchy integral formula is valid no longer. For real function this proposition is not real:

$$1/(1+ix), x\in\mathbf{R}$$

The convergence of real 2-arguments power series $f(x,y)$ is studied. As a fact the convergent radius (r_1, r_2) respectively for arguments (x,y) meet for sufficiently great m,n

$$\left|\frac{\partial^{m+n}f(x,y)}{(\partial x)^m(\partial y)^n}\right| < C'm!n!r_1^{-m}r_2^{-n}$$

C' is independent of m, n. The real derivatives of $\Gamma(s = x + yi), x > 0, y \in \mathbf{R}$ are studied for sufficiently great m, n:

$$|\Gamma^{(m,n)}(s)|_{s=x_0+iy_0}| > C(m+n)!|x_0 + y_0 i|^{-m-n}, C > 0$$

C is independent of m, n. From popular knowledge the function's expansion at $s + 0.0001, s > 0.1$ converges at $s(1 - e^{i\theta})$, θ is little and positive, the function must absolutely converges at this point, but this point is out of the absolutely convergent radius. The reason why the analysis by module norm is different is simply that there exists predisposition of minus operation cross the degrees, hence it doesn't produce limit of power series. This means $\Gamma(s)$ is locally analytic on $R(s) > 0$ but not entirely analytic in suitable circle in the analytic domain. There is another simple example

$$1/(1-x) = \sum_{i=0}^{\infty} x^n, |x| < 1, x = \alpha + y, 0 < \alpha < 1$$

$x = \alpha + y$ is substituted into the series and the series of y is obtained with its coefficient of x is infinite. In the other words: if an analytic power series is shifted in the classically convergent domain, a divergent series is possibly obtained.

It's defined that

$$\Gamma^*(s) = \frac{1}{1 - e^{i2\pi(s-1)}} \int_C x^{s-1} e^{-x} dx$$

The analytic properties of $\Gamma(s), \Gamma^*(s)$ at $s = 0$ are calculated. It's known that

$$s\Gamma(s) = \Gamma(s+1)$$

near $s = 0$

$$\Gamma(s) \neq \Gamma^*(s)$$

But for some s

$$s\Gamma^*(s) \neq \Gamma^*(s+1)$$

The reason of this difference from the classical complex analysis depends on the different norms: The first is real norm

$$||A||_R > |R(A)| + |Im(A)|, ||A||_1 = \inf(||A||_R)$$

If derivation is in consideration

$$||df(z)/dz||_R > ||\partial f(z)/\partial(R(z))||_1 + ||\partial f(z)/\partial Im(z)||_1 \tag{3.1}$$

The other is the module norm

$$||A||_m$$

Real norm is stronger than module norm. The integration theories depending on the different norms and thus on different absolute convergence conceptions become diverse. The *real absolute convergence value* $||f(z)||_b$ is defined as

$$||f(Z)||_R > ||\int_D f(Z)dZ||_b = \int_D |R(f(Z))| \cdot ||dZ|| + \int_D |Im(f(Z))| \cdot ||dZ||$$

X is defined in $\mathbf{C^n}$. This definition is on the true absolute convergence, which is possibly bounded means the integral limit is free of sequence of addition. As a fact the Stocks Formula operates on real function so that it's right to connect to it real norm with consideration of derivation. The norm of the derivation of $x^{\delta-1}e^{-x}$ in the integration is approximately

$$\int C'/r^{2-\delta}dr$$

or

$$\int\int C'r^{\delta-2}dxdy > c|\int\int z^{\delta-2}dzd\overline{z}|$$

It's clear that this value is unbounded.

The case for zeta is similar between $\zeta(s)$ and $\zeta^*(s)$. $\zeta^*(s)$ is analytic at $R(s) > 0$.

References

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