

# A Digital Signature Using Multivariate Functions on Quaternion Ring

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Masahiro Yagisawa †  
 † Resident in Yokohama-shi  
 Sakae-ku , Yokohama-shi, Japan

**SUMMARY:** We propose the digital signature scheme on non-commutative quaternion ring over finite fields in this paper. We generate the multivariate function of high degree  $F(X)$ . We construct the digital signature scheme using  $F(X)$ . Our system is immune from the Gröbner bases attacks because obtaining parameters of  $F(X)$  to be secret keys arrives at solving the multivariate algebraic equations that is one of NP complete problems.

**key words:** digital signature, multivariate algebraic equations, Gröbner bases attacks, quaternion, NP complete problems.

## 1. Introduction

Since Diffie and Hellman proposed the concept of the public key cryptosystem in 1976[1], various digital signature schemes were proposed.

Typical examples of digital signature are as follows.

- 1)The digital signature using RSA cryptosystem[2] based on factoring problem,
- 2)the ElGamal signature scheme [3] based on the discrete logarithm problem over finite fields,
- 3)the digital signature using elliptic curve cryptosystem[4] based on the discrete logarithm problem on the elliptic curve[5],[6],
- 4)the digital signature scheme based on multivariate public key cryptosystem (MPKC), such as the digital signature scheme using stepwise triangular scheme (STS), which is one of the basic trapdoors of MPKC[11], and so on.

Sato and Araki proposed a digital signature using non-commutative quaternion ring[7] which has been broken[8].

It is said that the problem of factoring large integers, the problem of solving discrete logarithms and the problem of computing elliptic curve discrete logarithms are efficiently solved in a polynomial time by the quantum computers.

It is thought that MPKC is immune from the attack of quantum computers. But MPKC proposed until now almost adopts multivariate quadratic equations because of avoiding the explosion of key length.

In the current paper, we propose the digital signature scheme using multivariate functions of high degree on non-commutative quaternion[9] ring  $H$  over finite fields  $Fq$ . The security of this system is based on the computational difficulty to solve the multivariate algebraic equations of high degree.

To break this cryptosystem it is thought that we must probably solve the multivariate algebraic equations of high degree that is equal to solving the NP complete

problem. Then it is thought that our system is immune from the attacks by quantum computers.

In the next section, we begin with the definition of the product  $AB$  between  $A$  and  $B$  on the non-commutative quaternion ring over  $Fq$ . In section 3, we generate the multivariate functions of high degree on the ring. In section 4, we describe the element expression of the multivariate functions of high degree. In section 5, we construct proposed digital signature scheme. In section 6, we verify the strength of our digital signature. We consider the size of the keys for our digital signature in section 7. In the last section, we provide concluding remarks.

## 2. The definition of the product $AB$

Let  $H$  be the quaternion ring over  $Fq$ . Here we define the product  $AB$  of  $A=(a_0, a_1, a_2, a_3)$  and  $B=(b_0, b_1, b_2, b_3)$  on quaternion ring  $H$  over  $Fq$  such that

$$\begin{aligned} AB &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 \text{ mod } q, \\ & a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2 \text{ mod } q, \\ & a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1 \text{ mod } q, \\ & a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0 \text{ mod } q). \end{aligned}$$

As we select the non-commutative quaternion ring as the basic ring, the modulus  $q$  needs to be more than 2 to keep non-commutative.

## 3. Multivariate functions of high degree

Let  $q$  be an odd prime. Let  $m, d$  and  $r$  be positive integers. We choose arbitrary parameters  $k_i \in Fq$  and  $A_j \in H (j=1, \dots, m)$  as secret keys. We define the multivariate function  $F(X)$  of high degree such that

$$F(X) = \sum_{i=1}^m [k_i \prod_{j=0}^d A_i^{r^j} X^{r^j}], \quad (1)$$

where  $X \in H$  is a variable. We determine the value of  $m$  later.

Next we choose an arbitrary element  $R \in H$  to be non-commutative to  $A_j (j=1, \dots, m)$ . We define a temporary multivariate function  $T(X)$  such that

$$T(X) = \sum_{i=1}^m [k_i \prod_{j=0}^d A_i^{r^j} R^{r^j} X^{r^j}]. \quad (2)$$

#### 4. The element expression of $F(X)$

Let  $s$  be

$$s = l + r + r^2 + \dots + r^d. \quad (3)$$

Let  $(f_0, f_1, f_2, f_3)$  be the element expression of  $F(X)$ . From (1),  $f_j (j=0, \dots, 3)$  is given such that

$$F(X) = (f_0, f_1, f_2, f_3), \quad (4)$$

$$f_j = \sum_{e_0 + \dots + e_3 = s} f_{j e_0 e_1 e_2 e_3} x_0^{e_0} x_1^{e_1} x_2^{e_2} x_3^{e_3} \mod q \quad (5)$$

with  $0 \leq e_0, e_1, e_2, e_3 \leq s$  and the coefficients  $f_{j e_0 e_1 e_2 e_3} \in \mathbf{F}_q$  to be published, where

$$X = (x_0, x_1, x_2, x_3) \in \mathbf{H},$$

$$x_i \in \mathbf{F}_q, (i = 0, \dots, 3).$$

$e_0, e_1, e_2$  and  $e_3$  are non-negative integers which satisfy  $e_0 + \dots + e_3 = s$ .

Then the number  $n$  of  $f_{j e_0 e_1 e_2 e_3}$  is

$$n = 4_4 H_s = 4_{s+3} C_3. \quad (6)$$

Let  $(t_0, t_1, t_2, t_3)$  be the element expression of  $T(X)$ . From (2),  $t_j (j=0, \dots, 3)$  is given such that

$$T(X) = (t_0, t_1, t_2, t_3), \quad (7)$$

$$t_j = \sum_{e_0 + \dots + e_3 = s} t_{j e_0 e_1 e_2 e_3} x_0^{e_0} x_1^{e_1} x_2^{e_2} x_3^{e_3} \mod q \quad (8)$$

with the coefficients  $t_{j e_0 e_1 e_2 e_3} \in \mathbf{F}_q$ .  $e_0, e_1, e_2$  and  $e_3$  are non-negative integers which satisfy  $e_0 + \dots + e_3 = s$ .

#### 5. Proposed digital signature scheme

We construct the digital signature scheme using  $F(X)$  and  $T(X)$  as follows.

Let's describe the procedure that user U sends to user V a signature  $S$  including a message  $E = (E_0, E_1, E_2, E_3) \in \mathbf{H}$ .

1) User U selects randomly  $A_i \in \mathbf{H} (i=1, \dots, m)$ ,  $R \in \mathbf{H}$  where  $R$  is non-commutative to  $A_i \in \mathbf{H} (i=1, \dots, m)$ .

2) User U generates  $F(X)$  and  $T(X)$  such that

$$F(X) = \sum_{i=1}^m [k_i \prod_{j=0}^d A_i^{r^j} X^{r^j}], \quad (9)$$

$$T(X) = \sum_{i=1}^m [k_i \prod_{j=0}^d A_i^{r^j} R^{r^j} X^{r^j}]. \quad (10)$$

3) User U calculates  $f_{j e_0 e_1 e_2 e_3}$  and  $t_{j e_0 e_1 e_2 e_3}$  from (9) and (10).

4) User U publishes the coefficients  $f_{j e_0 e_1 e_2 e_3}$  in the element expression of  $F(X)$  as user U's public keys beforehand.

5) User U sends  $S$  to User V such that

$$S = \{t_{j e_0 e_1 e_2 e_3}, R, E\}, \quad (11)$$

where  $E$  is non-commutative to  $R, A_i \in \mathbf{H} (i=1, \dots, m)$ .

6) User V calculates  $g$  from (7) and (8) as follows.

$$T(E) = (t_{E_0}, t_{E_1}, t_{E_2}, t_{E_3}) \in \mathbf{H}, \quad (12)$$

where

$$t_{E_j} = \sum_{e_0 + \dots + e_3 = s} t_{j e_0 e_1 e_2 e_3} E_0^{e_0} E_1^{e_1} E_2^{e_2} E_3^{e_3} \mod q, \quad (13)$$

$(j=0, 1, 2, 3)$ .  $e_0, e_1, e_2$  and  $e_3$  are non-negative integers which satisfy  $e_0 + \dots + e_3 = s$ .

Let  $g$  be

$$g = t_{E_0} + t_{E_1} + t_{E_2} + t_{E_3}. \quad (14)$$

7) User V confirms that  $F(RE) \neq T(E)$ .

If  $F(RE) = T(E)$ , then user V decides  $S$  to be not user U's signature.

The reason is given as follows.

The adversary can easily generate  $F(RX)$  such that

$$F(RX) = \sum_{i=1}^m [k_i \prod_{j=0}^d A_i^{r^j} (RX)^{r^j}].$$

$$F(X) \text{ satisfies } F(R^g) = F(R(R^{g-1})).$$

Then user V needs to confirm that  $F(RE) \neq T(E)$  to prevent the adversary from disguising  $F(RX)$  in  $T(X)$ .

8) User V calculates  $F(R^g)$  as follows.

Let  $R^g$  be

$$R^g = (b_0, b_1, b_2, b_3).$$

$$F(R^g) = \sum_{i=1}^m [k_i \prod_{j=0}^d A_i^{r^j} R^{g r^j}]$$

$$= F((b_0, b_1, b_2, b_3)) = (f_0', f_1', f_2', f_3') \quad (15)$$

where

$$f_j' = \sum_{e_0 + \dots + e_3 = s} f_{j e_0 e_1 e_2 e_3} b_0^{e_0} b_1^{e_1} b_2^{e_2} b_3^{e_3} \mod q, \quad (16)$$

$(j=0, 1, 2, 3)$ .  $e_0, e_1, e_2$  and  $e_3$  are non-negative integers which satisfy  $e_0 + \dots + e_3 = s$ .

9) User V also calculates  $T(R^{g-1})$  as follows.

Let  $R^{g-1}$  be

$$R^{g-1} = (c_0, c_1, c_2, c_3).$$

$$T(R^{g-1}) = \sum_{i=1}^m [k_i \prod_{j=0}^d A_i^{r^j} R^{(g-1)r^j}].$$

$$= \sum_{i=1}^m [k_i \prod_{j=0}^d A_i^{r^j} R^{g r^j}]$$

$$= T((c_0, c_1, c_2, c_3)) = (t_0', t_1', t_2', t_3') \quad (17)$$

where

$$t_j' = \sum_{e_0 + \dots + e_3 = s} t_{j e_0 e_1 e_2 e_3} c_0^{e_0} c_1^{e_1} c_2^{e_2} c_3^{e_3} \mod q, \quad (18)$$

$(j=0, 1, 2, 3)$ .  $e_0, e_1, e_2$  and  $e_3$  are non-negative integers which satisfy  $e_0 + \dots + e_3 = s$ .

10) If  $F(R^g) = T(R^{g-1})$  is true, user V considers  $S$  as user U's signature.

#### 6. Verification of the strength of our digital signature

Let's examine the strength of our digital signature. The strength of our digital signature depends on the strength of the multivariate functions described in section 3. In

other words, we mention the difficulty to obtain  $k_i \in \mathbf{Fq}$  and  $A_i \in \mathbf{H}$  ( $i=1, \dots, m$ ) from the value of coefficients  $f_{j e_0 e_1 e_2 e_3}$  of  $F(X)$  to be the public keys .

### 6.1 Multivariate algebraic equations from $F(X)$

Let  $A_j$  be  
 $A_i = (A_{i0}, A_{i1}, A_{i2}, A_{i3}) \quad (i=1, \dots, m)$  . (19)

All  $f_{j e_0 e_1 e_2 e_3}$  have the form

$$f_{j e_0 e_1 e_2 e_3} = \sum_{i=1}^m k_i \sum_{c_{ij0} + \dots + c_{ij3} = s} h_{ij e_0 \dots e_3 c_{ij0} \dots c_{ij3}} A_{i0}^{c_{ij0}} \dots A_{i3}^{c_{ij3}} \pmod q \quad (20)$$

$$(j = 0, \dots, 3; 0 \leq e_0, e_1, e_2, e_3 \leq s)$$

with the coefficients  $h_{ij e_0 \dots e_3 c_{ij0} \dots c_{ij3}} \in \mathbf{Fq}$  where  $c_{ij0}, c_{ij1}, c_{ij2}$  and  $c_{ij3}$  are non-negative integers which satisfy  $c_{ij0} + \dots + c_{ij3} = s$ .

From (20) we obtain  $n$  multivariate algebraic equations over  $\mathbf{Fq}$  where  $k_i$  and  $A_{ij} \in \mathbf{Fq}$  ( $i=1, \dots, m; j=0, \dots, 3$ ) are the variables i.e. unknown numbers.

The public keys are  $PK = \{q, r, d, f_{j e_0 e_1 e_2 e_3}\}$  ( $j=0, \dots, 3; 0 \leq e_0, \dots, e_3 \leq s$ ) and the secret keys are  $SK = \{k_i, A_i\}$  ( $i=1, \dots, m$ ) in our digital signature scheme.

### 6.2 Cryptanalysis using Gröbner bases

It is said that the Gröbner bases attacks is efficient for solving multivariate algebraic equations .We calculate the complexity  $G[10]$  to obtain the Gröbner bases for our multivariate algebraic equations on quaternion ring so that we confirm immunity of our digital signature scheme to the Gröbner bases attack .

We describe in the case of  $d=2, r=3$  as samples of lower degree equations.

$$s' : \text{degree of equations} = s + 1 = 1 + 3 + 3^2 + 1 = 14.$$

$$n : \text{the number of equations} = 4 \binom{s+3}{3} = 2240.$$

We select  $m$  so that the number of variables (i.e secret keys) is nearly equal to  $n$  , that is

$$m = \lceil 4 \binom{s+3}{3} / (4+1) \rceil = 448,$$

where  $\lceil * \rceil$  means the largest integer less than or the integer equal to  $*$ .

$$v : \text{the number of variables} = 5m = 2240$$

$$d_{reg} = s' + 1 = 15$$

$G = O(\binom{v}{n} G_{d_{reg}}^w) = O(2^{302})$  is more than  $2^{80}$  which is the standard for safety where  $w=2.39$ .

Our digital signature scheme is immune from the Gröbner bases attacks and from the differential attacks because of the equations of high degree in (20).

It is thought that the polynomial-time algorithm to break our digital signature scheme does not exist probably.

## 7. The Size of the keys

We consider the size of the system parameter  $q$  . We choose  $q = O(2^{20})$  so that the size of the space of  $F(R^8)$  or  $T(R^{8^1})$  is more than  $O(2^{80})$ .

In the case of  $d=2$  and  $r=3$  , the size of  $PK$  ,  $SK$  and  $S$  is 45kbits , 45kbits , 45kbits each.

## 8. Conclusion

We proposed the digital signature scheme using multivariate functions on non-commutative quaternion ring over  $\mathbf{Fq}$ . It is a computationally difficult problem to obtain the secret keys  $\{k_i, A_i\}$  from the public keys  $\{q, r, d, f_{j e_0 e_1 e_2 e_3}\}$  because the problem is one of NP complete problems. In order to ensure the safety, the size of  $q$  is to be more than 20 bits .

We can construct the same schemes on the other non-commutative ring ,for example matrix ring.

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