THE ANALYTIC PROPERTY FOR RIEMANN ZETA FUNCTION

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Abstract. This article discusses the analytic property of Riemann zeta function. The popular opinion is denied.

1. Introduction

$\zeta(s)$ is originally

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \Re(s) > 1$$

It is continuated by Riemann as:

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t - 1)dt \tag{1.1}$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people think this function is analytic except $s = 1$[1]. There is still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta^*(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, \Re(s) > 0 \tag{1.2}$$

This is a continuation of $\zeta(s)$. Someone deduced applaudably

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \frac{1}{1 - 2^{1-s}} \int_C t^{s-1}/(e^t + 1)dt \tag{1.3}$$

This expression is thought identical to Riemann's definition. In this article the analytic property is discussed.

2. Discussion

Theorem 2.1. *The second definition of $\zeta(s)$ has divergent derivative near $s = 0$.*

Proof.

$$F_m(s) := \sum_{n=1}^{m} (-1)^{n+1} n^{-s}, \Re(s) > 0$$

Set $s \in (0, 1)$.

$$F'_m(s) = \sum_{n>0,2|n+1}^{\infty} \ln(n)(sn^{-s-1} - n^{-s-2}s(s+1)\theta/2), 0 < \theta < 1$$

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$$F'_m(s) > \frac{1}{2}\int_{n=3}^{m-2} \ln(n)n^{-s-1}sdn - \sum_{n>0,2|n+1}^{\infty} \ln(n)n^{-s-2}s(s+1)\theta/2$$

$$\lim_{m\to\infty} F'_m(s) = \frac{1}{2}\int_{3}^{\infty} s\ln(x)x^{-s-1}dx - C_s, |C_s| < C$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} sxe^{-sx}dx - C_s$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} \frac{sxe^{-sx}d(sx)}{s} - C_s$$

It's easy to find when $s \to 0$ this term approaches to infinity. □

There is coming up sharp controversy, as is commonly known the $\zeta(s)$ hasn't infinity derivative in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition.

The first probe is on that the proposition for analytic function $f(x)$, the similarity of the middle value theorem, like

$$\exists z' \forall z(f(z) - f(0) = f'(z')z), |z'| \le |z|, |z| < r$$

is invalid. Here is a counterexample

$$f(x) = (1 + x)^a, a \in \mathbf{R}$$

If a is great it can be found a little $x_1, r : |x_1| < r$

$$\ln(f(x_1)) - \ln(f(\overline{x}_1)) = 2\pi i, f(x_1) = f(\overline{x}_1)$$

but the derivative zero is at

$$a(1 + x)^{a-1} = 0$$

It's impossible on $|x| \le 3r$.

The second probe is on the definition of integration of real function $f(x, y)$ on curve C.

Theorem 2.2. *$l \in [0, c]$ is any finite parametrization of piecewise smooth curve $C(l)$. l is divided into the collection of Δl. If the integration of the real $f(x, y)$ on $C(l)$ is defined as the limit of the sum of $f(x, y)\Delta C$ or $f(x, y)||\Delta C||$ when $\Delta l \to 0$. then this limit exists if $f(x, y)$ is continuous uniformly in a neighboring set of C.*

This is nothing special.

The third probe is on Cauchy Integral Formula that said derivable function is analytic. Integrations about the complex derivable function $f(x)$ in the considered domain meet

$$f(x) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z - x}dz$$

$$f^{(n)}(x) = \frac{n!}{2\pi i}\oint_{C'} \frac{f(z)}{(z - x)^{n+1}}dz$$

$$C' = re^{i\theta} - x, 0 \le \theta < 2\pi$$

Its famous proof of Cauchy's [2] is incorrect.

The first reason is that the limit of integral contours $r \to 0$ causes the integrated is not bounded (hence not continuous) uniformly in $0 < r < R$. The second, Cauchy's first integral formula is interpreted as the following in effective

$$\lim_{r\to 0}\lim_{\Delta x\to 0}\sum_{\Delta x(\Delta_r x)} g\Delta\theta = \lim_{\Delta x\to 0}\lim_{r\to 0}\sum_{\Delta x(\Delta_r x)} g\Delta\theta$$

$$x = \Re(z), y = \Im(z), (\Delta x, r) \to \Delta_r x$$

Suitable and valid connection between $\Delta\theta$ and $(\Delta x, \Delta y)$ has been defined. This implies that the convergence of the integrations for $0 < r < R$ is uniform for $0 < |\Delta x| < c$.

Now it is critical to define the conception of Analytic as expandable to power series. French mathematician J.Dieudonne use this definition consistently in his famous book "Foundations of Modern Analysis".

Definition 2.3. 2-dimensional power series $f(z)$ of real arguments as $\Re(z), \Im(z)$, its convergence is called wide convergence. In the 2-dimensional series, the same degree terms is combined as one term to form an 1-dimensional series, convergence of which is called narrow convergence. Complex smooth and narrowly convergent power series is narrowly analytic, Complex smooth and widely convergent power series is widely analytic.

Maybe somebody had said

Theorem 2.4. *A smooth complex function is analytic if and only if both its real part and its imaginary part have narrow convergent power series.*

It's obvious that narrowly analytic is analytic.

Obviously the analytic continuation is unique, however, if the term "entirely analytic function in a circle" means that its power series expansion at the center covers the whole circle , one may ask whether a circle of locally analytic function is also entirely analytic. This is a new problem after classical Cauchy integral formula is valid no longer. For real function this proposition is negative:

$$1/(1 + ix), x \in \mathbf{R}$$

The wide convergence of real 2-arguments power series $f(x, y)$ is studied. As a fact a convergent radius (r_1, r_2) respectively for arguments (x, y) meet for sufficiently great m, n

$$\left|\frac{\partial^{m+n} f(x,y)}{(\partial x)^m(\partial y)^n}\right| < C' m! n! r_1^{-m} r_2^{-n}$$

C' is independent of m, n. The real derivatives of $\Gamma(s = x + yi), x > 1, y \in \mathbf{R}$ are studied for sufficiently great m, n:

$$|\Gamma^{(m,n)}(s)|_{s=x_0+iy_0}| > C(m+n)!|x_0 + y_0 i|^{-m-n}, C > 0$$

Obviously the convergent areas for the power series of s or $(\Re(s), \Im(s))$ are distinct. The widely analytic is stronger.

3. Conclusion

I have several reasons to denied the popular knowledge.

The identity

$$\zeta^*(s) = \zeta(s), \Re(s) > 1$$

is not valid. Its deductive is like

$$\begin{aligned} &\int_0^\infty x^{s-1}/(e^x-1)dx \\ =&\int_0^\infty x^{s-1}\sum_{n=1}^\infty e^{-nx}dx \\ =&\sum_{n=0}^\infty \int_0^\infty x^{s-1}e^{-nx}dx \end{aligned}$$

For this exchange of limits the needed is the uniform convergence of the series, unfortunately, the convergence of which is not uniform.

Only the definition of $\zeta^*(s)$ is studied. At positive real s, the convergent radium of the supposed analytic $\zeta^*(s)$ is $|s|$, but it has two singularities $s = 1, 0$, which fact is enough denies the classical Cauchy integral formula and the declamation that $\zeta^*(s)$ is analytic on $0 < \Re(s) < 1.001$.

References

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