THE ANALYTIC PROPERTY FOR RIEMANN ZETA FUNCTION

WU SHENG-PING

ABSTRACT. This article discusses the analytic property of Riemann zeta function. The popular opinion is denied.

1. INTRODUCTION

$\zeta(s)$ is originally

$$\zeta^*(s) = \sum_{n=1}^{\infty} n^{-s}, \Re(s) > 1$$

It is continuated by Riemann as:

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t - 1)dt \tag{1.1}$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

$$(1 - e^{i2\pi s})\Gamma(s) := \int_C x^{s-1}e^{-x}dx$$

Most of people think this function is analytic except $s = 1$[1]. There is still another series for $\zeta(s)$

$$\zeta^*(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, \Re(s) > 0 \tag{1.2}$$

This is a continuation of the original $\zeta^*(s)$. One also can deduce that

$$\Gamma(s)\zeta(s)(1 - 2^{1-s}) = \int_0^{\infty} t^{s-1}/(e^t + 1)dt, \Re(s) > 0$$

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \frac{1}{1 - 2^{1-s}} \int_C t^{s-1}/(e^t + 1)dt \tag{1.3}$$

This expression is identical to Riemann's definition.

In this article the analytic properties of general complex variables especially of $\zeta(s)$ are discussed.

Date: December 10, 2010.

2000 *Mathematics Subject Classification.* Primary 11M06, Secondary 30B10 30B40.

Key words and phrases. Riemann zeta function, analytic continuation, Cauchy integral formula.

2. Discussion

The opinion of this article inclines to find the fault of the popular theory of complex variables.

The first probe is on that the proposition for analytic function $f(x)$, the similarity of the middle value theorem, like

$$\exists z'\forall z(f(z)-f(0)=f'(z')z), |z'|\leq|z|, |z|<r$$

is invalid. Here is a counterexample

$$f(x)=(1+x)^a, a\in\mathbf{R}$$

If a is great it can be found a little $x_1, r : |x_1|<r$

$$\ln(f(x_1))-\ln(f(\overline{x}_1))=2\pi i, f(x_1)=f(\overline{x}_1)$$

but the derivative zero is at

$$a(1+x)^{a-1}=0$$

It's impossible on $|x|\leq 3r$.

The second probe is on the definition of integration of real function $f(x,y)$ on curve C.

Theorem 2.1. *$l\in[0,c]$ is any continuous finite parametrization of piecewise smooth curve $C(l)$. l is divided into the collection of Δl. If the integration of the real $f(x,y)$ on $C(l)$ is defined as the limit of the sum of $f(x,y)\Delta C$ or $f(x,y)||\Delta C||$ when $\Delta l\to 0$. then this limit exists if $f(x,y)$ is continuous uniformly in a neighboring set of C.*

This is nothing special.

The third probe is on Cauchy Integral Formula that said derivable function is analytic. Integrations about the complex derivable function $f(x)$ in the considered domain is thought meeting

$$f(x)=\frac{1}{2\pi i}\oint_{C'}\frac{f(z)}{z-x}dz$$

$$f^{(n)}(x)=\frac{n!}{2\pi i}\oint_{C'}\frac{f(z)}{(z-x)^{n+1}}dz$$

$$C'=re^{i\theta}-x, 0\leq\theta<2\pi$$

Its famous proof of Cauchy's [2] is incorrect.

The first reason is that the limit of integral contours $r\to 0$ causes the integrated is not bounded (hence not continuous) uniformly in $0<r<R$. The second, Cauchy's first integral formula is interpreted as the following in effective

$$\lim_{r\to 0}\lim_{\Delta x\to 0}\sum_{\Delta x(\Delta_r x)} g\Delta\theta=\lim_{\Delta x\to 0}\lim_{r\to 0}\sum_{\Delta x(\Delta_r x)} g\Delta\theta$$

$$x=\Re(z), y=\Im(z), (\Delta x, r)\to\Delta_r x$$

Suitable and valid connection between $\Delta\theta$ and $(\Delta x,\Delta y)$ has been defined. This implies that the convergence of the integrations for $0<r<R$ is uniform for $0<|\Delta x|<c$.

Now it is critical to define the conception of Analytic as expandable to power series. French mathematician J.Dieudonne use this definition consistently in his famous book "Foundations of Modern Analysis".

After the *analytic function* is defined as power series locally expandable, the Cauchy Integral Formula for *analytic function* is all right, hence the endpoints of the biggest analytic convergent radium stops at the non-analytic point.

Definition 2.2. 2-dimensional power series $f(z)$ of real arguments as $\Re(z), \Im(z)$, its convergence is called wide convergence. In the 2-dimensional series, the same degree terms is combined as one term to form an 1-dimensional series, convergence of which is called narrow convergence. Complex smooth and narrowly convergent power series is narrowly analytic, Complex smooth and widely convergent power series is widely analytic.

Maybe somebody had said that

Theorem 2.3. *A smooth complex function is analytic if both its real part and its imaginary part have narrowly convergent power series expressions.*

3. Conclusion

Through calculation of derivatives, the function $(1-2^{1-s})\zeta^*(s)$ is analytic on $\Re(s)>1$ and the function $(1-2^{1-s})\zeta(s)$ is analytic in the whole complex plane. For the definition of Riemann's, Singularities exist on the zero points $S_0 := 1+it = s$ of factor $1-2^{1-s}$:

$$\zeta(s) = \frac{\int_0^\infty x^{s-1}/(1+e^x)dx}{\Gamma(s)(1-2^{1-s})}, S_0 = 1+it = s, 1-2^{1-S_0} = 0$$

As a fact, by being integrated within $\Re(s) = 1$ it's found that the $\Gamma(s)$ have zero points $s = S_0$ on $\Re(s) = 1$:
if

$$0 = \int_{-\infty}^{\infty} \cos(tx)e^x e^{-e^x} dx = \Re(\Gamma(it+1)), t \in \mathbf{R}$$

then

$$0 = \int_{-\infty}^{\infty} \cos(t(x+ia))e^{x+ia}e^{-e^{x+ia}} dx, 0 \le a < \pi/2$$

$$0 = e^{ta}\int_0^\infty x^{it}e^{-xe^{ia}}dx + e^{-ta}\int_0^\infty x^{-it}e^{-xe^{ia}}dx$$

it's derived on e^{ia} and it set that $a=0$ and the imaginary part is calculated

$$\Im(-it1^{-ti-1}(\Gamma(it+1) - \Gamma(-it+1))) = 0$$

This implies that a zero real half part is with the zero imaginary half part of $\Gamma(it+1 = S_0)$. S_0 is included in this zero points of $\Gamma(s)$ because

$$(1-2^{1-S_0})\Gamma(S_0)\zeta(S_0) = 0$$

$$(1-2^{1-S_0})\zeta(S_0) \neq 0$$

The second formula need an explicit proof. In fact, $\Gamma(s)\zeta(s)$ is not zero on $\Re(s) = 1$, if not

$$f(\delta, x) := e^{(\delta+1)x}/(e^{e^x} - 1), s = it' + 1 + \delta, \delta > 0, t' \in \mathbf{R}$$

$$\Gamma(s)\zeta(s) = \int_{-\infty}^{\infty} e^{it'} f(\delta, x)dx$$

$$A(\delta) := \int_{-\infty}^{\infty} \cos^2(t'x/2) f(\delta, x)dx$$

$$\lim_{\delta\to 0} A(\delta) = \lim_{\delta\to 0}\int_{-\infty}^{\infty} \cos^2(t'x/2) f(\delta, x+\ln 2)dx$$
$$= \lim_{\delta\to 0}[2^\delta A(\delta) + 2^\delta \int_{-\infty}^{\infty} \frac{\cos^2(t'x/2)e^{\delta+1}dx}{1+e^{e^x}}]$$
$$0 = \int_{-\infty}^{\infty} \frac{\cos^2(t'x/2)de^x}{1+e^{e^x}} > 0$$

The classical Cauchy integral formula implies that first order complex derivable function is analytic. Its counterexample can be found through the solution of this second order differential equation

$$\Delta f(x,y) = 0$$

$f(x,y)$ is unnecessary to be smooth, and by Cauchy-Riemann formula, $f_y + if_x$ is obtained as a function derivable complexly. A direct counter-examples of Cauchy Integral Formula is the multi-leaf function $z^{1.1}$ at $z = 0$ neighboring, or the following one

$$\Gamma(s) = 1/(s(s+1)) + F(s)$$

$F(s)$ is Cauchy's analytic on $\Re(s) > -2$

$$\Gamma(s+1) = s\Gamma(s)$$
$$\ln(s) = \ln(\Gamma(s+1)) - \ln(\Gamma(s))$$
$$\frac{1}{s} = \frac{\Gamma'(s+1)}{\Gamma(s+1)} - \frac{\Gamma'(s)}{\Gamma(s)}$$

It's integrated circling $s = -1$

$$2 = F(0) + F(-1), \Gamma(2)' - \Gamma'(1) = 2$$

This identity is wrong. As the popular knowledge $\Gamma(s)$ is analytic except non-positive integer. This evinces that the quotient of two analytic functions is not always analytic even if the denominator is not zero.

References

[1] E.C. Titchmarsh, The theory of the Riemann zeta function, Oxford University Press, 2nd ed, 1986.

[2] M. A. Lavrentieff, B. V. Shabat, Methods of functions of a complex variable, Russia, 2002

Wuhan university, Wuhan, China.
E-mail address: hiyaho@126.com