# THE ANALYTIC PROPERTY FOR RIEMANN ZETA FUNCTION

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Abstract. This article discusses the analytic property of Riemann zeta function. The popular opinion is denied.

## 1. Introduction

$\zeta(s)$ [1]is defined by Riemann as:

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t - 1)dt \tag{1.1}$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people think this function is analytic except $s = 1$[1]. There is still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta^*(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, \Re(s) > 0 \tag{1.2}$$

Someone deduced applaudably

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \frac{1}{1 - 2^{1-s}} \int_C t^{s-1}/(e^t + 1)dt \tag{1.3}$$

This expression is thought identical to Riemann's definition. In this article the analytic property is discussed.

## 2. Discussion

**Theorem 2.1.** *The second definition of $\zeta(s)$ has divergent derivative near $s = 0$.*

*Proof.*

$$F_m(s) := \sum_{n=1}^{m} (-1)^{n+1} n^{-s}, \Re(s) > 0$$

Set $s \in (0, 1)$.

$$F'_m(s) = \sum_{n>0,2|n+1}^{\infty} \ln(n)(sn^{-s-1} - n^{-s-2}s(s+1)\theta/2), 0 < \theta < 1$$

$$F'_m(s) > \frac{1}{2} \int_{n=3}^{m-2} \ln(n)n^{-s-1}sdn - \sum_{n>0,2|n+1}^{\infty} \ln(n)n^{-s-2}s(s+1)\theta/2$$

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$$\lim_{m\to\infty} F'_m(s) = \frac{1}{2}\int_3^\infty s\ln(x)x^{-s-1}dx - C_s, |C_s| < C$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^\infty sxe^{-sx}dx - C_s$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^\infty \frac{sxe^{-sx}d(sx)}{s} - C_s$$

It's easy to find when $s \to 0$ this term approaches to infinity. □

There is coming up sharp controversy, as is commonly known the $\zeta(s)$ hasn't infinity derivative in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition.

The first probe is on that the proposition for analytic function $f(x)$, the similarity of the middle value theorem, like

$$\exists z' \forall z (f(z) - f(0) = f'(z')z), |z'| \le |z|, |z| < r$$

is invalid. Here is a counterexample

$$f(x) = (1 + x)^a, a \in \mathbf{R}$$

If $a$ is great it can be found a little $x_1, r : |x_1| < r$

$$\ln(f(x_1)) - \ln(f(\overline{x}_1)) = 2\pi i, f(x_1) = f(\overline{x}_1)$$

but the derivative zero is at

$$a(1 + x)^{a-1} = 0$$

It's impossible on $|x| \le 3r$.

The second probe is on the definition of integration of real function $f(x, y)$ on curve $C$.

**Theorem 2.2.** *$l \in [0, c]$ is the parametrization of the length of piecewise smooth curve $C(l)$. $l$ is divided into the collection of $\Delta l$. If the integration of the real $f(x, y)$ on $C(l)$ is defined as the limit of the sum of $f(x, y)\Delta C$ or $f(x, y)||\Delta C||$ when $\Delta l \to 0$. then this limit exists if $f(x, y)$ is continuous uniformly in a neighboring set of $C$.*

This is nothing special.

The third probe is on Cauchy Integral Formula that said derivable function is analytic. Integrations about the complex derivable function $f(x)$ in the considered domain meet

$$f(x) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z - x}dz$$

$$f^{(n)}(x) = \frac{n!}{2\pi i}\oint_{C'} \frac{f(z)}{(z - x)^{n+1}}dz$$

$$C' = re^{i\theta} - x, 0 \le \theta < 2\pi$$

Its famous proof of Cauchy's [2] is incorrect.

The first reason is that the limit of integral contours $r \to 0$ causes the integrated is unbounded (hence not continuous) uniformly in $0 < r < R$. Cauchy's first integral formula is interpreted as the following in effective

$$\lim_{r\to 0}\lim_{\Delta\theta\to 0}\sum_{\Delta\theta} g\Delta\theta = \lim_{\Delta\theta\to 0}\lim_{r\to 0}\sum_{\Delta\theta} g\Delta\theta$$

This implies uniform convergence of $\lim_{\Delta\theta\to 0}$ in $0 < r < R$. $\lim_{\Delta\theta\to 0}$ means integration.

Now it is critical to define the conception of Analytic as expandable to power series. French mathematician J.Dieudonne use this definition consistently in his famous book "Foundations of Modern Analysis".

**Definition 2.3.** 2-dimensional power series $f(z)$ of real arguments as $\Re(z), \Im(z)$, its convergence is called wide convergence. In the 2-dimensional series, the same degree terms is combined as one term to form an 1-dimensional series, convergence of which is called narrow convergence. Complex smooth and narrowly convergent power series is narrowly analytic, Complex smooth and widely convergent power series is widely analytic.

Maybe somebody had said

**Theorem 2.4.** *A smooth complex function is analytic if and only if both its real part and its imaginary part have narrow convergent power series.*

It's obvious that narrowly analytic is analytic.

## 3. Conclusion

I have several reasons to denied the popular knowledge.

Obviously the analytic continuation is unique, however, if the term "entirely analytic function in a circle" means that its power series expansion at the center covers the whole circle , one may ask whether a circle of locally analytic function is also entirely analytic. This is a new problem after classical Cauchy integral formula is valid no longer. For real function this proposition is negative:

$$1/(1 + ix), x \in \mathbf{R}$$

The wide convergence of real 2-arguments power series $f(x, y)$ is studied. As a fact a convergent radius $(r_1, r_2)$ respectively for arguments $(x, y)$ meet for sufficiently great $m, n$

$$\left|\frac{\partial^{m+n} f(x, y)}{(\partial x)^m (\partial y)^n}\right| < C' m!n! r_1^{-m} r_2^{-n}$$

$C'$ is independent of $m, n$. The real derivatives of $\Gamma(s = x + yi), x > 0, y \in \mathbf{R}$ are studied for sufficiently great $m, n$:

$$|\Gamma^{(m,n)}(s)|_{s=x_0+iy_0}| > C(m + n)!|x_0 + y_0 i|^{-m-n}, C > 0$$

Obviously the convergent areas for the power series of $s$ or $(\Re(s), \Im(s))$ are distinct. The widely analytic is stronger.

The integration by parts is studied

$$\int_a^b df(x)g(x) = \int_a^b g df(x) + \int_a^b f(x)dg(x)$$

Its definitional micro sum is like

$$\lim_{\Delta x\to 0} \sum_{\Delta x} \Delta f g = \lim_{\Delta x\to 0} \sum_{\Delta x} (g\Delta f + f\Delta g + \Delta f \cdot \Delta g)$$

This case inclines to

$$\lim_{\Delta x\to 0} \sum_{\Delta x} \Delta f \cdot \Delta g = 0 \tag{3.1}$$

If derivation of the functions $f, g$ is bounded the identity 3.1 is right, otherwise, it's hard to say anything. Here is an counterexample

$$\int_0^x e^{-x} dx^{1/2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1/2}}{n!(n+1/2)}$$

$$x^{1/2}e^{-x} + \int_0^x x^{1/2}e^{-x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1/2}}{n!} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n+1/2}}{(n-1)!(n+1/2)}$$

Obviously

$$\int_0^x e^{-x} dx^{1/2} \neq x^{1/2}e^{-x} + \int_0^x x^{1/2}e^{-x} dx$$

$\int_{C_2} x^{s-1}e^{-x}dx$ is studied by the method of integration by parts, and the singularity like that in

$$dg(x)|_{C_2(\Re(x))}$$

is identified at $re^{i\pi/2}$. When $C_2$ is parameterized by it's length the integration is without singularity, but the integral definition of

$$\int_{C_2} g_1(x)d\Re(x) + g_2(x)d\Im(x)$$

should be compact to any parametrization of $C_2$.

$\zeta(s), \Gamma(s)$ on $\Re(s) \leq 1$ is of very bad properties and not all analytic.

## References

[1] E.C. Titchmarsh, The theory of the Riemann zeta function, Oxford University Press, 2nd ed, 1986.

[2] M. A. Lavrentieff, B. V. Shabat, Methods of functions of a complex variable, Russia, 2002

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