THE ANALYTIC PROPERTY FOR RIEMANN ZETA FUNCTION

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Abstract. This article discusses the analytic property of Riemann zeta function. The popular opinion is denied.

1. Introduction

$\zeta(s)$ is originally

$$\zeta^*(s) = \sum_{n=1}^{\infty} n^{-s}, \Re(s) > 1$$

It is continuated by Riemann as:

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t - 1)dt \tag{1.1}$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people think this function is analytic except $s = 1$[1]. There is thought still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta^*(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, \Re(s) > 0 \tag{1.2}$$

This is a continuation of the original $\zeta(s)$. Someone deduced applaudably

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \frac{1}{1 - 2^{1-s}} \int_C t^{s-1}/(e^t + 1)dt \tag{1.3}$$

This expression is thought identical to Riemann's definition. In this article the analytic property is discussed.

2. Discussion

Theorem 2.1. *The second definition of $\zeta(s)$ has divergent derivative near $s = 0$.*

Proof.

$$F_m(s) := \sum_{n=1}^{m} (-1)^{n+1} n^{-s}, \Re(s) > 0$$

Set $s \in (0, 1)$.

$$F'_m(s) = \sum_{n>0, 2|n+1}^{\infty} \ln(n)(sn^{-s-1} - n^{-s-2}s(s+1)\theta/2), 0 < \theta < 1$$

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$$F'_m(s) > \frac{1}{2}\int_{n=3}^{m-2} \ln(n)n^{-s-1}sdn - \sum_{n>0,2|n+1}^{\infty} \ln(n)n^{-s-2}s(s+1)\theta/2$$

$$\lim_{m\to\infty} F'_m(s) = \frac{1}{2}\int_{3}^{\infty} s\ln(x)x^{-s-1}dx - C_s, |C_s| < C$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} sxe^{-sx}dx - C_s$$

$$\lim_{m\to\infty} F'_m(s) > \frac{1}{2}\int_{\ln(3)}^{\infty} \frac{sxe^{-sx}d(sx)}{s} - C_s$$

It's easy to find when $s \to 0$ this term approaches to infinity. □

There is coming up sharp controversy, as is commonly known the $\zeta(s)$ hasn't infinity derivative in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition.

The first probe is on that the proposition for analytic function $f(x)$, the similarity of the middle value theorem, like

$$\exists z'\forall z(f(z) - f(0) = f'(z')z), |z'| \leq |z|, |z| < r$$

is invalid. Here is a counterexample

$$f(x) = (1+x)^a, a \in \mathbf{R}$$

If a is great it can be found a little $x_1, r : |x_1| < r$

$$\ln(f(x_1)) - \ln(f(\overline{x}_1)) = 2\pi i, f(x_1) = f(\overline{x}_1)$$

but the derivative zero is at

$$a(1+x)^{a-1} = 0$$

It's impossible on $|x| \leq 3r$.

The second probe is on the definition of integration of real function $f(x, y)$ on curve C.

Theorem 2.2. *$l \in [0, c]$ is any continuous finite parametrization of piecewise smooth curve $C(l)$. l is divided into the collection of Δl. If the integration of the real $f(x, y)$ on $C(l)$ is defined as the limit of the sum of $f(x, y)\Delta C$ or $f(x, y)||\Delta C||$ when $\Delta l \to 0$. then this limit exists if $f(x, y)$ is continuous uniformly in a neighboring set of C.*

This is nothing special.

The third probe is on Cauchy Integral Formula that said derivable function is analytic. Integrations about the complex derivable function $f(x)$ in the considered domain is thought meeting

$$f(x) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z-x}dz$$

$$f^{(n)}(x) = \frac{n!}{2\pi i}\oint_{C'} \frac{f(z)}{(z-x)^{n+1}}dz$$

$$C' = re^{i\theta} - x, 0 \leq \theta < 2\pi$$

Its famous proof of Cauchy's [2] is incorrect.

The first reason is that the limit of integral contours $r \to 0$ causes the integrated is not bounded (hence not continuous) uniformly in $0 < r < R$. The second, Cauchy's first integral formula is interpreted as the following in effective

$$\lim_{r\to 0}\lim_{\Delta x\to 0}\sum_{\Delta x(\Delta_r x)} g\Delta\theta = \lim_{\Delta x\to 0}\lim_{r\to 0}\sum_{\Delta x(\Delta_r x)} g\Delta\theta$$

$$x = \Re(z), y = \Im(z), (\Delta x, r) \to \Delta_r x$$

Suitable and valid connection between $\Delta\theta$ and $(\Delta x, \Delta y)$ has been defined. This implies that the convergence of the integrations for $0 < r < R$ is uniform for $0 < |\Delta x| < c$.

Now it is critical to define the conception of Analytic as expandable to power series. French mathematician J.Dieudonne use this definition consistently in his famous book "Foundations of Modern Analysis".

Definition 2.3. 2-dimensional power series $f(z)$ of real arguments as $\Re(z), \Im(z)$, its convergence is called wide convergence. In the 2-dimensional series, the same degree terms is combined as one term to form an 1-dimensional series, convergence of which is called narrow convergence. Complex smooth and narrowly convergent power series is narrowly analytic, Complex smooth and widely convergent power series is widely analytic.

Maybe somebody had said

Theorem 2.4. *A smooth complex function is analytic if and only if both its real part and its imaginary part have narrow convergent power series.*

It's obvious that narrowly analytic is equivalent to analytic.

The wide convergence of real 2-arguments power series $f(x, y)$ is studied. As a fact a convergent radius (r_1, r_2) respectively for arguments (x, y) meet for sufficiently great m, n

$$|\frac{\partial^{m+n} f(x,y)}{(\partial x)^m(\partial y)^n}| < C' m!n! r_1^{-m} r_2^{-n}$$

C' is independent of m, n. The real derivatives of $\Gamma(s = x + yi), x > 1, y \in \mathbf{R}$ are studied for sufficiently great m, n:

$$|\Gamma^{(m,n)}(s)|_{s=x_0+iy_0}| > C(m+n)!|x_0 + y_0 i|^{-m-n}, C > 0$$

Obviously the convergent areas for the power series of s or $(\Re(s), \Im(s))$ are different. The widely analytic is stronger.

3. Conclusion

I have several reasons to denied the popular knowledge.

The identity

$$\zeta^*(s) = \zeta(s), \Re(s) > 1$$

is invalid. Its deductive is like

$$\int_0^\infty x^{s-1}/(e^x - 1)dx$$

$$= \int_{x\to 0}^\infty x^{s-1}\sum_{n=1}^\infty e^{-nx}dx$$

$$= \sum_{n=1}^{\infty} \int_{0}^{\infty} x^{s-1} e^{-nx} dx$$

For this exchange of limits the needed is the uniform convergence of the series, unfortunately, the convergence of which is not uniform. Calculation evinces the last step is wrong:

$$\sum_{n=1}^{\infty} \int_{0}^{a} x e^{-nx} dx$$
$$= \lim_{x \to 0} -x \ln(1 - e^{-x})$$
$$= \lim_{\delta \to 0} \ln(1-\delta) \ln \delta$$
$$= \lim_{\delta \to 0} -\delta \ln \delta$$
$$= \lim_{\delta \to 0} - \ln \delta^{\delta}$$
$$\to \infty$$

Thus the result is obtained. However, somebody argued that

$$\int_{x\to 0}^{\infty} x^{s-1}/(e^x - 1) dx = \sum_{n=1}^{N} \int_{x \to 0}^{\infty} x^{s-1} e^{-nx} dx + \int_{0}^{\infty} e^{-Nx} x^{s-1}/(e^x - 1) dx$$

and by limit the result is proven. The problem is that this formula is ready for the limit, so that it must be valid uniformly for all $N \in \mathbf{Z}$, or in other words

$$\forall \epsilon > 0 \exists a > 0 \forall N (|\sum_{n=1}^{N} \int_{0}^{a} x^{s-1} e^{-nx} dx| < \epsilon)$$

This is not different from the previous view point.

Obviously, $(s-1)\zeta^*(s)$ is analytic on $\Re(s) > 0$.

The classical Cauchy integral formula implies that first order complex derivable function is analytic. Its counterexample can be found through the solution of this second order differential equation

$$\Delta f(x, y) = 0$$

$f(x, y)$ is unnecessary to be smooth, and by Cauchy-Riemann formula, $g(x, y) + if(x, y)$ is obtained as a function derivable complexly.

For real analytic function, locally analytic everywhere doesn't mean global analytic , which function can be expressed by a power series in the whole domain. Here is a example

$$\frac{1}{1+ix}$$

How about analytic complex function on this aspect is an unsettled problem by me.

Whether a smooth complex function is analytic is an unsettled problem by me.

References

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