THE ANALYTIC PROPERTY FOR RIEMANN ZETA FUNCTION

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ABSTRACT. This article discusses the analytic property of Riemann zeta function. The popular opinion is denied.

1. INTRODUCTION

$\zeta(s)$ is originally

$$\zeta^*(s) = \sum_{n=1}^{\infty} n^{-s}, \Re(s) > 1$$

It is continuated by Riemann as:

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t - 1)dt \tag{1.1}$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people think this function is analytic except $s = 1$[1]. There is still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta^*(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, \Re(s) > 0 \tag{1.2}$$

This is a continuation of the original $\zeta(s)$. Someone deduced that

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta^*(s) = \frac{1}{1 - 2^{1-s}} \int_C t^{s-1}/(e^t + 1)dt \tag{1.3}$$

This expression is thought identical to Riemann's definition. In this article the analytic property is discussed.

2. DISCUSSION

The opinion of this article inclines to find the fault of the popular theory of complex variables.

The first probe is on that the proposition for analytic function $f(x)$, the similarity of the middle value theorem, like

$$\exists z' \forall z (f(z) - f(0) = f'(z')z), |z'| \le |z|, |z| < r$$

is invalid. Here is a counterexample

$$f(x) = (1 + x)^a, a \in \mathbf{R}$$

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If a is great it can be found a little $x_1, r : |x_1| < r$

$$\ln(f(x_1)) - \ln(f(\overline{x}_1)) = 2\pi i, f(x_1) = f(\overline{x}_1)$$

but the derivative zero is at

$$a(1+x)^{a-1} = 0$$

It's impossible on $|x| \leq 3r$.

The second probe is on the definition of integration of real function $f(x, y)$ on curve C.

Theorem 2.1. *$l \in [0, c]$ is any continuous finite parametrization of piecewise smooth curve $C(l)$. l is divided into the collection of Δl. If the integration of the real $f(x, y)$ on $C(l)$ is defined as the limit of the sum of $f(x, y)\Delta C$ or $f(x, y)||\Delta C||$ when $\Delta l \to 0$. then this limit exists if $f(x, y)$ is continuous uniformly in a neighboring set of C.*

This is nothing special.

The third probe is on Cauchy Integral Formula that said derivable function is analytic. Integrations about the complex derivable function $f(x)$ in the considered domain is thought meeting

$$f(x) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z - x} dz$$

$$f^{(n)}(x) = \frac{n!}{2\pi i}\oint_{C'} \frac{f(z)}{(z - x)^{n+1}} dz$$

$$C' = re^{i\theta} - x, 0 \leq \theta < 2\pi$$

Its famous proof of Cauchy's [2] is incorrect.

The first reason is that the limit of integral contours $r \to 0$ causes the integrated is not bounded (hence not continuous) uniformly in $0 < r < R$. The second, Cauchy's first integral formula is interpreted as the following in effective

$$\lim_{r\to 0}\lim_{\Delta x\to 0}\sum_{\Delta x(\Delta_r x)} g\Delta\theta = \lim_{\Delta x\to 0}\lim_{r\to 0}\sum_{\Delta x(\Delta_r x)} g\Delta\theta$$

$$x = \Re(z), y = \Im(z), (\Delta x, r) \to \Delta_r x$$

Suitable and valid connection between $\Delta\theta$ and $(\Delta x, \Delta y)$ has been defined. This implies that the convergence of the integrations for $0 < r < R$ is uniform for $0 < |\Delta x| < c$.

Now it is critical to define the conception of Analytic as expandable to power series. French mathematician J.Dieudonne use this definition consistently in his famous book "Foundations of Modern Analysis".

After the *analytic function* is defined as power series locally expandable, the Cauchy Integral Formula for *analytic function* is all right, hence the endpoints of the biggest analytic convergent radium stops at the non-analytic point.

3. Conclusion

The classical Cauchy integral formula implies that first order complex derivable function is analytic. Its counterexample can be found through the solution of this second order differential equation

$$\Delta f(x, y) = 0$$

$f(x,y)$ is unnecessary to be smooth, and by Cauchy-Riemann formula, $f_y(x,y) + if_x(x,y)$ is obtained as a function derivable complexly (ie. Cauchy's analytic). A direct counter-example of Cauchy Integral Formula is the following

Gamma function can be defined as

$$(1 - e^{i2\pi s})\Gamma(s) := \int_{C,r=1} x^{s-1} e^{-x} dx$$

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx, s > 0$$

the expansion of it

$$\Gamma(s+1) - 1/(s+1) = \sum_0^\infty s^i (\Gamma^{(i)}(1)/i! - (-1)^i)$$

In fact this expansion has a singularity (infinity) near $s = -1$.

$$|\Gamma^{(i)}(1)/i! - (-1)^i| > C \int_0^1 dx |\ln^i x|(-e^{-x} + 1)/i!$$

$$> C' \int_0^a dx x^i (1 - e^{-e^{-x}}) e^{-x}/i! > C'' \int_0^a dx x^i e^{-x}/i!$$

$$> C''' \int_0^\infty dx x^i e^{-x}/i!$$

$$C, C', C'', C''' > 0, a = 2$$

Despite of the popular knowledge of the analytic $\Gamma(s)$, it's not.

References

[1] E.C. Titchmarsh, The theory of the Riemann zeta function, Oxford University Press, 2nd ed, 1986.

[2] M. A. Lavrentieff, B. V. Shabat, Methods of functions of a complex variable, Russia, 2002

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