THE ANALYTIC PROPERTY FOR RIEMANN ZETA FUNCTION

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Abstract. This article discusses the analytic property of Riemann zeta function. The popular opinion is denied.

1. Introduction

$\zeta(s)$ is originally

$$\zeta^*(s) = \sum_{n=1}^{\infty} n^{-s}, \Re(s) > 1$$

It is continued by Riemann as:

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t - 1)dt \tag{1.1}$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people think this function is analytic except $s = 1$[1]. There is thought still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta^*(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, \Re(s) > 0 \tag{1.2}$$

This is a continuation of the original $\zeta(s)$. Someone deduced applaudably

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \frac{1}{1 - 2^{1-s}} \int_C t^{s-1}/(e^t + 1)dt \tag{1.3}$$

This expression is thought identical to Riemann's definition. In this article the analytic property is discussed.

2. Discussion

Theorem 2.1. *The second definition of $\zeta(s)$ has divergent derivative near $s = 0$.*

Proof.

$$F(s) := \sum_{n=5}^{\infty} (-1)^{n+1} n^{-s}, \Re(s) > 0$$

Set $s \in (0, 0.1)$.

$$-F'(s) = \sum_{n=5,}^{\infty} (-1)^{n+1} \ln(n) n^{-s}$$

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$$= \sum_{n=5,2|n+1}^{\infty} \ln(n)(sn^{-s-1} - n^{-s-2}s(s+1))\theta), 0 < \theta < 1$$

$$> \frac{1}{2}\int_{n=5}^{\infty} \ln(n)n^{-s-1}sdn - \sum_{n=5,2|n+1}^{\infty} \ln(n)n^{-s-2}s(s+1)\theta$$

$$= \frac{1}{2}\int_{5}^{\infty} s\ln(x)x^{-s-1}dx - C_s, |C_s| < C$$

$$> \frac{1}{2}\int_{\ln(5)}^{\infty} sxe^{-sx}dx - C_s$$

$$> \frac{1}{2}\int_{\ln(5)}^{\infty} \frac{sxe^{-sx}d(sx)}{s} - C_s$$

It's easy to find when $s \to 0$ this term approaches to infinity. □

There is coming up sharp controversy, as is commonly known the $\zeta(s)$ hasn't infinity derivative in near $s = 0$. But in this article the opinion inclines to find the fault of the Riemann's definition.

The first probe is on that the proposition for analytic function $f(x)$, the similarity of the middle value theorem, like

$$\exists z'\forall z(f(z) - f(0) = f'(z')z), |z'| \leq |z|, |z| < r$$

is invalid. Here is a counterexample

$$f(x) = (1+x)^a, a \in \mathbf{R}$$

If a is great it can be found a little $x_1, r : |x_1| < r$

$$\ln(f(x_1)) - \ln(f(\overline{x}_1)) = 2\pi i, f(x_1) = f(\overline{x}_1)$$

but the derivative zero is at

$$a(1+x)^{a-1} = 0$$

It's impossible on $|x| \leq 3r$.

The second probe is on the definition of integration of real function $f(x, y)$ on curve C.

Theorem 2.2. *$l \in [0, c]$ is any continuous finite parametrization of piecewise smooth curve $C(l)$. l is divided into the collection of Δl. If the integration of the real $f(x, y)$ on $C(l)$ is defined as the limit of the sum of $f(x, y)\Delta C$ or $f(x, y)||\Delta C||$ when $\Delta l \to 0$. then this limit exists if $f(x, y)$ is continuous uniformly in a neighboring set of C.*

This is nothing special.

The third probe is on Cauchy Integral Formula that said derivable function is analytic. Integrations about the complex derivable function $f(x)$ in the considered domain is thought meeting

$$f(x) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z-x}dz$$

$$f^{(n)}(x) = \frac{n!}{2\pi i}\oint_{C'} \frac{f(z)}{(z-x)^{n+1}}dz$$

$$C' = re^{i\theta} - x, 0 \leq \theta < 2\pi$$

Its famous proof of Cauchy's [2] is incorrect.

The first reason is that the limit of integral contours $r \to 0$ causes the integrated is not bounded (hence not continuous) uniformly in $0 < r < R$. The second, Cauchy's first integral formula is interpreted as the following in effective

$$\lim_{r\to 0}\lim_{\Delta x\to 0}\sum_{\Delta x(\Delta_r x)} g\Delta\theta = \lim_{\Delta x\to 0}\lim_{r\to 0}\sum_{\Delta x(\Delta_r x)} g\Delta\theta$$

$$x = \Re(z), y = \Im(z), (\Delta x, r) \to \Delta_r x$$

Suitable and valid connection between $\Delta\theta$ and $(\Delta x, \Delta y)$ has been defined. This implies that the convergence of the integrations for $0 < r < R$ is uniform for $0 < |\Delta x| < c$.

Now it is critical to define the conception of Analytic as expandable to power series. French mathematician J.Dieudonne use this definition consistently in his famous book "Foundations of Modern Analysis".

Definition 2.3. 2-dimensional power series $f(z)$ of real arguments as $\Re(z), \Im(z)$, its convergence is called wide convergence. In the 2-dimensional series, the same degree terms is combined as one term to form an 1-dimensional series, convergence of which is called narrow convergence. Complex smooth and narrowly convergent power series is narrowly analytic, Complex smooth and widely convergent power series is widely analytic.

Maybe somebody had said

Theorem 2.4. *A smooth complex function is analytic if and only if both its real part and its imaginary part have narrow convergent power series expressions.*

It's obvious that narrowly analytic is equivalent to analytic.

The analytic property of $\Gamma(s)$ is discussed.

$$\Gamma^{(2k)}(S) > \int_{x^{-1/2}}^{x} \ln^{2k} x x^{s-1} e^{-x} dx, s > 1$$

$$> C\int_{x^{-1/2}}^{x} \ln^{2k} x x^{\delta-1} dx = C\int_{-\ln x/2}^{\ln x} x^{2k} e^{\delta x} dx, \delta \to 0^+, C = C_x > 0$$

$$= \frac{C}{\delta^{2k+1}}\int_{-1/4}^{1/2} x^{2k} e^x dx = \frac{C}{\delta^{2k+1}}\sum_{n=0}^{2k}\frac{(-1)^n x^{2k-n} e^x (2k)!}{(2k-n)!}|_{-1/4}^{1/2}$$

$$> \frac{C'_\delta (2k)!}{\delta^{2k+1}}, C'_\delta > 0$$

This means the convergent radium of $\Gamma(s)$ is close to zero at any point on $s > 1$, in the other words: non-analytic.

3. Conclusion

I have several reasons to denied the popular knowledge.

The identity

$$\zeta^*(s) = \zeta(s), s > 1$$

is invalid on $1 < s < 2$ although it's valid on $\Re(s) > 2$. The popular deductive is like

$$\int_0^\infty x^{s-1}/(e^x - 1)dx = \int_{x\to 0}^\infty x^{s-1}\sum_{n=1}^\infty e^{-nx} dx$$

$$= \sum_{n=1}^{\infty} \int_{x\to 0}^{\infty} x^{s-1} e^{-nx} dx$$

The problem is that the limits exchange presents, which need the series uniformly convergent, unfortunately, it's not. Somebody argued that

$$\int_{x\to 0}^{\infty} x^{s-1} \sum_{n=1}^{\infty} e^{-nx} dx = \sum_{n=1}^{N} \int_{x\to 0}^{\infty} x^{s-1} e^{-nx} dx + \int_{0}^{\infty} x^{s-1} e^{-Nx}/(e^x - 1) dx$$

One can use this and make limit to prove the results. However, because it's ready for the limit this formula must be valid uniformly for all N, i.e.

$$\forall \epsilon > 0 \exists a > 0 \forall N (\sum_{n=1}^{N} \int_0^a x^{s-1} e^{-nx} dx < \epsilon)$$

This is not different from the previous view point. The following difference is presumed approaching to zero, and it is calculated for little x and $0 < \delta < 1$

$$\lim_{x\to 0} \sum_{n=1}^{\infty} \int_0^x x^{\delta} e^{-nx} dx$$

$$= \delta \lim_{x\to 0} \sum_{n=1}^{\infty} \int_0^x x^{\delta-1} e^{-nx}/n dx$$

$$> \delta \lim_{x\to 0} x^{\delta-1} \sum_{n=1}^{\infty} \int_0^x e^{-nx}/n dx$$

$$> \delta \lim_{x\to 0} x^{\delta-1} \sum_{n=1}^{\infty} \int_0^x x^{1-\delta} e^{-nx}/n dx$$

$$= \delta(1-\delta) \lim_{x\to 0} x^{\delta-1} \sum_{n=1}^{\infty} \int_0^x x^{-\delta} e^{-nx}/n^2 dx$$

$$> \delta(1-\delta) \lim_{x\to 0} x^{\delta-1-\delta} \sum_{n=1}^{\infty} \int_0^x e^{-nx}/n^2 dx$$

$$> \delta(1-\delta) \lim_{x\to 0} x^{1-1} \sum_{n=1}^{\infty} e^{-nx}/n^2$$

$$= \delta(1-\delta) \sum_{n=1}^{\infty} 1/n^2$$

There is a direct counter-example at $s \in [1,2]$. The three functions

$$A: \quad (1-2^{1-s})\Gamma(s)\zeta(s), \quad B: \quad \Gamma(s), \quad C: \quad \Gamma(s) - (1-2^{1-s})\Gamma(s)\zeta^*(s)$$

has positive derivatives of second order with the zero derivatives at

$$A: \quad z_A, \quad B: \quad z_B, \quad C: \quad z_C,$$

Because

$$\int_0^{\infty} \frac{\ln x x^{s-1}}{e^x+1} dx > \frac{e}{e+1} \int_0^{\infty} \frac{\ln x x^{s-1}}{e^x} dx, \int_0^{\infty} \frac{\ln x x^{s-1}}{e^x(e^x+1)} dx > \frac{e}{e+1} \int_0^{\infty} \frac{\ln x x^{s-1}}{e^{2x}} dx$$

it's valid that

$$z_A \le z_B, z_c \le z_B$$

if the identity

$$\zeta(s) = \zeta^*(s)$$

is valid. However, from this identity, at $s = z_B$, A, C have the derivatives with different signs that's a contradictory, or they all are with zero derivatives, which case implies $(1-2^{1-s})\zeta^*(s)$ is also with zero derivative that's impossible.

There is also a calculation at $s = 3/2$

$$\int_0^\infty x^{1/2}/(1+e^x)dx = \int_0^1 (-\ln x)^{1/2}/(1+x)dx$$
$$< (\int_0^1 -\ln x/(1+x)^2 dx)^{1/2} = ((\ln 2 + 1/2 + 1/8)/3)^{1/2}$$
$$= 0.662859$$
$$\Gamma(3/2) = \sqrt{\pi}/2 = 0.8862269$$
$$(1-2^{-1/2})\zeta(3/2) < 0.74795658$$
$$(1-2^{-1/2})\zeta^*(3/2) = 0.765651$$

Obviously $\zeta^*(s)$ is analytic on $\Re(s) > 0$.

The classical Cauchy integral formula implies that first order complex derivable function is analytic. Its counterexample can be found through the solution of this second order differential equation

$$\Delta f(x,y) = 0$$

$f(x,y)$ is unnecessary to be smooth, and by Cauchy-Riemann formula, $g(x,y) + if(x,y)$ is obtained as a function derivable complexly.

However, after the *analytic function* is defined as power series expandable locally, the Cauchy Integral Formula for *analytic function* is all right, hence the endpoints of the biggest analytic convergent radium stops at the non-analytic point.

References

[1] E.C. Titchmarsh, The theory of the Riemann zeta function, Oxford University Press, 2nd ed, 1986.

[2] M. A. Lavrentieff, B. V. Shabat, Methods of functions of a complex variable, Russia, 2002

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