THE ANALYTIC PROPERTY FOR RIEMANN ZETA FUNCTION

WU SHENG-PING

Abstract. This article discusses the analytic property of Riemann zeta function. The popular opinion is denied.

1. Introduction

$\zeta(s)$ is originally

$$\zeta^*(s) = \sum_{n=1}^{\infty} n^{-s}, \Re(s) > 1$$

It is continuated by Riemann as:

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \int_{C=C_1+C_2+C_3} t^{s-1}/(e^t - 1)dt \tag{1.1}$$

$$C_1 = (-\infty, r]e^{2i\pi}, C_2 = re^{i\theta}, \theta = (2\pi, \pi], C_3 = (r, \infty), 0 < r < 2\pi$$

Most of people think this function is analytic except $s = 1$[1]. There is still another series for $\zeta(s)$ that 's called the second definition in this article.

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}, \Re(s) > 0 \tag{1.2}$$

This is a continuation of the original $\zeta(s)$. Someone deduced that

$$(1 - e^{i2\pi(s-1)})\Gamma(s)\zeta(s) = \frac{1}{1 - 2^{1-s}} \int_C t^{s-1}/(e^t + 1)dt \tag{1.3}$$

This expression is thought identical to Riemann's definition. In this article the analytic property is discussed.

2. Discussion

The opinion of this article inclines to find the fault of the popular theory of complex variables.

The first probe is on that the proposition for analytic function $f(x)$, the similarity of the middle value theorem, like

$$\exists z' \forall z (f(z) - f(0) = f'(z')z), |z'| \le |z|, |z| < r$$

is invalid. Here is a counterexample

$$f(x) = (1 + x)^a, a \in \mathbf{R}$$

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If a is great it can be found a little $x_1, r : |x_1| < r$

$$\ln(f(x_1)) - \ln(f(\overline{x}_1)) = 2\pi i, f(x_1) = f(\overline{x}_1)$$

but the derivative zero is at

$$a(1+x)^{a-1} = 0$$

It's impossible on $|x| \leq 3r$.

The second probe is on Cauchy Integral Formula that said the complex derivable function $f(x)$ in the considered domain is thought meeting

$$f(x) = \frac{1}{2\pi i} \oint_{C'} \frac{f(z)}{z-x} dz$$

$$f^{(n)}(x) = \frac{n!}{2\pi i} \oint_{C'} \frac{f(z)}{(z-x)^{n+1}} dz$$

$$C' = re^{i\theta} - x, 0 \leq \theta < 2\pi$$

Its famous proof of Cauchy's [2] is incorrect. We can verify this

$$(f(z) - f(0))/z)' = f'(z)/z + \frac{1}{2}(f(z)/z^2 - f(0)/z^2)$$

This term is not easy to prove analytic in meaning of cauchy's. In Cauchy's proof the integration the little circle is not right to rely on the reasoning of complex middle value theorem

$$\lim_{r\to 0} \int_{C_r} dz(f(z)/z - f(0)/z) = \lim_{r\to 0} \int_{C_r} dz(f_1(ax, by)x + f_2(a'x, b'y)y)/z$$

Now it is critical to define the conception of Analytic as expandable to power series. French mathematician J.Dieudonne use this definition consistently in his famous book "Foundations of Modern Analysis".

After the *analytic function* is defined as power series locally expandable, the Cauchy Integral Formula for *analytic function* is all right, hence the endpoints of the biggest analytic convergent radium stops at the non-analytic point.

3. Conclusion

The classical Cauchy integral formula implies that first order complex derivable function is analytic. Its counterexample can be found through the solution of this second order differential equation

$$\Delta f(x, y) = 0$$

$f(x, y)$ is unnecessary to be smooth, and by Cauchy-Riemann formula, $f_y(x, y) + if_x(x, y)$ is obtained as a function derivable complexly (ie. Cauchy's analytic). A direct counter-example of Cauchy Integral Formula is the following

Gamma function can be defined as

$$(1 - e^{i2\pi s})\Gamma(s) := \int_{C, r=1} x^{s-1} e^{-x} dx$$

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx, s > 0$$

As the normal knowledge Gamma function is analytic (conservative) except the non-positive point. We will look into the zeta function, for the conservative definition of analytic the analytic convergent circle ends in singularity or infinity. we check the expansion of Cauchy's analytic function extending to the whole plane

$$\Gamma(s+2)\zeta(s+2) - 1/(s+1) = \int_0^\infty \frac{x^{s+1}}{e^x - 1} dx - 1/(s+1), \Re(s) > -1$$

Considering the expansion in $s = 0$

$$\int_0^1 \frac{\ln^i xx}{e^x - 1} dx - (-1)^i i! = \int_0^1 dx (\ln^i x + B_1 x \ln^i x + a B_2 x^2 \ln^i x/2) - (-1)^i i!, |a| < C'$$

B_n is Bernoulli number

$$= \int_0^1 dx (B_1 x \ln^i x + c B_2 x^2 \ln^i x/2)$$

$$|\int_0^1 dx (B_1 x \ln^i x + c B_2 x^2 \ln^i x/2)|$$

As i is great enough

$$> C i! 2^{-i-1}$$

So that it has convergent radium less than 2, but in popular knowledge the radium should be infinity.

References

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[2] M. A. Lavrentieff, B. V. Shabat, Methods of functions of a complex variable, Russia, 2002

Wuhan university, Wuhan, China.
E-mail address: hiyaho@126.com