Interplay between (Im)perfectness, Synchrony and Connectivity: The Case of Reliable Message Transmission

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ABSTRACT. We consider the distributed setting, where a subset of nodes are under the control of a malicious adversary with unbounded computational powers. The problem of simulating a directed reliable channel from a sender **S** to a receiver **R** in the absence of a true physical point to point channel is fundamental to the area of Distributed Computing and is popularly known as "Unconditionally Reliable Message Transmission" (URMT).
In this work, we distinguish between two variants of URMT - *Monte Carlo* and *Las Vegas*. The former protocols allow **R** to output an incorrect message with negligibly small probability whereas the latter only allows **R** to abort the protocol with a small probability but never output an incorrect message.
We establish a novel hierarchy with respect to the connectivity requirements for URMT protocols to be possible over directed networks, under two extremes w.r.t timing model: all the edges are either *synchronous* or *asynchronous*. We show that the minimum connectivity requirements for the existence of Las Vegas URMT protocols over synchronous networks is same as that of Monte Carlo URMT protocols over asynchronous networks - a surprising equivalence between two very different models. Furthermore, the higher connectivity requirements for Las Vegas URMT over asynchronous networks match exactly with that of zero-error (perfect) protocols over synchronous networks. While it is well-known that, in synchronous networks, the minimum connectivity requirements for Monte Carlo URMT protocols to exist is strictly less than those where for perfect protocols, we establish the fact that the connectivity requirements for the case of Las Vegas URMT are strictly more than that of Monte Carlo URMT.
We also show that, for the "easier" randomized variant (ones demanding lesser minimum connectivity requirements compared to the perfect ones) the number of *critical edges* are higher than that of the perfect protocols, in the worst case. Hence, establishing an interesting interplay, for the case of URMT.

1 Introduction

Most of the distributed computing protocols assume that every pair of participating nodes share a reliable channel which is usually not true, in practice. In the Unconditionally Reliable Message Transmission (URMT) problem, two non-faulty players, the sender **S** and the receiver **R** are part of a communication network modeled as a digraph over n players/nodes influenced by an unbounded adversary that may corrupt some subset of these n players/nodes. **S** has a message that he wishes to send to **R**; the challenge is to design a protocol such that **R** correctly obtains **S**'s message with arbitrarily small error probability, irrespective of what the adversary (maliciously) does to disrupt the protocol. Note that by "unconditional", we mean that the adversary is of unbounded computational power and therefore modern cryptographic tools for verifying the integrity of the data are irrelevant.

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Analogous to randomized sequential algorithms, one may distinguish between two variants of URMT, namely, *Monte Carlo* and *Las Vegas*. In the former variant **R** outputs the sender's message with high probability and may produce an incorrect output with small probability; in the latter, **R** outputs the sender's message with high probability and with small probability may abort the protocol but in no case does the receiver terminate with an incorrect output.

2 Model and Definitions

In this section, we present the model we work with followed by rigorous definitions for (various variants of) URMT.

2.1 Network Model

We model the underlying network as a directed graph $\mathcal{N} = (V, \mathcal{E})$, where V is the set of nodes and $\mathcal{E} \subseteq V \times V$ is the set of all directed edges in the network. A sender $\mathbf{S} \in V$ and a receiver $\mathbf{R} \in V$ are two distinguished nodes. Such an abstraction is well motivated in practice because not every communication channel admits bi-directional communication or we would want not to use certain channels as they are too costly. We assume the secure channels setting, i.e., all the edges are secure, reliable and authenticated. We assume that every node can be modeled as an interactive probabilistic Turing Machine and is aware of the topology of the network of which it is a part i.e. it knows $\mathcal{N}$. An interaction between a set of players in known as *protocol*. Modeling a protocol as a set of interactive Turing machines is a popular abstraction technique, described at length in [3]. An *execution of a protocol* is defined as a run of a protocol with inputs and coin tosses from any subset of participating nodes.

2.2 Fault Model

Modern cryptography assumes the availability of Public Key Infrastructure (PKI) and digital signature schemes. However their existence is based on the conjectured hardness of some problems like integer factorization [16], discrete logarithm [10], lattice based problems [15] to name a few. It is not reasonable to base security of a real-life system on conjectured hardness (which might be proven incorrect in the course of time). Also, advent of new computing paradigms, such as quantum computing has already rendered quite a few of such techniques ineffective. For instance Shor in [18] showed that integer factorization and discrete logarithm problems can be solved in polynomial time, given access to a quantum computer. Hence, it is worthwhile to look at *information theoretically secure* schemes, which assume the worst case that the adversary has unbounded computational powers [1, 4].

We model faults in the network by a fictitious centralized entity called the *adversary* which has unbounded computing power as done in [11, 1]. A single "snapshot" of faults in the network can be described as $B \subseteq V \backslash \{\mathbf{S}, \mathbf{R}\}$ which means that all the nodes in the set B are faulty. We assume that **S** and **R** are non-faulty, for otherwise reliable message transmission need not happen. We denote the set of all such B's by $\mathbb{A}$ and refer to it as an *adversary structure*. The adversary structure is *monotone*: if $B_1 \in \mathbb{A}$ then $\forall B_2 \subset B_1,\ B_2 \in \mathbb{A}$.

We note that $\mathbb{A}$ can be uniquely represented by listing the elements in its *maximal basis* $\overline{\mathbb{A}} = \{B \mid B \in \mathbb{A}, \nexists X \in \mathbb{A} \text{ s.t. } B \subset X\}$. Abusing the standard notation, we assume that $\mathbb{A}$ itself is a maximal basis. We refer to $\mathbb{A}$ as a *t-threshold adversary* if $\mathbb{A} = \{B \mid B \subseteq V \setminus \{\mathbf{S}, \mathbf{R}\}$ and $|B| = t\}$. We only deal with cases where $|\mathbb{A}| \geq 2$, since otherwise the problems are trivial.

We allow Byzantine corruption, i.e., all nodes in the set $B \in \mathbb{A}$ corrupted by the adversary can deviate arbitrarily from the designated protocol. Adversary may also deviate from the assigned protocol in one of the following constraint manners:-

1. *Fail-stop*: A node corrupted in fail-stop fashion may stop sending and receiving messages at any point of time during an execution of a protocol.
2. *Passive*: Adversary may eavesdrop on the passively corrupt nodes.
3. *Omission*: Nodes corrupted in omission fashion may omit (sending or receiving) certain messages and adversary may eavesdrop on them. Such a corruption is stronger than both, the passive and the fail-stop corruption.

Note that all three corruptions described above are weaker than that of Byzantine corruption and are a special case of it.

We consider the case of *adaptive* adversary – it can choose which nodes to corrupt during an execution of a protocol based on its view, as long as the set of nodes corrupted during the entire execution is a member of $\mathbb{A}$. Results in this paper also hold for the case of *static adversary*, one in which adversary chooses which nodes to corrupt in an execution of a protocol prior to it's initiation. One may also consider the case of *mobile adversary* in which at any given time during an execution of a protocol adversary can corrupt at most one member of $\mathbb{A}$. We remark that our results do not hold for the mobile adversary setting.

We assume that the adversary knows the topology of the network as well as the protocol specification. We further make a conservative assumption that the adversary knows the message sender **S** has chosen to send to **R**. The results we prove in this paper hold good even if we do not make this assumption, but with a slight change in our definition of URMT (see [8]).

2.3 Timing Model

Protocols running over directed networks tolerating an adversary rely heavily on the information of the timing of the events within the system. We consider two extremes w.r.t timing model, i.e., all the edges in the network are either *synchronous* or *asynchronous*. Former case is referred to as synchronous networks and the latter as asynchronous networks.

In synchronous networks, a protocol is executed in a sequence of *rounds* where in each round, a player can send messages to his out-neighbours, receive the messages sent in that round by his in-neighbours and performs local computation on the received messages, in that order. Readers may find rigorous description of the model in [1].

In asynchronous networks, there is no fixed upper bound on the timings of events. In order to model the computation is such networks, we assume that the adversary is additionally equipped with the ability to schedule all the messages exchanged over the network while remaining oblivious to the messages being exchanged. Computation in such networks proceed in a sequence of steps, order of which is controlled by the adversary. In each

step a single node is active. The node is activated by receiving a message; it then performs an internal computation, and possibly sends messages on its outgoing channels. For, more details we refer the readers to [2, 7].

2.4 Reliability

Let the message space be a large finite field $\langle \mathbb{F}, +, \cdot \rangle$. By f^{-1}, we denote additive inverse of any $f \in \mathbb{F}$. All the computations are done in this field.

We refer to *Las Vegas* URMT as URMT$_{LV}$ and *Monte Carlo* URMT as URMT$_{MC}$. We may also use URMT without any subscript to refer to both the variants together. We now formally define what we mean by a protocol being URMT$_{LV}$ or URMT$_{MC}$. All the probabilities are taken over the coin tosses of all the participating honest nodes and the adversary.

Definition 1. *[$(\mathbb{A}, \delta)$-URMT$_{MC}$] Let $\delta < \frac{1}{2}$. We say that a protocol for transmitting messages in a network $\mathcal{N}$ from $\boldsymbol{S}$ to $\boldsymbol{R}$ is $(\mathbb{A}, \delta)$-URMT$_{MC}$ if for all valid Byzantine corruptions of any $B \in \mathbb{A}$ and $\forall \mathbf{m} \in \mathbb{F}$, the probability that $\boldsymbol{R}$ outputs $\mathbf{m}$ given that $\boldsymbol{S}$ has sent $\mathbf{m}$, is at least $(1 - \delta)$. Otherwise $\boldsymbol{R}$ outputs $\mathbf{m}' \neq \mathbf{m}$ or does not terminate.*

Definition 2. *[$(\mathbb{A}, \delta)$-URMT$_{LV}$] Let $\delta < \frac{1}{2}$. We say that a protocol for transmitting messages in a network $\mathcal{N}$ from $\boldsymbol{S}$ to $\boldsymbol{R}$ is $(\mathbb{A}, \delta)$-URMT$_{LV}$ if for all valid Byzantine corruptions of any $B \in \mathbb{A}$ and $\forall \mathbf{m} \in \mathbb{F}$, the probability that $\boldsymbol{R}$ outputs $\mathbf{m}$ given that $\boldsymbol{S}$ has sent $\mathbf{m}$, is at least $(1 - \delta)$. Otherwise, $\boldsymbol{R}$ outputs a special symbol $\perp$ $(\notin \mathbb{F})$ or does not terminate.*

Definition 3. *[$\mathbb{A}$-PRMT] We say that a protocol for transmitting messages in a network $\mathcal{N}$ from $\boldsymbol{S}$ to $\boldsymbol{R}$ is $\mathbb{A}$-PRMT if for all valid Byzantine corruptions of any $B \in \mathbb{A}$ and $\forall m \in \mathbb{F}$, the probability that $\boldsymbol{R}$ outputs $\mathbf{m}$ when $\boldsymbol{S}$ has sent $\mathbf{m}$ is 1.*

We refer to $(\mathbb{A}, \delta)$-URMT and $\mathbb{A}$-PRMT as (t, δ)-URMT and t-PRMT respectively, when $\mathbb{A}$ is a t-threshold adversary.

2.5 Preliminaries

Following definitions are helpful while we present sufficiency proofs in the subsequent sections.

Definition 4. *[Strong path] A sequence of vertices $v_1, v_2, v_3, \ldots, v_k$ is said to be a strong path from v_1 to v_k in the network $\mathcal{N} = (V, \mathcal{E})$ if for each $1 \leq i < k$, $(v_i, v_{i+1}) \in \mathcal{E}$. Note that, a strong path is just a path in its usual sense. Furthermore, we assume that there vacuously exists a strong path from a node to itself.*

Definition 5. *[Weak path] A sequence of vertices $v_1, v_2, v_3, \ldots, v_k$ is said to be a weak path from v_1 to v_k in the network $\mathcal{N} = (V, \mathcal{E})$ if for each $1 \leq i < k$, $(v_i, v_{i+1}) \in \mathcal{E}$ or $(v_{i+1}, v_i) \in \mathcal{E}$.*

Definition 6. *[Blocked node] A node u along a weak path p is called a blocked node if its out-degree along p is 0.*

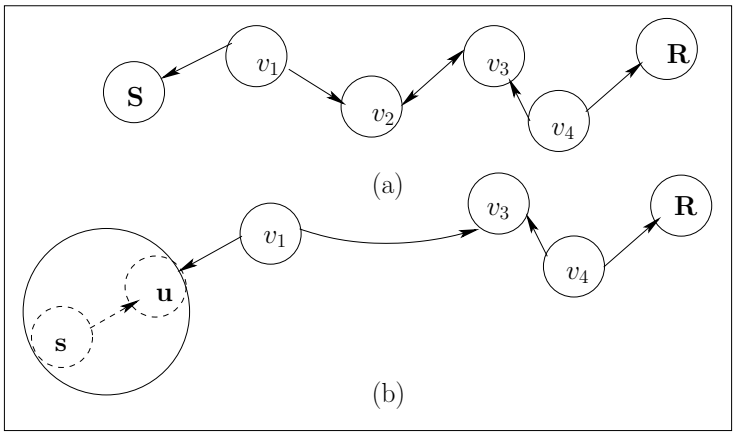


Figure 1: (a) A weak path between **S** and **R**. (b) We view it as a sequence of alternating blocked nodes and head nodes starting with a head node **s** (the virtual sender) and ending in a blocked node **R**. Nodes **s**, v_1 and v_4 are head nodes; **u** and v_3 are blocked nodes such that there is a strong path from **s** to **u**; v_1 to **u** and v_3; and v_4 to v_3 and **R**.

DEFINITION 7.*[Head node] A node y along a weak path p is called a head node if it is an intermediate node with out-degree 2 or a terminal node with out-degree 1.*

Every weak path p between **S** and **R** can be viewed as an alternating sequence of blocked nodes u_i's and head nodes y_i's starting with **S** as a head node denoted by y_0 and ending into **R** as a blocked denoted by u_{n+1} i.e. $\exists\ n > 0$, path p can be viewed as $y_0, u_1, y_1, u_2, y_2, \ldots, u_n, y_n, u_{n+1}$ such that each y_i has a strong path to u_i and u_{i+1} along p. The case when **S** is not a head node (i.e. it is a blocked node) along p, the following described simulation within the node **S** can enusre that a virtual sender **s** is a head node: (i) **S** simulates two nodes, **s** and **u** and a directed edge (**s**,**u**). (ii) The incoming path to **S** along p (which made **S** a blocked node) is now an incoming path to **u**.
Analogously, it can always be ensured that **R** is a blocked node. We elaborate our representation of weak path with an example in Figure 1. Such a representation of a weak path comes handy in giving easy to understand sufficiency proofs.

DEFINITION 8.*[Critical edge] In a digraph G for which URMT protocol exists, an edge is said to be critical if the deletion of that edge renders URMT impossible.*

DEFINITION 9.*[Authentication function] Let $K_1, K_2, K_3 \in_R \mathbb{F} \times \mathbb{F} \times \mathbb{F}$ and $m \in \mathbb{F}$. Authentication function χ is defined as $\chi(m; K_1, K_2, K_3) = (m + K_1, (m + K_1) \cdot K_2 + K_3)$.*

The authentication function *authenticates* any message $m \in \mathbb{F}$. We refer to K_1, K_2 and K_3 as *keys*. If three randomly chosen keys (unknown to the adversary) are established between two nodes u and v such that $\exists$ path p from u to v then authentication function is used as follows: (a) Say, $x = m + K_1$ and $y = x \cdot K_2 + K_3$; u sends $\langle x, y \rangle$ (a two tuple) to v along path p. (b) Say node v receives $\langle x', y' \rangle$. v verifies if $y' \stackrel{?}{=} x' \cdot K_1 + K_2$. If the verification passes then $x' = x$ with probability at least $\frac{|\mathbb{F}|-1}{|\mathbb{F}|}$, or otherwise v can deduce with certainty

that p is a faulty path. (Proofs for the same appear in [14]). Note that, authentication function described above ensures that no information regarding m is revealed to the adversary sitting on the path p. We now describe an alternative authentication function χ defined as $\chi(m; K_1, K_2) = (m, m \cdot K_1 + K_2)$, which when used instead of one described using three keys meets all the properties described above but one – that is it reveals m to the adversary sitting on p.

3 Related Work

The problem of PRMT was formulated by Dolev et. al in [6]. They showed that PRMT tolerating t-threshold adaptive byzantine adversary when underlying network can be abstracted as n synchronous, all bi-directional or all uni-directional channels from **S** and **R** is possible iff $n > 2t$. f PRMT was formulated by Dolev et. al in [6]. They showed that PRMT tolerating t-threshold adaptive byzantine adversary when underlying network can be abstracted as n synchronous, all bi-directional or all uni-directional channels from **S** and **R** is possible iff $n > 2t$. The problem of URMT was first defined by Franklin et. al. in [8]. They considered the problem of Monte Carlo URMT over undirected networks tolerating threshold adaptive Byzantine adversary and showed that t-URMT is possible iff t-PRMT is possible. In [5], Desmedt et. al. gave efficient protocols for URMT (which also achieve perfect privacy) in directed networks while abstracting the network as a collection of disjoint wires between **S** and **R**. Arpita et. al, Kannan et. al, and Kannan in [13], [20], and [19] respectively, generalize the result to the case of threshold adaptive mixed adversary when underlying network is all bi-directional channels between **S** and **R**. A more general setting is considered in [19, 22, 17], which is also the setting in which we work in this paper, where the underlying network is abstracted as a directed graph and every node is as powerful as an interactive probabilistic Turing Machine. The characterization of directed synchronous networks for the possibility of Monte Carlo URMT tolerating non-threshold adaptive mixed adversary is done in [19, 22], a simplified characterization in the case of threshold adaptive Byzantine adversary follows in [17].

To the best of our knowledge there is no literature which deals with URMT over asynchronous networks. We initiate the study in this direction. The theorem presented in this work, establishing equivalence between the connectivity requirements for the possibility of synchronous Las Vegas URMT and that of asynchronous Monte Carlo URMT appears (without proof) as a brief announcement in [12].

Franklin and Wright in [9], Wang and Desmedt in [23], and Desmedt and Wang in [5] study the problem of Monte Carlo URMT tolerating Byzantine adversary in synchronous multi-recipient (multi-cast) model. In such a model, a message sent by a node is simultaneously received by all it's neighbours. They assume that all the edges are bi-directional and all the paths between **S** to **R** are disjoint. They refer to it as "multicast line" model. Srinathan et. al. in [21] consider the problem of Monte Carlo URMT in synchronous directed hyper-graphs tolerating threshold adaptive mixed adversary. Directed hyper-graphs are strict generalization of the multi-cast line model i.e. edges may be directed and the paths between **S** and **R** need not be disjoint.

4 Our Contributions

We initiate and study of exact characterization of URMT$_{LV}$ over synchronous and URMT$_{MC}$ over asynchronous, directed networks. Though these problems seem unrelated, interestingly, we show that former is possible iff latter is possible. We also initiate the study of URMT$_{LV}$ in asynchronous (directed) networks and show that in asynchronous networks, minimum connectivity requirements for it to be possible are no less than that required for PRMT.

We further improve our insights in the problem by studying on how sparse can a digraph that permits URMT be. Specifically, we say that an edge is *critical* if its removal renders the graph insufficiently connected for URMT protocols (though before its removal the connectivity was sufficient). Ironically, it turns out that for perfect protocols, the number of critical edges is always $O(n)$ where as for the "easier" randomized protocols, we give a family of digraphs with $\Omega(n^2)$ critical edges! We remark that an earlier attempt in [17] to give such a family of digraphs for the case of URMT$_{MC}$ protocols is incorrect and we correct the same; we also give similar families of digraphs (with $\Omega(n^2)$ critical edges) for synchronous URMT$_{LV}$ (and asynchronous URMT$_{MC}$) protocols.

5 Organization of paper

We characterize directed networks for the possibility (and impossibility) of (i) Las Vegas URMT in synchronous networks (ii) Monte Carlo URMT in asynchronous networks (iii) Las Vegas URMT in asynchronous networks, tolerating a non-threshold adaptive byzantinely corrupt adversary structure $\mathbb{A}$ in Section 6, 7 and 8 respectively. Dealing with arbitrary sized adversary structure is hard and non-intuitive, so we take the following described two step approach in each of these sections. We show that, in order to tolerate an adversary structure $\mathbb{A}$, it is sufficient to tolerate all its two sized subsets. Surprisingly, this turns out to be true for all three problems of URMT discussed in this paper. We then give a necessary and sufficient condition for the respective URMT problems tolerating a two-sized adversary structure to be possible (in principle).

In Section 9, we study the concept of *critical edges*. We first show that the family of digraphs over n nodes proposed in [17] for synchronous URMT$_{MC}$ claimed to have $\Omega(n^2)$ critical edges has only $O(n)$ critical edges. Then we present another family which has indeed $\Omega(n^2)$ critical edges. Further in the section, we give a family of digraphs with $\Omega(n^2)$ critical edges w.r.t. the two variants of URMT - synchronous URMT$_{LV}$ and asynchronous URMT$_{MC}$.

6 Characterizing synchronous networks for $(\mathbb{A}, \delta)$-URMT$_{LV}$

In this section, we deal with the possibility and impossibility of Las Vegas URMT protocols from a sender **S** to a receiver **R**, tolerating an adversary structure $\mathbb{A}^*$, when the underlying network can be abstracted as a directed graph, all its edges being synchronous. We refer to this variant of URMT as $(\mathbb{A}, \delta)$-URMT$_{LV}$, formally defined in Definition 2.

It is easier to deal with fixed size adversary structures. In the following theorem we show that in the case of synchronous networks, the problem of characterizing networks for

*adversary model assumed in this work is non-threshold, adaptive and Byzantine, as described earlier

(the (im)possibility of) $(\mathbb{A}, \delta)$-URMT$_{LV}$ reduces to the problem of characterizing networks for (B, δ)-URMT$_{LV}$, where $|B| = 2$ (similar reductions can be found in [22, 17]).

THEOREM 10. *In a directed synchronous network $\mathcal{N}$, $(\mathbb{A}, \delta)$-URMT$_{LV}$ protocol is possible if and only if for every adversary structure $B \subseteq \mathbb{A}$ such that $|B| = 2$, (B, δ)-URMT$_{LV}$ protocol is possible.*

PROOF. *Necessity*: Obvious. *Sufficiency:* We show how to construct a protocol tolerating an adversary structure of larger size from protocols tolerating adversary structures of smaller size without increasing the probability of error. Therefore if protocols tolerating adversary structures of size two are available, we can inductively construct protocol tolerating any arbitrary sized adversary structure.

Let $f \in \mathbb{F}$ be any element **S** intends to send to **R**. Let $\mathcal{A} \subseteq \mathbb{A}$. Consider three $\lceil \frac{2|\mathcal{A}|}{3} \rceil$-sized subsets of $\mathcal{A}$, namely $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$, such that each element of $\mathcal{A}$ occurs in at least two distinct $\mathcal{A}_i$'s. For $i \in \{1, 2, 3\}$, let Y_i be an $(\mathcal{A}_i, \delta)$-URMT$_{LV}$ protocol. Using Y_i as sub-protocol, we first construct Z_i which is an $(\mathcal{A}, \frac{\delta}{2})$-URMT$_{LV}$ protocol. This is done by repeating each Y_i sufficiently many times, keeping the **S**'s input same as f, in order to amplify the probability of success. We then use Z_i's to construct a protocol Γ which is an $(\mathcal{A}, \delta)$-URMT$_{LV}$ protocol (as proved in the following lemma) as follows:

- For each $i \in \{1, 2, 3\}$, sub-protocol Z_i is run on f.
- **R** outputs the majority of the outcomes of the three sub-protocols and in case there is no majority, it outputs $\perp$.

∎

LEMMA 11. *For the directed synchronous network $\mathcal{N}$, the protocol Γ constructed above is an $(\mathcal{A}, \delta)$-URMT$_{LV}$ protocol.*

PROOF. Any set $B \in \mathcal{A}$ is present in at least two subsets among $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$; say the two subsets are $\mathcal{A}_2$ and $\mathcal{A}_3$. Hence the outcome of the two sub-protocols Z_2 and Z_3 is correct with at least $1 - \frac{\delta}{2}$ probability each. Since **R** outputs the majority of the outcomes, its output is correct if both the sub-protocols produce the correct outcome which happens with at least $(1 - \frac{\delta}{2})^2$ probability. Hence the error probability is upper bounded by $\delta - \frac{\delta^2}{4}$ or δ. Additionally, it is easy to see that **R** would never output an incorrect message. ∎

Having reduced the problem of URMT$_{LV}$ in a synchronous network tolerating an adversary structure to the problem of URMT$_{LV}$ tolerating all its 2-sized subsets, we now proceed to characterize directed synchronous networks in which URMT$_{LV}$ tolerating adversary structure $B = \{B_1, B_2\}$ is possible (where $B_1, B_2 \in \mathbb{A}$).

THEOREM 12. *In a directed synchronous network $\mathcal{N}$, (B, δ)-URMT$_{LV}$ protocol is possible if and only if for each $\alpha \in \{1, 2\}$, there exists a weak path q_α avoiding nodes in $B_1 \cup B_2$ such that every node u along the path q_α has a strong path to **R** avoiding all nodes in $B_{\overline{\alpha}}$[†] (Paths q_1, q_2 need not be distinct.)*

We prove the theorem in the following sub-sections.

[†]We denote $\overline{1} = 2$ and vice-versa.

6.1 Sufficiency

For a directed synchronous network $\mathcal{N}$, which satisfies the conditions given in Theorem 12, we show how to construct a protocol Π tolerating the adversary structure $B = \{B_1, B_2\}$. Let m be the message **S** intends to send. If either q_1 or q_2 is a strong path from **S** to **R**, **S** trivially sends m along that path. When this is not the case, we construct two sub-protocols Π_1 and Π_2. For each $i \in \{1, 2\}$, protocol Π_i uses the honest weak path q_i. We give a construction for Π_1, and the construction of Π_2 follows by symmetry. For convenience of writing the protocol, we first represent weak path q_1 as $y_0, u_1, y_1, u_2, y_2, \ldots, u_n, y_n, u_{n+1}$ as explained in Chapter 1 (right after the definitions of weak path, head node and blocked node). We denote **S** and **R** interchangeably as y_0 and u_{n+1} respectively. Π_1 proceeds in the following steps:

1. **S** sends m to u_1 along q_1. For $1 \leq k \leq n$, node y_k chooses 3^k random keys namely $K_{k,1}, K_{k,2}, \ldots, K_{k,3^k}$ and sends those to u_k and u_{k+1}.
2. Node u_1 receives m from **S** and keys $K_{1,1}, K_{1,2}, K_{1,3}$ from y_1, along q_1. It calculates $(\psi_{1,1}, \phi_{1,1}) = \chi(m; K_{1,1}, K_{1,2}, K_{1,3})$ and sends those to **R** along a strong path avoiding B_2 in some fixed round r_{u_1}.
   For $1 < k \leq n$, u_k receives 3^{k-1} keys from y_{k-1} and 3^k keys from y_k. It *authenticates* the keys received from y_{k-1} with the keys received from y_k and sends it to **R** along a strong path avoiding B_2 in some fixed round r_{u_k}. Formally, u_k calculates, $\forall j\ 1 \leq j \leq 3^{k-1}$, $(\psi_{k,j}, \phi_{k,j}) = \chi(K_{k-1,j}; K_{k,3j-2}, K_{k,3j-1}, K_{k,3j})$.
3. **R** receives $\{K'_{n,1}, K'_{n,2}, \ldots, K'_{n,3^n}\}$ from y_n along q_1. $\mathcal{N}$ being a synchronous network, **R** knows exactly the round number, say r'_{u_k}, in which it will receive messages that u_k sent to it in round r_{u_k}. If **R** does not receive valid messages from u_k in round r'_{u_k}, it assumes that B_1 is faulty and stops. Else if it receives $\forall k\ \forall j\ 1 \leq k \leq n, 1 \leq j \leq 3^{k-1}$, $(\psi'_{k,j}, \phi'_{k,j})$, the protocol proceeds as follows.

   for k in n to 2

   R verifies $\forall j$, $\phi'_{k,j} \stackrel{?}{=} \psi'_{k,j} \cdot K'_{k,3j-1} + K'_{k,3j}$. If the verification fails for any j, **R** concludes that B_1 is faulty and stops. Otherwise, **R** recovers $\forall j$, $K'_{k-1,j}$ as $\psi'_{k,j} + {K_{k,3j-2}}^{-1}$.

   If at the end of the loop, **R** has recovered $K'_{1,1}, K'_{1,2}, K'_{1,3}$ then **R** verifies whether $\phi'_{1,1} \stackrel{?}{=} \psi'_{1,1} \cdot K'_{1,2} + K'_{1,3}$. If the verification passes, **R** recovers $m_1 = \psi'_{1,1} + {K'_{1,1}}^{-1}$ as the message. i

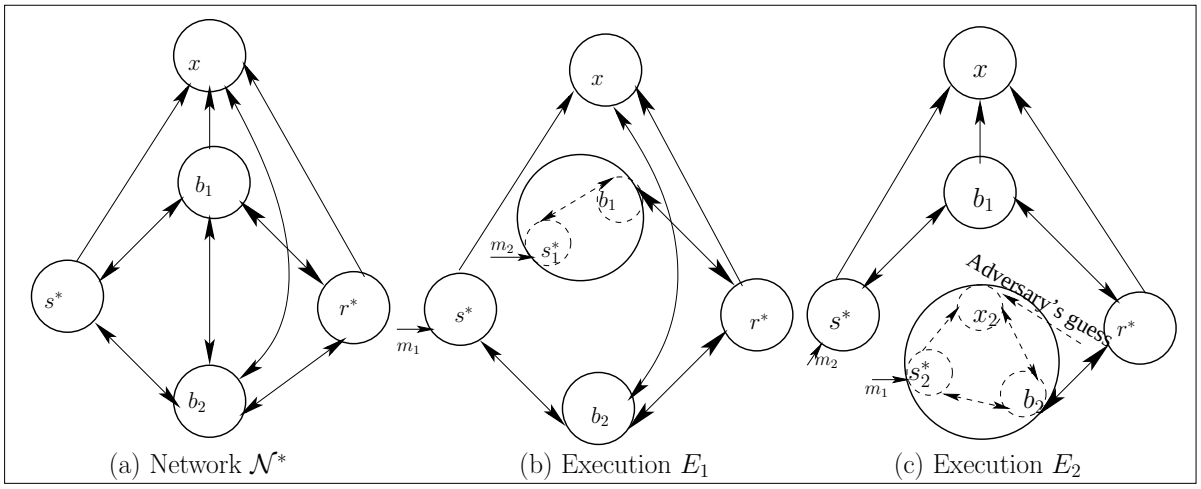
This completes the description of Π_1. The protocols Π_1 and Π_2 are run on network $\mathcal{N}$. **R** takes one of the following actions based on the outcomes of these protocols: (a) If **R** detects that B_i is corrupt in Π_i, it outputs whatever message it recovered from $\Pi_{\bar{i}}$. (b) If **R** recovers messages from each of the Π_i's and both the messages are same, it outputs that message. (c) If messages recovered through Π_1 and Π_2 are different, it outputs $\perp$. This completes Π. In the following lemma we prove that this is a $(B, \frac{1}{|\mathbb{F}|})$-URMT$_{LV}$ protocol.

LEMMA 13. *Π, as constructed above, is a $(B, \frac{1}{|\mathbb{F}|})$-URMT$_{LV}$ protocol.*

PROOF. We analyze the protocol case wise: (a) For some i, **R** concludes through Π_i that B_i is faulty, and outputs whatever it recovers from $\Pi_{\bar{i}}$. For each i, none of the nodes in $B_{\bar{i}}$

participate in the protocol Π_i. Hence, if some verification fails during Π_i, B_i has to be faulty, and $\Pi_{\bar{i}}$ should recover the correct message m. (b) For each $i \in \{1,2\}$, all verifications in Π_i pass. Case (i) $m_i = m_{\bar{i}}$, **R** outputs m_i. Since one of m_i or $m_{\bar{i}}$ has to be same as m, **R**'s output is correct. This happens with $\leq \frac{1}{|\mathbb{F}|}$ probability.

6.2 Necessity



(a) Network $\mathcal{N}^*$ (b) Execution E_1 (c) Execution E_2

Figure 2: (a) The directed network $\mathcal{N}^*$ (b) Adversary strategy when b_1 is faulty (simulates a sender with input m_2, namely s_1^*; snaps down all the communication with s^*, x, b_2). (c) Adversary strategy when b_2 is faulty (simulates a sender with input m_1, namely s_2^*; Simulates node x, namely x_2. To complete the simulation of x, adversary feeds x_2 with random elements. 'adversary's guess' is what r^* sends to x in E_1 and coin tosses with respect to all nodes (described in the lemma 15) are such that r^* is bound to output m_1 in E_2.

Let $\mathcal{N}$ be a network that does not satisfy the conditions of Theorem 12. We show that in such a network $(\{B_1, B_2\}, \delta)$-URMT$_{LV}$ from **S** to **R** is impossible. Without loss of generality, let us assume that the two sets comprising the adversary structure are disjoint[‡]. Let the path q_1[§] be not present between **S** and **R** in $\mathcal{N}$. Hence, every weak path between **S** and **R** avoiding nodes in $B_1 \cup B_2$ has at least one node w such that every strong path from w to **R** passes through B_2.

We first consider the simple network $\mathcal{N}^* = (V^*, \mathcal{E}^*)$ shown in Figure 2(a) consisting of five nodes s^*, r^*, b_1, b_2 and x where s^* is the sender and r^* is the receiver and show that $(\{\{b_1\}, \{b_2\}\}, \delta)$-URMT$_{LV}$ from s^* to r^* is impossible in Lemma 14. We then show that the digraph $\mathcal{N}$ can be partitioned into disjoint sets whose connectivity properties are similar to the connectivity between nodes of digraph $\mathcal{N}^*$ in Lemma 16. We then prove in Lemma 17 that if $(\{B_1, B_2\}, \delta)$-URMT$_{LV}$ from **S** to **R** is possible in $\mathcal{N}$ then $(\{\{b_1\}, \{b_2\}\}, \delta)$-URMT$_{LV}$

[‡]In case $B_1 \cap B_2 \neq \phi$, adversary strategy to fail any protocol in $\mathcal{N}$ includes *fail-stopping* the nodes in the intersection.

[§]The case when the path q_2 is not present from **S** to **R** can be handled analogously.

from s^* to r^* is also possible in $\mathcal{N}^*$, which is a contradiction. Hence, the conditions mentioned in Theorem 12 are necessary.

LEMMA 14. *In the synchronous network $\mathcal{N}^*$, shown in Figure 2(a), $(\{\{b_1\},\{b_2\}\},\delta)$-URMT$_{LV}$ ($\delta < 1/2$) from s^* to r^* is impossible.*

PROOF. We assume that a protocol Π^* exists in $\mathcal{N}^*$ which is a $(\{\{b_1\},\{b_2\}\},\delta)$-URMT$_{LV}$ protocol. We describe an adversary strategy $\mathcal{S}$ to fail any protocol Π^* and prove it's correctness in the following lemma. Adversary chooses any two messages $m_1, m_2 \in \mathbb{F}$, $m_1 \neq m_2$. When s^* intends to send m_i, the adversary corrupts the set b_i and snaps all communication with the nodes: $b_{\bar{i}}$, x and s^*.

When adversary corrupts b_1, it simulates a local copy of s^* on input m_2, say s_1^*. At the beginning of each round, b_1 receives messages from r^* and from the simulated s_1^*, does local computation and sends out messages to r^* and s_1^*. During the same round, s_1^* receives messages from b_1, its state is updated and messages are sent out for the next round.

When adversary corrupts b_2, it simulates a local copy of s^* on input m_1 and a local copy of x, say s_2^* and x_2 respectively. It handles the simulated s_2^* and x_2 locally in the same manner as it handled the simulated s_1^* when b_1 was corrupted. For simulation of x to happen, x_2 is to be fed with some input on behalf of r^*, since an edge $(r^*, x) \in \mathcal{E}^*$. Adversary guesses the messages sent along this edge and feeds those to x_2 round by round until the protocol terminates. Note that the node x has no strong path to r^* and hence does not have any influence on $\mathbf{R}$'s output.

For the sake of clarity, a pictorial view of the adversary strategy is shown in the Figure 2(b), (c). We prove that the above strategy fails every protocol Π^* in the following Lemma.

LEMMA 15. *With the adversary strategy $\mathcal{S}$, no protocol Π^* is a $\{\{b_1\},\{b_2\}\}$-URMT$_{LV}$ protocol.*

PROOF. Before proceeding to the proof we introduce the following notations w.r.t. to an execution E_i of the protocol Π^*: (a) The vector $\vec{C}_i = (c^i_{s^*}, c^i_{r^*}, c^i_{b_1}, c^i_{b_2}, c^i_x)$ which denotes the coin tosses input to nodes, where c^i_n denotes the coin tosses of node n. (b) The view of a node n, $view_n(E_i)$, which comprises of the internal coin tosses c^i_n of node n and the messages it receives during execution E_i. We now consider the following two executions:

1. There exists an execution E_1 of Π^* such that s^* chooses to send m_1 and r^* outputs m_1 when the random tosses used by s_1^* are denoted by a string r, for otherwise Π^* won't be a $\{\{b_1\},\{b_2\}\}$-URMT$_{LV}$ protocol.
2. Execution E_2: s^* chooses to send m_2. Coin tosses $\vec{C}_2$ of nodes are such that $c^2_{b_1} = c^1_{b_1}$, $c^2_{r^*} = c^1_{r^*}$ and $c^2_{s^*} = r$. Coin tosses of s_2^* and x_2 are $c^1_{s^*}$ and c^1_x respectively. Messages fed to x_2 by the adversary matches exactly the messages sent by r^* to x in E_1.

For the above mentioned executions E_1 and E_2, $view_{r^*}(E_1) = view_{r^*}(E_2)$. Hence r^* halts with output m_1 in execution E_2, violating the condition of Π^* being a URMT$_{LV}$ protocol.

LEMMA 16. *The set of nodes V in the network $\mathcal{N}$ can be partitioned into 5 disjoint sets $S^*, R^*, B_1' \subseteq B_1, B_2$ and X' such that $\mathbf{S} \in S^*$, $\mathbf{R} \in R^*$ and an edge exists from a node in $\vec{L}[i]$ to a node in $\vec{L}[j]$ only if $(\vec{l}[i], \vec{l}[j]) \in \mathcal{E}^*$ where $\vec{L} = [S^*, R^*, B_1', B_2, X']$ and $\vec{l} = [s^*, r^*, b_1, b_2, x]$ are two ordered lists, $\vec{l}[i]$ (resp. $\vec{L}[i]$) denotes the i^{th} element of the list $\vec{l}$ (resp. $\vec{L}$).*

PROOF. In the network $\mathcal{N}$, every weak path between **S** and **R** avoiding $B_1 \cup B_2$ has at least one node w such that every strong path from w to **R** passes through B_2.

We partition the non-faulty nodes $H = V \setminus \{B_1 \cup B_2\}$ into 3 disjoint sets namely: R^*, S^* and X defined as follows. $R^* = \{w \mid w \in H$ and $\exists$ a weak path p between w and **R** s.t all the nodes in p have a strong path to **R** avoiding nodes in $B_2\}$. $S^* = \{w \mid w \in H \setminus R^*$ and w has a strong path to **R** avoiding $B_2\}$. $X = H \setminus \{S^* \cup R^*\}$. Clearly, $\mathbf{R} \in R^*$ and $\mathbf{S} \in S^*$. Moreover, if any node $w \in X$ has a strong path to **R**, it passes through some node in B_2. We now divide the set B_1 into two disjoint sets namely: B_1' and B_1^X. $B_1' = \{u \mid u \in B_1$ and u has a strong path to **R** avoiding $B_2\}$. $B_1^X = B_1 \setminus B_1'$. We consider two sets X and B_1^X together as a set X' i.e. $X' = X \cup B_1^X$.

It trivially follows from the definitions above that $\nexists\ (u, v) \in \mathcal{E}$ such that $u \in X'$ and $v \in S^* \cup R^* \cup B_1'$, otherwise there would be a path from a node in X' to **R** avoiding B_2. Also, there cannot exist any directed edge between a node in S^* and a node in R^*. Note the only edges missing from $\mathcal{N}^*$ are $(x, s^*), (x, r^*), (x, b_1)$ and $(s^*, r^*), (r^*, s^*)$. Hence, proved.

LEMMA 17. *In the directed synchronous network $\mathcal{N} = (V, \mathcal{E})$, $(\{B_1, B_2\}, \delta)$-URMT$_{LV}$ is possible from* **S** *to* **R** *only if $(\{\{b_1\}, \{b_2\}\}, \delta)$-URMT$_{LV}$ is possible from s^* to r^* in the network $\mathcal{N}^*$.*

PROOF. We show how a $(\{B_1, B_2\}, \delta)$-URMT$_{LV}$ protocol $\mathcal{P}$ on $\mathcal{N}$ can be simulated on $\mathcal{N}^*$ to obtain a $(\{\{b_1\}, \{b_2\}\}, \delta)$-URMT$_{LV}$ protocol. We simulate a virtual network $\mathcal{N}$ over $\mathcal{N}^*$ such that $\mathcal{P}$ when run over the virtual $\mathcal{N}$ is a $(\{\{b_1\}, \{b_2\}\}, \delta)$-URMT$_{LV}$ protocol. Simulation runs as follows: Node $\vec{l}[i]$ simulates all the nodes in the set $\vec{L}[i]$. Edges in $\mathcal{N}$ lie in one of the following two categories: (i) Those amongst the nodes within a $\vec{L}[i]$. Such edges can be simulated by $\vec{l}[i]$ internally. (ii) Edges from a node in $\vec{L}[i]$ to a node in $\vec{L}[j]$. We proved in the previous lemma that, an edges exists from a node in $\vec{L}[i]$ to a node in $\vec{L}[j]$ only if there exists an edge $(\vec{l}[i], \vec{l}[j]) \in \mathcal{E}^*$. Hence, such an edge can be simulated on edge $(\vec{l}[i], \vec{l}[j])$. Now, protocol $\mathcal{P}$ is run on the virtual $\mathcal{N}$ which is a $(\{\{b_1\}, \{b_2\}\})$-URMT$_{LV}$ protocol from s^* to r^* in $\mathcal{N}^*$.

From Lemma 14 we know that $(\{\{b_1\}, \{b_2\}\}, \delta)$-URMT$_{LV}$ is impossible from s^* to r^* in the network $\mathcal{N}^*$. We arrive at a contradiction. Hence, the conditions mentioned in Theorem 12 are necessary.

7 Characterizing asynchronous networks for $(\mathbb{A}, \delta)$-URMT$_{MC}$

In this section, we deal with the possibility and impossibility of Monte Carlo URMT protocols from a sender **S** to a receiver **R**, tolerating an adversary structure $\mathbb{A}$ which is non-threshold adaptive Byzantine adversary, when the underlying network can be abstracted as a directed graph, all its edges being asynchronous. We refer to this variant of URMT as $(\mathbb{A}, \delta)$-URMT$_{MC}$, formally defined in Definition 1.

It is easier to deal with fixed size adversary structures. In the following theorem we show that in the case of synchronous networks, the problem of characterizing networks for (the (im)possibility of) $(\mathbb{A}, \delta)$-URMT$_{MC}$ reduces to the problem of characterizing networks for (B, δ)-URMT$_{MC}$, where $|B| = 2$ (similar reductions can be found in [22, 17]).

THEOREM 18. *In a directed asynchronous network $\mathcal{N}$, $(\mathbb{A}, \delta)$-URMT$_{MC}$ protocol is possible if and only if for every adversary structure $B \subseteq \mathbb{A}$ such that $|B| = 2$, (B, δ)-URMT$_{MC}$ protocol is possible.*

PROOF. *Necessity*: Obvious. *Sufficiency:* The proof takes an approach similar to the one taken in the proof of Theorem 10. However, since the network is asynchronous, the way we build a protocol tolerating larger sized adversary structure from protocols tolerating smaller sized ones changes. Let $f \in \mathbb{F}$ be any element **S** intends to send to **R**. Consider $\mathcal{A}$ and its three subsets $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ as described in Theorem 10. For $i \in \{1, 2, 3\}$, let Z_i be an $(\mathcal{A}_i, \frac{\delta}{2})$-URMT$_{MC}$ protocol which can be constructed easily by repeating $(\mathcal{A}_i, \delta)$-URMT$_{MC}$ sufficiently many times, keeping **S**'s input same as f, in order to amplify the probability of success. The protocol η which is an $(\mathcal{A}, \delta)$-URMT$_{MC}$ protocol (as proved in the following lemma) is constructed as follows:

- For each $i \in \{1, 2, 3\}$, sub-protocols Z_i are run in parallel on f.
- **R** waits until two of the three Z_i sub-protocols terminate with same output and outputs that as the message.

LEMMA 19. *For the directed synchronous network $\mathcal{N}$, the protocol η constructed above is a $(\mathcal{A}, \delta)$-URMT$_{MC}$ protocol.*

PROOF. Any set $B \in \mathcal{A}$ is present in at least two subsets among $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$; say the two subsets are $\mathcal{A}_2$ and $\mathcal{A}_3$. Hence the two sub-protocols Z_2 and Z_3 terminate with the correct output with at least $1 - \frac{\delta}{2}$ probability each. As **R** waits until two of the three Z_i sub-protocols terminate with same output, η fails only if at least one of Z_2 and Z_3 terminates with an incorrect message or does not terminate at all. Since this happens with at most $1 - (1 - \frac{\delta}{2})^2$ probability, η is an $(\mathcal{A}, \delta - \frac{\delta^2}{4})$-URMT$_{MC}$ i.e. an $(\mathcal{A}, \delta)$-URMT$_{MC}$ protocol.

Having reduced the problem of URMT$_{MC}$ in an asynchronous network tolerating an adversary structure to the problem of URMT$_{MC}$ tolerating all its 2-sized subsets, we now proceed to characterize directed asynchronous network in which URMT$_{MC}$ tolerating adversary structure $B = \{B_1, B_2\}$ is possible (where $B_1, B_2 \in \mathbb{A}$).

THEOREM 20. *In a directed asynchronous network $\mathcal{N}$, (B, δ)-URMT$_{MC}$ protocol is possible if and only if for each $\alpha \in \{1, 2\}$, there exists a weak path q_α avoiding nodes in $B_1 \cup B_2$ such that every node u along the path q_α has a strong path to **R** avoiding all nodes in $B_{\overline{\alpha}}$. (Paths q_1, q_2 need not be distinct.)*

We give the sufficiency and the necessity proofs in the following sub-sections.

7.1 Sufficiency

The protocol for the sufficiency proof of above theorem is constructed in a manner similar to the synchronous Las Vegas protocol in Section 6.1. However, there are some important differences.

For a directed asynchronous network $\mathcal{N}$, which satisfies the conditions given in Theorem 20, we show how to construct a protocol Π tolerating the adversary structure $B = \{B_1, B_2\}$. If either q_1 or q_2 is a strong path from **S** to **R**, **S** trivially sends m along that path and **R** is bound to receive it. When this is not the case, we construct two sub-protocols Π_1

and Π_2. For each $i \in \{1,2\}$, protocol Π_i uses the honest weak path q_i. We give a construction of Π_1, and the construction of Π_2 follows by symmetry. We represent weak path q_1 as $y_0, u_1, y_1, \ldots, u_n, y_n, u_{n+1}$ as explained in Section 2 (right after the definitions of weak path, head node and blocked node). Π_1 proceeds in the following steps:

1. **S** sends m to u_1 along q_1. For $1 \leq k \leq n$, node y_k chooses 3^k random keys namely $K_{k,1}, K_{k,2}, \ldots, K_{k,3^k}$ and sends those to u_k and u_{k+1}.
2. Node u_1 waits for m to arrive from **S** and keys $K_{1,1}, K_{1,2}, K_{1,3}$ to arrive from y_1 along q_1. It calculates $(\psi_{1,1}, \phi_{1,1}) = \chi(m; K_{1,1}, K_{1,2}, K_{1,3})$ and sends it to **R** along a strong path avoiding B_2.
   For $1 < k \leq n$, u_k waits for 3^{k-1} keys to arrive from y_{k-1} and 3^k keys to arrive from y_k along q_1[¶]. It authenticates the keys received from y_{k-1} with the keys received from y_k and sends it to **R** along a strong path avoiding B_2. Formally, u_k calculates
   $\forall j\ 1 \leq j \leq 3^{k-1}\ (\psi_{k,j}, \phi_{k,j}) = \chi(K_{k-1,j}; K_{k,3j-2}, K_{k,3j-1}, K_{k,3j})$.
3. **R** waits for $\{K'_{n,1}, K'_{n,2}, \ldots, K'_{n,3^n}\}$ to arrive from y_n. **R** runs the following loop:

   for k in n to 2

   R waits until it receives $\forall j\ 1 \leq j \leq 3^{k-1},\ (\psi'_{k,j}, \phi'_{k,j})$ from u_k[†]. If **R** does receive, it verifies $\forall j,\ \phi'_{k,j} \stackrel{?}{=} \psi'_{k,j} \cdot K'_{k,3j-1} + K'_{k,3j}$. If the verification fails for any j, **R** concludes that B_1 is faulty and stops. Otherwise, **R** recovers $K'_{k-1,j}$ as $\psi'_{k,j} + {K_{k,3j-2}}^{-1}$, for every j.

   If at the end of the loop, **R** has recovered $K'_{1,1}, K'_{1,2}$ and $K'_{1,3}$ then **R** waits to receive $(\psi'_{1,1}, \phi'_{1,1})$ and verifies if $\phi'_{1,1} \stackrel{?}{=} \psi'_{1,1} \cdot K'_{1,2} + K'_{1,3}$. If the verification passes, **R** recovers $m_1 = \psi'_{1,1} + {K'_{1,1}}^{-1}$ as the message.

This completes the description of Π_1. The protocols Π_1 and Π_2 are run in parallel in the asynchronous network $\mathcal{N}$. **R** takes one of the following actions based on the outcomes of these protocols: (a) If for any $i \in \{1,2\}$, Π_i concludes that B_i is faulty, **R** waits for $\Pi_{\bar{i}}$ to terminate, and outputs $m_{\bar{i}}$ as message. (b) If for any $i \in \{1,2\}$, Π_i halts with m_i as message, **R** outputs that as message without waiting for the protocol $\Pi_{\bar{i}}$ to terminate. Above is a $(B, \frac{1}{|\mathbb{F}|})$-URMT$_{MC}$ protocol as proved in the following lemma.

LEMMA 21. *Π, as constructed above, is a $(B, \frac{1}{|\mathbb{F}|})$-URMT$_{MC}$ protocol.*

PROOF. We analyze the protocol case wise: (a) For some i, Π_i outputs that B_i is faulty, and **R** outputs what it recovers from $\Pi_{\bar{i}}$. None of the nodes in B_i participate in the protocol $\Pi_{\bar{i}}$. Therefore, some verification fails during Π_i only if B_i is faulty. Hence, $\Pi_{\bar{i}}$ is bound to terminate with $m_{\bar{i}} = m$. (b) For some i, Π_i terminates successfully with output m_i. Probability that $m_i \neq m$ is at most $\frac{1}{|\mathbb{F}|}$. Hence, **R**'s output is correct with probability at least $\frac{|\mathbb{F}|-1}{|\mathbb{F}|}$.

[¶]As weak path q_1 comprises only of honest nodes, every u_i receives the keys (or message) eventually.

[†]As these messages are delivered along faulty paths, they may never arrive. However, since Π_1 and Π_2 are run in parallel (as mentioned in the sequel) and **R** waits for only one of them to terminate, the protocol Π always terminates.

7.2 Necessity

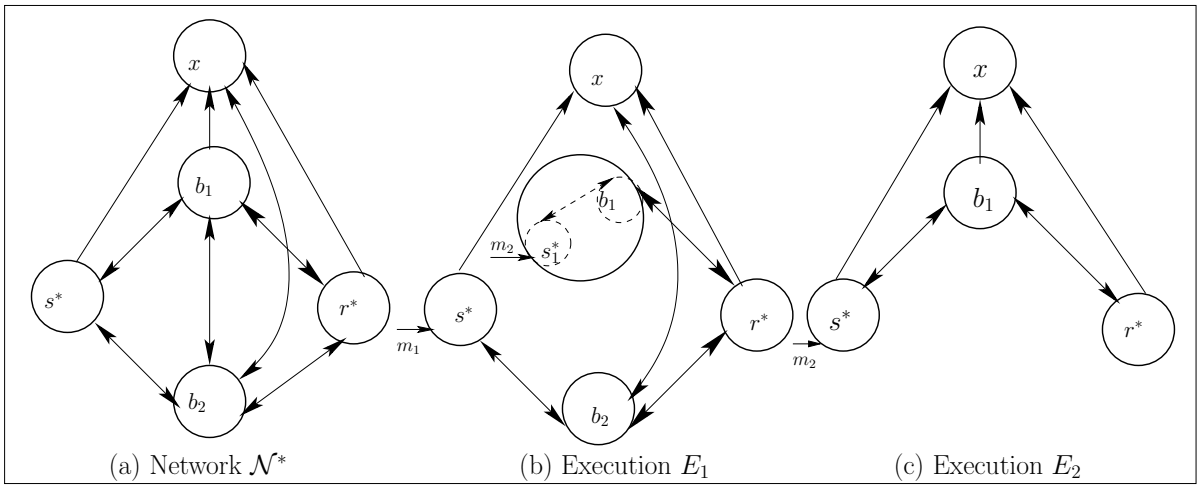


Figure 3: (a) The directed network $\mathcal{N}^*$. (b) Adversary strategy when b_1 is faulty (simulates a local copy of sender with input m_2 and snaps down communication with s^* and x; delays outgoing messages from b_2 beyond some threshold time period T). (c) Adversary strategy when b_2 is faulty (b_2 fail-stops).

Let $\mathcal{N}$ be a network that does not satisfy the condition of 20. We show that in such a network $(\{B_1, B_2\}, \delta)$-URMT$_{MC}$ from **S** to **R** is impossible. Without loss of generality, we assume that the sets B_1 and B_2 are disjoint and path q_1 is not present between **S** and **R** in $\mathcal{N}$ (Reasons for the assumptions are clearly stated in Section 6.2). Hence, every weak path between **S** and **R** avoiding $B_1 \cup B_2$ has at least one node w such that every strong path from w to **R** passes through B_2. We again consider the simple network $\mathcal{N}^* = (V^*, \mathcal{E}^*)$ shown in Figure 5(a) consisting of five nodes s^*, r^*, b_1, b_2 and x where s^* is the sender and r^* is the receiver. However, this time the edges between nodes are asynchronous. We show that $(\{\{b_1\}, \{b_2\}\}, \delta)$-URMT$_{MC}$ from s^* to r^* is impossible in $\mathcal{N}^*$ in Lemma 22. We then need to show that the digraph $\mathcal{N}$ can be partitioned into disjoint sets whose connectivity properties are similar to the connectivity between nodes of digraph $\mathcal{N}^*$, which we have already proved in Lemma 16. Now, if $(\{B_1, B_2\}, \delta)$-URMT$_{MC}$ from **S** to **R** is possible in $\mathcal{N}$ then $(\{\{b_1\}, \{b_2\}\})$-URMT$_{MC}$ from s^* to r^* is possible in $\mathcal{N}^*$, which leads us to a contradiction (We need not prove this separately as the proof given in Lemma 17 works even when both $\mathcal{N}$ and $\mathcal{N}^*$ are asynchronous networks). Hence, the conditions mentioned in Theorem 20 are necessary.

LEMMA 22. *In the asynchronous network $\mathcal{N}^*$, shown in figure 5(a), $(\{\{b_1\}, \{b_2\}\}, \delta)$-URMT$_{MC}$ ($\delta < 1/2$) from s^* to r^* is impossible.*

PROOF. We assume that a protocol Π^* exists in $\mathcal{N}^*$ which is a $(\{\{b_1\}, \{b_2\}\}, \delta)$-URMT$_{MC}$ protocol from s^* to r^*. This implies that there exists a finite time instant T such that for every message m, r^* outputs m before instant T in at least half of the executions in which s^* has chosen to send m. We now describe an adversary strategy to fail any such protocol Π^*.

Adversary chooses any two messages $m_1, m_2 \in \mathbb{F}, m_1 \neq m_2$ and ensures that the probability that r^* outputs m_1 given s^* has sent m_1 is $< \frac{1}{2}$. When s^* intends to send m_i, it corrupts the node b_i, for $i \in \{1, 2\}$. When b_2 is corrupt, adversary simply fail-stops it. When b_1 is corrupt, adversary does the following:

- Simulates a local copy of s^* on input m_2, say s_1^*.
- Snaps all it's communication with the nodes: s^*, x and b_2.
- Delays all the outgoing messages from b_2 beyond the time instant T. Schedules events between the rest of the nodes (and simulated nodes) i.e. s_1^*, r^*, b_1 and x as it schedules events between s^*, r^*, b_1 and x respectively when s^*'s input is m_2 (Note that in an asynchronous network, the adversary is additionally equipped with the ability to schedule messages).

We now show that for every execution in which s^*'s input is m_2, there is an execution in which s^*'s input is m_1 such that the view at r^* is same in both the executions. Consider the following two executions:-

1. Execution E_1: s^*'s input is m_2 and coin tosses of s, b_1 and r^* are c_1, c_2 and c_3 respectively. Note that in this execution the coin tosses of x do not affect the output at r^* as there is no strong path from x to r^*.
2. Execution E_2: s^*'s input is m_1 and coin tosses of s_1^*, b_1 and r^* are c_1, c_2 and c_3 respectively. Note that in such an execution the coin tosses of s^* and x do not affect the output at r^*.

For the sake of clarity, a pictorial view of the adversary strategy is shown in the Figure 5(b), (c). It follows that for every such E_1 there is a corresponding E_2 and $View_{r^*}(E_1) = View_{r^*}(E_2)$. This ensure, the probability that r^* outputs m_2 given s^* has sent m_1 is same as the probability that r^* outputs m_2 given s^* has sent m_2, which is $> \frac{1}{2}$. Hence, a contradiction.

We now present one of the main results of this paper i.e. synchronous Las Vegas protocols are possible if and only if asynchronous Monte Carlo protocols.

COROLLARY 23. *In a directed network $\mathcal{N} = (V, \mathcal{E})$, a synchronous $(\mathbb{A}, \delta)$-URMT$_{LV}$ protocol exists if and only if a protocol exists for asynchronous $(\mathbb{A}, \delta)$-URMT$_{MC}$.*

PROOF. Follows from Theorem 10, 12 and 18, 20. ∎

8 Characterizing asynchronous networks for $(\mathbb{A}, \delta)$-URMT$_{LV}$

In this chapter, we deal with the possibility and impossibility of Las Vegas URMT protocols from a sender **S** to a receiver **R**, tolerating an adversary structure $\mathbb{A}$ which is a non-threshold adaptive Byzantine adversary, when the underlying network can be abstracted as a directed graph, all its edges being asynchronous.

In the following theorem we show that in the case of asynchronous networks, the problem of characterizing networks for (the (im)possibility of) $(\mathbb{A}, \delta)$-URMT$_{LV}$ reduces to the problem of characterizing networks for (B, δ)-URMT$_{LV}$, where $|B| = 2$.

THEOREM 24. *In a directed asynchronous network $\mathcal{N}$, $(\mathbb{A}, \delta)$-URMT$_{LV}$ protocol is possible if and only if for every adversary structure $B \subseteq \mathbb{A}$ such that $|B| = 2$, (B, δ)-URMT$_{LV}$ protocol is possible.*

PROOF. Similar to the proof of Theorem 18, hence omitted. ∎

Having reduced the problem of URMT$_{LV}$ in an asynchronous network tolerating an adversary structure to the problem of URMT$_{LV}$ tolerating all its 2-sized subsets, we now proceed to characterize directed asynchronous networks in which URMT$_{LV}$ tolerating adversary structure $B = \{B_1, B_2\}$ is possible (where $B_1, B_2 \in \mathbb{A}$).

THEOREM 25. *In a directed asynchronous network $\mathcal{N}$, (B, δ)-URMT$_{LV}$ protocol is possible if and only if there exists a strong path from* **S** *to* **R** *avoiding nodes in $B_1 \cup B_2$.*

PROOF. *Sufficiency*: Let f be the field element **S** intends to send. Send f to **R** along the strong path avoiding nodes in $B_1 \cup B_2$. Since, the path does not contain any corrupt nodes, f is eventually received by **R**. ∎

We give the necessity proof of the above theorem in the following sub-section.

8.1 Necessity

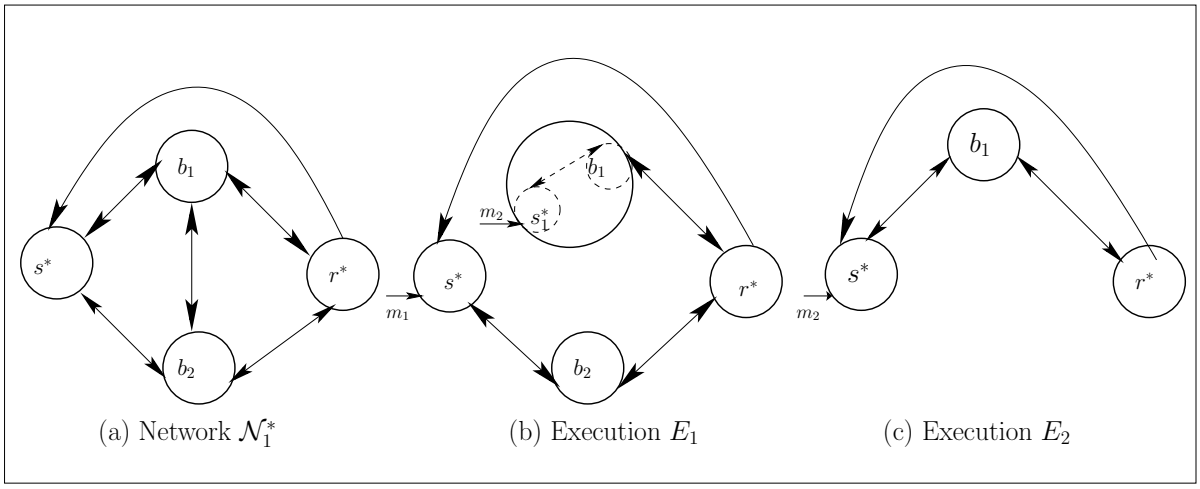


Figure 4: (a) The directed network $\mathcal{N}_1^*$ (b) Adversary strategy when b_1 is faulty (c) Adversary strategy when b_2 is faulty.

The proof in this section is along similar lines to the necessity proofs earlier. Let $\mathcal{N}$ be a network that does not satisfy the condition mentioned in Theorem 25. We first consider the simple asynchronous network $\mathcal{N}_1^* = (V_1^*, \mathcal{E}_1^*)$ with $V_1^* = \{s^*, r^*, b_1, b_2\}$ and $\mathcal{E}_1^* = (V_1^* \times V_1^*) \setminus \{(s^*, r^*)\}$ as shown in Figure 4(a) and show that $(\{\{b_1\}, \{b_2\}\})$-URMT$_{LV}$ from s^* to r^* is impossible in Lemma 26. We then show that the digraph $\mathcal{N}$ can be partitioned into four disjoint sets whose connectivity properties are similar to the connectivity between nodes of digraph $\mathcal{N}_1^*$ in Lemma 27. Finally in Lemma 28, we show that if $(\{B_1, B_2\}), \delta)$-URMT$_{LV}$ from **S** to **R** is possible in $\mathcal{N}$ then $(\{\{b_1\}, \{b_2\}\})$-URMT$_{LV}$ from s^* to r^* is possible in $\mathcal{N}_1^*$, which is a contradiction. Hence, the conditions mentioned in Theorem 25 are necessary.

LEMMA 26. *In the asynchronous network $\mathcal{N}_1^*$, $(\{\{b_1\},\{b_2\}\},\delta)$-URMT$_{LV}$ ($\delta < 1/2$) from s^* to r^* is impossible.*

PROOF. We assume that a protocol Π^* exists in $\mathcal{N}_1^*$ which is a $(\{\{b_1\},\{b_2\}\},\delta)$-URMT$_{LV}$ protocol from s^* to r^*. This implies that there exists a finite time instant T such that for every message m, r^* outputs m before instant T in at least half of the executions in which s^* has chosen to send m. We use $\vec{C}_i$ and $view_n(E_i)$ as defined in the proof of Lemma 14.

We now describe an adversary strategy to fail any protocol Π^*. Adversary chooses any two messages $m_1, m_2 \in \mathbb{F}$, $m_1 \neq m_2$. When s^* intends to send m_i, the adversary corrupts the set $\{b_i\}$, for $i \in \{1,2\}$. When b_2 is corrupt, adversary fail-stops it. When b_1 is corrupt, adversary does the following:-

- Simulates a local copy of s^* on input m_2, say s_1^*.
- Snaps all its communication with nodes s^* and b_2.
- Delays all outgoing messages from b_2 beyond the time instant T. Schedules events between s_1^*, b_1, and r^* as it does between s^*, b_1, and r^* respectively when s^*'s input is m_2.

We now consider the following two executions of Π^* with the above mentioned adversary strategy.

1. Execution E_1: s^*'s input is m_1 and r^* outputs m_1. (For Π^* to be a valid $(\{\{b_1\},\{b_2\}\})$-URMT$_{LV}$ protocol, such an execution exists). Let the coin tosses of s_1^* be c.
2. Execution E_2: s^*'s input is m_2. Coin tosses $\vec{C}_2$ of nodes are such that $c^2_{s^*} = c$, $c^2_{r^*} = c^1_{r^*}$, $c^2_{b_1} = c^1_{b_1}$.

Above mentioned adversary strategy ensures that $view_{r^*}(E_2) = view_{r^*}(E_1)$. Therefore r^* halts with output m_1 in E_2 which is a contradiction on the existence of Π^* since Las Vegas protocols do not allow incorrect output.

We now consider network $\mathcal{N} = (V, \mathcal{E})$ which does not satisfy the conditions of Theorem 25.

LEMMA 27. *The set of nodes V in the network $\mathcal{N}$ can be partitioned into 4 disjoint sets S^*, B_1, B_2 and R^* such that $\mathbf{S} \in S^*$, $\mathbf{R} \in R^*$ and an edge exists from a node in $\vec{L}[i]$ to a node in $\vec{L}[j]$ only if $(\vec{l}[i], \vec{l}[j]) \in \mathcal{E}_1^*$ where $\vec{L} = [S^*, B_1, B_2, R^*]$ and $\vec{l} = [s^*, b_1, b_2, r^*]$ are two ordered lists, $\vec{l}[i]$ (resp. $\vec{L}[i]$) denotes the i^{th} element of the list $\vec{l}$ (resp. $\vec{L}$).*

PROOF. We partition the non-faulty nodes in $H = V \setminus \{B_1 \cup B_2\}$ into 2 disjoint sets S^* and R^*. Let R^* denote the set of all nodes in H having a strong path to $\mathbf{R}$ avoiding nodes in $B_1 \cup B_2$. Let $S^* = V \setminus \{R^* \cup B_1 \cup B_2\}$. There does not exist a directed edge from a node in S^* to a node in R^* by the definition of R^*. Since, $\mathcal{E}_1^* = (V_1^* \times V_1^*) \setminus \{(s^*, r^*)\}$, an edge exists from a node in $\vec{L}[i]$ to a node in $\vec{L}[j]$ only if there exists an edge $(\vec{l}[i], \vec{l}[j]) \in \mathcal{E}_1^*$.

LEMMA 28. *In the directed asynchronous network $\mathcal{N} = (V, \mathcal{E})$, $(\{B_1, B_2\}, \delta)$-URMT$_{LV}$ is possible from $\mathbf{S}$ to $\mathbf{R}$ only if $(\{\{b_1\},\{b_2\}\},\delta)$-URMT$_{LV}$ is possible from s^* to r^* in the network $\mathcal{N}_1^*$.*

PROOF. Proof is on the lines similar to the proof of Lemma 17. We show how a $(\{B_1, B_2\}, \delta)$-URMT$_{LV}$ protocol $\mathcal{P}$ on $\mathcal{N}$ can be simulated on $\mathcal{N}_1^*$ to obtain a $(\{\{b_1\},\{b_2\}\},\delta)$-URMT$_{LV}$ protocol. We simulate a virtual network $\mathcal{N}$ over $\mathcal{N}_1^*$ such that $\mathcal{P}$ when run over the virtual $\mathcal{N}$ is a $(\{\{b_1\},\{b_2\}\},\delta)$-URMT$_{LV}$ protocol. Simulation runs as follows: Node $\vec{l}[i]$ simulates all the nodes in the set $\vec{L}[i]$. Edges in $\mathcal{N}$ lie in one of the following two categories: (i) Those

amongst the nodes within a $\vec{L}[i]$. Such edges can be simulated by $\vec{l}[i]$ internally. (ii) Edges from a node in $\vec{L}[i]$ to a node in $\vec{L}[j]$. We proved in the previous lemma that, an edges exists from a node in $\vec{L}[i]$ to a node in $\vec{L}[j]$ only if there exists an edge $(\vec{l}[i], \vec{l}[j]) \in \mathcal{E}^*$. Hence, such an edge can be simulated on edge $(\vec{l}[i], \vec{l}[j])$. Now, protocol $\mathcal{P}$ is run on the virtual $\mathcal{N}$ which is a $(\{\{b_1\}, \{b_2\}\})$-URMT$_{LV}$ protocol from s^* to r^* in $\mathcal{N}_1^*$.

From Lemma 26 we know that $(\{\{b_1\}, \{b_2\}\}, \delta)$-URMT$_{LV}$ is impossible from s^* to r^* in the network $\mathcal{N}_1^*$. Using Lemma 28, we arrive at a contradiction. Hence, the conditions mentioned in Theorem 25 are necessary.

We now present the second main result of this paper i.e. the minimum connectivity requirements for the case of asynchronous Las Vegas protocols is same as that of the perfect protocols.

THEOREM 29. *In a directed synchronous (or asynchronous) network $\mathcal{N} = (V, \mathcal{E})$, $\mathbb{A}$-PRMT from $\mathbf{S}$ to $\mathbf{R}$ is possible if and only if for all $B_1, B_2 \in \mathbb{A}$ there exists a strong path from $\mathbf{S}$ to $\mathbf{R}$ avoiding nodes in $B_1 \cup B_2$.*

PROOF. Follows from [6]. ∎

COROLLARY 30. *In a directed network $\mathcal{N} = (V, \mathcal{E})$, an asynchronous $(\mathbb{A}, \delta)$-URMT$_{LV}$ protocol exists if and only if a protocol exists for synchronous (or asynchronous) $\mathbb{A}$-PRMT.*

PROOF. Follows from Theorem 24, 25 and 29. ∎

9 Critical edges

We say that an edge is *critical* if its removal renders the graph insufficiently connected for URMT protocols (though before its removal the connectivity was sufficient). Ironically, it turns out that for perfect protocols, the number of critical edges is always $O(n)$ where as for the "easier" randomized protocols (demanding lesser connectivity requirements), we give a family of digraphs with $\Omega(n^2)$ critical edges! We remark that an earlier attempt in [17] to give such a family of digraphs for the case of synchronous Monte Carlo URMT protocols is incorrect and we correct the same; we also give similar families of digraphs (with $\Omega(n^2)$ critical edges) for synchronous Las Vegas (and asynchronous Monte Carlo) protocols. In this chapter, we only deal with t-threshold adversary.

Characterization of synchronous networks in which (t, δ)-URMT$_{MC}$ is possible follows from [22]. In [17], Bhavani et. al. proposed a family of graphs($t > 0$), $G_t = (V, \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3)$ where $V =$
$\{\mathbf{S}, v_1, \ldots, v_{t+1}, u_1, \ldots, u_t, \mathbf{R}\}$ and $\mathcal{E}_1 = \bigcup_1^{t+1}\{(\mathbf{S}, v_i), (v_i, \mathbf{R})\}$; $\mathcal{E}_2 = \bigcup_{i=1}^{t}\{(\mathbf{S}, u_i), (\mathbf{R}, u_i)\}$; $\mathcal{E}_3 = \bigcup_{i=1}^{t}\{(u_i, v_1), \ldots, (u_i, v_{t+1})\}$ as shown in Figure 5(a) and claimed that G_t has $\Omega(n^2)$ critical edges w.r.t synchronous (t, δ)-URMT$_{MC}$. In the following theorem, we prove that this is not the case.

THEOREM 31. *G_t has only $\Theta(n)$ critical edges w.r.t synchronous (t, δ)-URMT$_{MC}$.*

PROOF. Consider the subgraph G_t' of G_t given by $G_t' = (V, \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3')$, where $\mathcal{E}_3' = \bigcup_{i=1}^{t}\{u_i, v_i\}$. According to [17], for (t, δ)-URMT$_{MC}$ to be impossible in G_t', there exists B_1, B_2 such that $|B_1|, |B_2| \leq t$ and every weak path from $\mathbf{S}$ to $\mathbf{R}$ avoiding nodes in $B_1 \cup B_2$ must

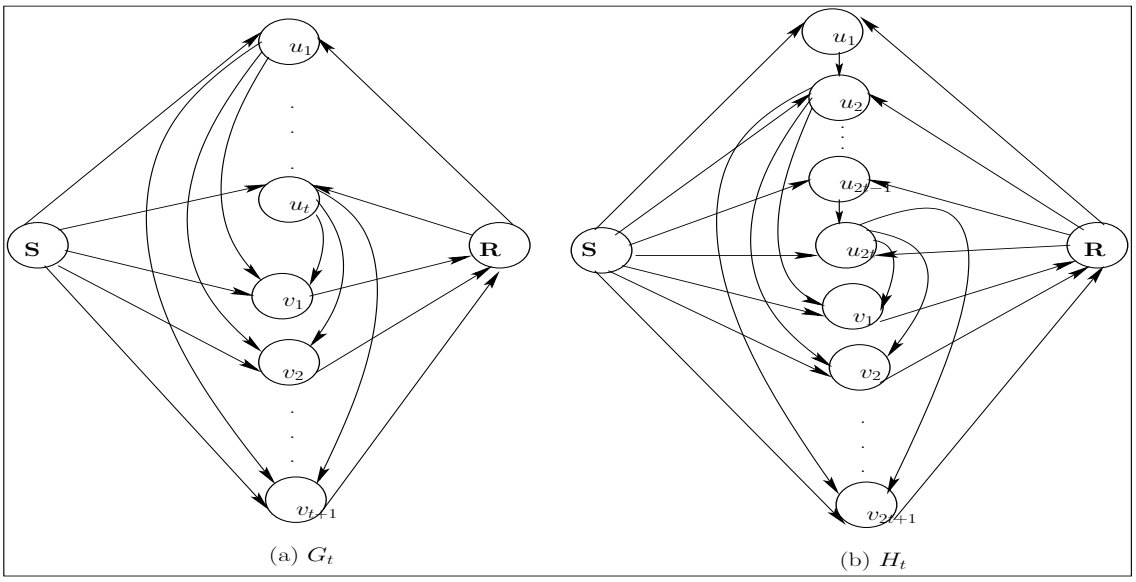


Figure 5: (a) Graph G_t, (b) Graph H_t

have at least one node x such that every strong path from x to $\mathbf{R}$ passes through nodes in B_1 and in B_2. We first show that no such sets B_1, B_2 exists for G'_t. If there exists a strong path from $\mathbf{S}$ to $\mathbf{R}$ avoiding nodes in $B_1 \cup B_2$, URMT$_{MC}$ is trivially possible. Hence, $\forall i\ 1 \leq i \leq t+1,\ v_i \in B_1$ or $v_i \in B_2$. As $|B_1| + |B_2| \leq 2t$, at least one u_i has to be honest. As this leaves an honest weak path from $\mathbf{S}$ to $\mathbf{R}$ i.e. $\mathbf{S} \rightarrow u_i \leftarrow \mathbf{R}$ with u_i having a strong path to $\mathbf{R}$ via v_i. For the impossibility of synchronous (t,δ)-URMT$_{MC}$, node v_i must belong to both B_1 and B_2. This would imply that another $u_{i'}$ $(i' \neq i)$ is honest and hence $v_{i'}$ must belong to both B_1 and B_2. Repeating the inductive arguing for another $t-2$ times, we can show that $B_1 = B_2 = \{v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_t}\}$ for some $\{\alpha_1, \alpha_2, \ldots, \alpha_t\} \subset \{1, 2, 3, \ldots, t+1\}$. But this leaves a strong honest path from $\mathbf{S}$ to $\mathbf{R}$. Hence, no B_1, B_2 exists such that (B,δ)-URMT$_{MC}$ is impossible in G'_t i.e. (t,δ)-URMT$_{MC}$ is possible in G'_t. Since G'_t has $O(n)$ edges, this proves an upper bound of $O(n)$ on the number of critical edges in G_t. Hence, the claim in [17] that G_t has $\Omega(n^2)$ critical edges is wrong. Moreover, since deleting any one edge $(\mathbf{S}, v_i)$ in G_t leaves only $2t$ disjoint weak paths between $\mathbf{S}$ and $\mathbf{R}$, G_t has $\Omega(n)$ critical edges. It therefore follows that G_t has $\Theta(n)$ critical edges.

We now propose a family of graphs with $\Omega(n^2)$ critical edges. For all $t > 0$, consider $H_t = (\mathrm{V}^1, \bigcup_{i=1}^{4} \mathcal{E}_i^1)$ with $\mathrm{V}^1 = \{\mathbf{S}, v_1, \ldots, v_{2t+1}, u_1, \ldots, u_{2t}, \mathbf{R}\}$ and $\mathcal{E}_1^1 = \bigcup_1^{2t+1}\{(\mathbf{S}, v_i), (v_i, \mathbf{R})\}$; $\mathcal{E}_2^1 = \bigcup_{i=1}^{2t}\{(\mathbf{S}, u_i), (\mathbf{R}, u_i)\}$; $\mathcal{E}_3^1 = \bigcup_{i=1}^{t}\{(u_{2i-1}, u_{2i}))\}$; $\mathcal{E}_4^1 = \bigcup_{i=1}^{t}\{(u_{2i}, v_1), \ldots, (u_{2i}, v_{2t+1})\}$ as shown in Figure 5(b). Here, number of nodes in graph H_t is $n = 4t+3$.

THEOREM 32. *H_t has $\Omega(n^2)$ critical edges w.r.t synchronous $(2t,\delta)$-URMT$_{MC}$.*

PROOF. $(2t,\delta)$-URMT$_{MC}$ is possible in H_t (Follows from [17]). Suppose we delete an edge $e = (u_{2i}, v_j) \in \mathcal{E}_4^1$. Consider $B_1 = \bigcup_{k=1}^{2t}\{u_k\} \cup \{v_j\} - \{u_{2i-1}\}, B_2 = \bigcup_{k=1}^{2t+1}\{v_k\} - \{v_j\}$. Only honest weak path left is $\mathbf{S} \rightarrow u_{2i-1} \leftarrow \mathbf{R}$. Every strong path from u_{2i-1} to $\mathbf{R}$ passes through both B_1 and B_2. This renders $(\{B_1, B_2\},\delta)$-URMT$_{MC}$ impossible, hence $(2t,\delta)$-URMT$_{MC}$ is

impossible. Hence, H_t has $\Omega(|\mathcal{E}_3^1|)$ or $\Omega(n^2)$ critical edges.

9.1 Critical Edges for asynchronous Monte Carlo and synchronous Las Vegas

In this section, we show existence of a family of digraphs with $\Omega(n^2)$ critical edges w.r.t asynchronous (t,δ)-URMT$_{MC}$ and synchronous (t,δ)-URMT$_{LV}$. Since, characterization of synchronous networks for the possibility of URMT$_{LV}$ is same as that of asynchronous networks for URMT$_{MC}$(proved in Collorary 23). We can deduce that, any given graph (meeting the sufficiency conditions) has same number of critical edges w.r.t both the aforementioned variants. Hence, a family of digraph with $\Omega(n^2)$ critical edges w.r.t asynchronous (t,δ)-URMT$_{MC}$ has same number of critical edges w.r.t (t,δ)-URMT$_{LV}$ too. We now propose a family of digraphs with $\Omega(n^2)$ critical edges w.r.t asynchronous (t,δ)-URMT$_{MC}$.

THEOREM 33. *G_t has $\Omega(n^2)$ critical edges w.r.t asynchronous (t,δ)-URMT$_{MC}$.*

PROOF. (t,δ)-URMT$_{MC}$ is possible in asynchronous network G_t (Follows from Theorem 18, 20). Suppose we delete some edge $e = (u_i, v_j) \in \mathcal{E}_3$. Consider $B_1 = \bigcup_{k=1}^{t}\{u_k\} \cup \{v_j\} - \{u_i\}$, $B_2 = \bigcup_{k=1}^{t+1}\{v_k\} - \{v_j\}$. Only honest weak path left is $\mathbf{S} \rightarrow u_i \leftarrow \mathbf{R}$. All the strong paths from u_i to $\mathbf{R}$ pass through B_2. This renders $(\{B_1, B_2\}, \delta)$-URMT$_{MC}$ impossible, hence (t,δ)-URMT$_{MC}$ is impossible. Hence, $\Omega(|\mathcal{E}_3|)$ or $\Omega(n^2)$ critical edges.

10 Discussion and Open Problems

In this work we focus on the characterization of networks for possibility and impossibility of (i) synchronous Las Vegas (ii)asynchronous Monte Carlo (iii)asynchronous Las Vegas protocols. We establish the fact that the minimum connectivity requirements for the possibility of synchronous Las Vegas is same as asynchronous Monte Carlo and asynchronous Las Vegas is same as synchronous perfect. It turns out that the minimum connectivity requirements synchronous Monte Carlo (characterized in [17]) is less than synchronous Las Vegas, which in turns is less than asynchronous Las Vegas. To summarize, our results establish the following hierarchy with respect to the connectivity requirements,
synchronous URMT$_{MC}$ $<$ synchronous URMT$_{LV}$ $=$ asynchronous URMT$_{MC}$ $<$ asynchronous URMT$_{LV}$ $=$ synchronous perfect $=$ asynchronous perfect.

It is known, for any graph over n nodes, the number of critical edges for synchronous perfect protocols tolerating threshold adversary is $O(n)$. We present family of digraphs for (i) synchronous Monte Carlo (ii) synchronous Las Vegas (iii) asynchronous Las Vegas, protocols with $\Omega(n^2)$ critical edges. Hence, drawing attention to a surprising interplay between the randomized versus perfect protocols, that is, randomized protocols demanding lesser (minimum) connectivity compared to the case of perfect protocols sometimes have higher number of critical edges.

10.1 Impact on real-life problems

1. Since, the main focus of the work was possibility and impossibility of various protocols, the protocols presented here remain exponential. We do not even rule out the

possibility of exponential lower bounds. Nonetheless, for the case when the size of adversary structure is small, we present the following interesting use-case which exploits the matching connectivity requirements of synchronous URMT$_{LV}$ and asynchronous URMT$_{MC}$. Suppose a hardware vendor has to design a network such that URMT$_{MC}$ between given two nodes is possible even when the network is asynchronous and deploy hardware for it. Testing of asynchronous networks is a tedious and time consuming job. So, he may rather test if the hardware achieves URMT$_{LV}$ when the network is synchronous, and be rest assured. This also motivates us to explore more such equivalence (between synchronous Las Vegas and asynchronous Monte Carlo variants of same problem) w.r.t. other Distributed Computing problems.

2. We initiate the study of URMT$_{LV}$ over directed networks. We show that in the case of synchronous networks, minimum connectivity requirements obey the following hierarchy, URMT$_{MC}$ $<$ URMT$_{LV}$ $<$ perfect protocols. One advantage of running a Las Vegas protocol (if it is possible) is that the receiver knows when it has received an incorrect output. So, if an application demands reliability in Las Vegas sense and it so turns out that the network is insufficiently connected for perfect protocols, but meets the sufficiency conditions for the Las Vegas protocols, one might write a Las Vegas protocol. Such a situation may arise for applications with critical reliability requirements. For instance, while communicating the amount of money to be transferred from a bank account to another, reliability is Las Vegas sense is the appropriate model. A Monte Carlo protocol is inappropriate for the situation.

10.2 Open Problems

We briefly discuss a few related open problems:

1. There are real life networks where only directed hyper-graphs can appropriately model the underlying network. Hence, it is useful to study the problem of URMT in directed hyper-graphs under various timing models, fault models, etc. It would be interesting to see if the equivalence w.r.t respect to matching connectivity requirements, shown for directed graphs in Corollary 23, 30 extend to the case of directed hyper-graphs as well.
2. Let $\mathcal{N}$ be the underlying network, and $\mathbb{A}$ be an adaptive non-threshold byzantine adversary and $\mathbf{S}$ and $\mathbf{R}$ be sending and receiving nodes respectively, which are part of $\mathcal{N}$. Given $< \mathcal{N}, \mathbf{S}, \mathbf{R}, \mathbb{A} >$ as input, a decision problem is to find out whether synchronous URMT$_{LV}$ (or asynchronous URMT$_{MC}$) is possible or not. It is a non-trivial open problem to prove these decision problem to be certain complexity class hard.
3. As we focused on characterizing directed networks in which URMT is possible, our protocols are inefficient in the worst case (synchronous Las Vegas and asynchronous Monte Carlo protocols). It remains open to give efficient protocols or establish lower bounds. We strongly believe, latter is the case.
4. We considered the problem of reliable uni-cast wherein there is only one receiver. In general, one may consider the problem of *reliable multi-cast* wherein there are multiple recipients for the message, the sender sends. The characterization of networks for

the (im)possibility of reliable mult-cast is non-trivial because the problem definition demands all the receivers to agree on the same message when the sender is faulty. Since, in this work, we only deal with the problems with size of receiver's set being one, makes the agreement problem trivial.

5. We only considered adaptive non-threshold byzantine adversary. Characterization for the case of mobile adversary doesn't trivially follow from the adaptive case. A mobile adversary is much more powerful and damaging than its corresponding adaptive counterpart.
   We also leave the case of mixed adversary open which is well-motivated in practice. In typical network, certain nodes may be strongly protected and others may be weakly protected. An adversary may only be able to fail-stop or eavesdrop a strongly protected node, while he may corrupt a weakly protected node in Byzantine fashion. Characterization for the case of mixed adversary would be a strict generalization of the case considered here that is, the case of Byzantine adversary.

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