A Family of Implementation-Friendly BN Elliptic Curves

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Abstract. For the last decade, elliptic curve cryptography (ECC) has gained increasing interest in industry and in the academic community. This is especially due to the high level of security it provides with relatively small keys and to its ability to create very efficient and multifunctional cryptographic schemes by means of bilinear pairings.

Among the possible choices of pairing-friendly elliptic curves, Barreto-Naehrig (BN) curves arguably constitute one of the most versatile families. In this paper, we further expand the potential of this family. We describe BN curves that are not only computationally very simple to generate, but also specially suitable for efficient implementation on a very broad range of scenarios. We also present implementation results of the optimal ate pairing using such a curve defined over a 254-bit prime field.

1 Introduction

Since the introduction of elliptic curves to cryptography [27, 32], they have raised increasing interest as a useful tool for creating efficient asymmetric schemes (e.g., digital signatures) with a high security level per bit of the used keys. Hence, they are able to cope with important (albeit challenging) requirements of modern systems such as reductions on the usage of processing power, storage, bandwidth and energy. Furthermore, elliptic curves are the foundation of Pairing-based Cryptography (PBC), which has been responsible for the development of state-of-the-art applications such as identity-based encryption [9] and certificateless public key cryptography [1]. Inherently, feasible pairing-based cryptography requires pairing-friendly curves.

Barreto-Naehrig (BN) curves arguably constitute one of the most versatile classes of pairing-friendly elliptic curves. Among other things, they are known [6] to (this list may not be complete):

- facilitate the deployment of bilinear pairings at the 128-bit security level [15];
- enable all kinds of pairing-based cryptographic schemes and protocols (including short signatures) [19];
- be plentiful and easily found [35, Section 2.1.1];
- support a sextic twist, so pairing arguments can be defined over relatively small finite fields $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ respectively [24];
- be amenable to twofold or threefold pairing compression [36];
- attain high efficiency for all pairing computation algorithms known, including the Tate [40], ate [24], eil [23], R-ate [30], Xate [38] and optimal [44] pairings;
- admit optimizations based on endomorphisms and homomorphisms for all groups involved [18, 20], thereby enabling fast non-pairing operations as well;
- be suitable for software and hardware implementations on a wide range of platforms [16, 21].

Recent research has focused on certain individual curves to attain exceptional performance gains [37, 8]. This is essential since pairings are usually the most computationally expensive operation in any pairing-based cryptographic scheme. On the other hand, one may argue that only targeting fast pairings is insufficient, and may lead to annoying or unacceptable inefficiencies on certain highly constrained platforms like smart cards or wireless sensor networks. Indeed, because of the intrinsic high cost of pairings, many protocols are already designed to rely on them only when the corresponding protocol parties are assumed to have plentiful computational resources (e.g. servers or clusters) while constrained parties only need to perform non-pairing operations [4, 10, 31, 45]. In such scenarios, curve parameters leading to fast (but still proportionally slow) pairings at the price of deteriorating performance elsewhere would be harmful rather than helpful.

A different line of research is that of obtaining parameterized curves with prescribed properties to avoid computationally expensive tests during curve generation. Simplification of curve parameter testing is even more important. Certain attacks can be prevented by checking that the purported BN curve contained in a given digital certificate does indeed exhibit the expected properties before using that certificate. This procedure is commonplace for non-pairing-friendly curves, but the special-purpose nature of BN curves exacerbates the amount of necessary computations. By adopting a curve where certain properties are guaranteed to hold, the testing overhead would be greatly reduced. It could then be carried out on much simpler platforms; e.g. a lightweight certificate server would only need plain integer arithmetic up to primality checking (and no elliptic curve arithmetic support) to attest the well-formedness of the curves.

Constructing the right twist of the curve over the base field without resorting to any elliptic curve arithmetic has been carried out successfully [39]. In contrast,

the related tasks of choosing suitable representations for all extension fields involved and selecting the correct twist $E'(\mathbb{F}_{p^2})$ have received limited attention in the literature. They would still seem to need square and cube detection in extension fields and full group arithmetic in that twist. We note that extension fields are usually chosen *a priori*, based on features of supporting libraries and oblivious to the peculiar nature of BN curves.

Our contribution in this paper is the definition of a (rather large) subclass of BN curves that is particularly suitable for efficient construction, parameter checking and implementation, while retaining a very simple description. The proposed subclass favors efficiency of all typical arithmetic operations needed to instantiate cryptographic protocols on the broadest possible landscape, i.e. in the sense of targeting the widest possible range of platforms and applications. In particular, our construction automatically yields the right twist $E'(\mathbb{F}_{p^2})$ entirely avoiding curve arithmetic for that purpose. It gives to field representations an overall unity that provides several optimization opportunities. Our proposal has intersections with other interesting curve families that occur in the literature (e.g. [38, 43]), offering additional benefits in those cases.

We stress that it is not the purpose of this paper to evaluate optimization techniques that are exclusive to a particular platform, nor to focus on the operation of pairing computation itself or on techniques that are only available on a narrow set of circumstances. Our goal is rather to explore a simple yet comprehensive theoretical setting that avoids most if not all general drawbacks and implementation hindrances, while favoring complete pairing-based cryptosystems.

To underline the flexible nature of our proposal and to reproduce the operation counts, we have implemented in Java the optimal ate pairing on one of the proposed curves (254 bits). We have focused on employing the best known techniques and algorithms for each operation as listed in Section 4. To compare to other work [37, 8], the target metric was chosen to be the equivalent number of base-field multiplications for an entire pairing. The implementation is available online[3].

The remainder of this paper is organized as follows. We introduce theoretical concepts related to bilinear maps and BN curves in Section 2. We describe the proposed implementation-friendly family of BN curves and discuss its features in Section 3. Concrete examples tailored for practical deployment are suggested in Section 4. We conclude in Section 5.

2 Preliminaries

Let p be a prime and let $e > 0$. The conjugates of $a \in \mathbb{F}_{p^e}$ are the elements a^{p^i}, $0 \leqslant i < e$. The norm of $a \in \mathbb{F}_{p^e}$ is defined to be the product of all its conjugates, $N(a) := \prod_i a^{p^i}$. Whenever $p \equiv 3 \pmod{4}$, the finite field $\mathbb{F}_{p^2}$ can be represented as $\mathbb{F}_p[i]/(i^2 + 1)$. In analogy to the complex numbers, the non-trivial conjugate of the field element $\gamma = \alpha + i\beta \in \mathbb{F}_{p^2}$ is $\bar{\gamma} := \gamma^p = \alpha - i\beta$.

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[3] http://code.google.com/p/bnpairings/

Given three groups $\mathbb{G}_1$, $\mathbb{G}_2$, and $\mathbb{G}_T$ of the same prime order n, a pairing is a feasibly computable, non-degenerate bilinear map $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$. Usually $\mathbb{G}_1$ and $\mathbb{G}_2$ are written additively, while $\mathbb{G}_T$ is written multiplicatively. In practice, the pairing groups $\mathbb{G}_1$ and $\mathbb{G}_2$ are most commonly determined by the eigenspaces of the Frobenius endomorphism ϕ_p on some elliptic curve $E/\mathbb{F}_p$ of embedding degree $k > 1$. Specifically, $\mathbb{G}_1$ is taken to be the 1-eigenspace $E[n] \cap \ker(\phi_p - [1]) = E(\mathbb{F}_p)[n]$. The group $\mathbb{G}_2$ is taken to be the preimage $E'(\mathbb{F}_{p^e})[n]$ of the p-eigenspace $E[n] \cap \ker(\phi_p - [p]) \subseteq E(\mathbb{F}_{p^k})[n]$ under a twisting isomorphism $\psi : E' \to E$, $(x, y) \mapsto (\mu^2 x, \mu^3 y)$ for some $\mu \in \mathbb{F}_{p^k}$. In particular, $e = k/d$ where the curve $E'/\mathbb{F}_{p^e}$ is the unique twist of largest possible twist degree $d \mid k$ for which n divides $\#E'(\mathbb{F}_{p^e})$ (see [24] for details). This means that e is as small as possible. Typical pairing algorithms are based on Miller's algorithm [33] with a number of optimizations [3, 24, 30, 38, 44], most notably optimal pairings [44] which have loop order of length $\lceil \lg n \rceil / \varphi(k)$ in general, where φ is Euler's totient function. Note that this compares well with the original Tate pairing which has loop order of length $\lceil \lg n \rceil$.

A Barreto-Naehrig (BN) curve [5] is an elliptic curve $E_b : y^2 = x^3 + b$ defined over a finite field $\mathbb{F}_p$ with the following properties. It has prime order $n = \#E(\mathbb{F}_p)$, and the primes p and n are given by $p = p(u) = 36u^4 + 36u^3 + 24u^2 + 6u + 1$ and $n = n(u) = 36u^4 + 36u^3 + 18u^2 + 6u + 1$ for some $u \in \mathbb{Z}$. We drop the subscript and simply write E when the specific equation coefficient b is irrelevant to the discussion or clear from context. For primes p of the above polynomial form the condition $p \equiv 3 \pmod 4$ holds if and only if u is odd. The BN field $\mathbb{F}_p$ contains a primitive cube root of unity $\zeta(u) = 18u^3 + 18u^2 + 9u + 1$ as one can check by straightforward inspection. BN curves have embedding degree $k = 12$ and admit a sextic twist ($d = 6$), so that one can set $\mathbb{G}_2 = E'(\mathbb{F}_{p^2})[n]$, and there exists an optimal ate pairing with loop order $\omega = |6u + 2|$.

Since BN curves have j-invariant 0, it is relatively easy to find them when compared to pairing-friendly curves from other families (see [17] for an extensive survey). In particular, there is no need to resort to the CM method explicitly. To generate a BN curve, one chooses an integer u until p and n as given by the above polynomials are prime. The size of u is selected such that it yields a desired size for p and n. To find a corresponding curve, one chooses $b \in \mathbb{F}_p$ so that the curve $E : y^2 = x^3 + b$ has order n. For these steps, we need primality tests, possibly square detection and square root computations in $\mathbb{F}_p$ to obtain a point in $E(\mathbb{F}_p)$, and finally a scalar multiplication to check for order n.

The corresponding twist $E'/\mathbb{F}_{p^2}$ is usually selected by finding a non-square and non-cube $\xi \in \mathbb{F}_{p^2}$ and then checking via scalar multiplication whether the curve $E' : y^2 = x^3 + b'$ given by $b' = b/\xi$ or by $b' = b/\xi^5$ has order divisible by n. The element ξ can be used to represent the field extensions of $\mathbb{F}_{p^2}$ contained in $\mathbb{F}_{p^{12}}$ since the polynomial $z^r - \xi$ is irreducible over $\mathbb{F}_{p^{2h}}$ for $r \in \{2, 3, 6\}$ and $h \in \{1, 2, 3\}$ whenever $\gcd(h, r) = 1$ [35, Lemma 2.14].

Example 1. Let $p^e \equiv 1 \pmod 6$. For each $\xi \in \mathbb{F}_{p^e}$ that is neither a square nor a cube, one can represent $\mathbb{F}_{p^{6e}}$ as a tower extension of $\mathbb{F}_{p^e}$ in the following three different ways:

- $\mathbb{F}_{p^{6e}} = \mathbb{F}_{p^e}[u]/(u^6 - \xi)$;
- $\mathbb{F}_{p^{6e}} = \mathbb{F}_{p^{2e}}[v]/(v^3 - \xi)$ with $\mathbb{F}_{p^{2e}} = \mathbb{F}_{p^e}[s]/(s^2 - \xi)$;
- $\mathbb{F}_{p^{6e}} = \mathbb{F}_{p^{3e}}[w]/(w^2 - \xi)$ with $\mathbb{F}_{p^{3e}} = \mathbb{F}_{p^e}[t]/(t^3 - \xi)$.

The components of an element from $\mathbb{F}_{p^{6e}}$ in any of these can be extracted directly without the need to perform expensive computations. Thus: $a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5 \leftrightarrow (a_0 + a_3 s) + (a_1 + a_4 s)v + (a_2 + a_5 s)v^2 \leftrightarrow (a_0 + a_2 t + a_4 t^2) + (a_1 + a_3 t + a_5 t^2)w$, for $a_i \in \mathbb{F}_{p^e}$. This shows that the suggested setting automatically yields so-called "compositum" or tower-friendly fields [7, 22], with their associated efficiency gains. □

We will propose a subfamily of BN curves that does away with the square and cube detection tests usually needed when deciding how to represent the finite fields extensions that occur in a typical implementation of pairing-based protocols. The following lemma captures an important property of the class of elliptic curves to which BN curves belong:

Lemma 1. ([35, Lemma 2.7]) *Let $E : y^2 = x^3 + b$ be an elliptic curve over $\mathbb{F}_p$ of order $n = \#E(\mathbb{F}_p)$ such that $2 \nmid n$ and $3 \nmid n$. Then b is neither a square nor a cube in $\mathbb{F}_p$.*

Proof. For any $\gamma, \delta \in \mathbb{F}_p$, the point $(0, \gamma)$ on $E : y^2 = x^3 + \gamma^2$ has order 3 and hence $3 \mid n$, while the point $(-\delta, 0)$ on $E : y^2 = x^3 + \delta^3$ has order 2 and hence $2 \mid n$, either way contradicting the assumption that $2 \nmid n$ and $3 \nmid n$. □

As a consequence, we arrive at the following useful result:

Lemma 2. *Let $\xi \in \mathbb{F}_{p^e}^*$ and let $b = N(\xi) \in \mathbb{F}_p$. If $E : y^2 = x^3 + b$ has order $n = \#E(\mathbb{F}_p)$ with $2 \nmid n$ and $3 \nmid n$, then ξ is neither a square nor a cube in $\mathbb{F}_{p^e}$.*

Proof. Assume that ξ is a square or a cube in $\mathbb{F}_{p^e}$, i.e. $\xi = \gamma^r$ for some $\gamma \in \mathbb{F}_{p^e}$ and $r \in \{2, 3\}$. Then $b = N(\xi) = N(\gamma^r) = N(\gamma)^r$, i.e. b is a square or a cube in $\mathbb{F}_p$, contradicting Lemma 1. □

This means that square or cube detection is not necessary in either $\mathbb{F}_p$ or $\mathbb{F}_{p^e}$. In particular, the element ξ specified in Lemma 2 can be used to define all extensions of $\mathbb{F}_{p^e}$ that are of interest to pairing implementation and to facilitate changes of representations in field towers, as shown in Example 1. We remark that this choice of representation for finite field extensions may favor the implementation of other families of pairing-friendly elliptic curves (see [17]). Pursuing this possibility, however, transcends the scope of this paper.

The next result addresses the matter of avoiding order computation and arithmetic on the sextic twist $E'(\mathbb{F}_{p^2})$ by revealing immediately which twist has the correct order. To that end we need one more property:

Lemma 3. *Let $p \equiv 1 \pmod 3$ be a prime. For any $\xi \in \mathbb{F}_{p^2}$, let $b = N(\xi) = \xi\bar{\xi}$. Then b/ξ^5 is a cube.*

Proof. We first notice that $b = \xi\bar{\xi} = \xi\xi^p = \xi^{p+1}$ and thus $b/\xi^5 = \xi^{p-4}$. Since $p-4$ is divisible by 3, we see that b/ξ^5 is a cube. $\qquad\square$

We are finally in a position to state the following theorem:

Theorem 1. *Given a BN curve of form $E : y^2 = x^3 + b$ with $b = N(\xi)$ for some $\xi \in \mathbb{F}_{p^2}$, the particular sextic twist $E' : y^2 = x^3 + \bar{\xi}$ satisfies $\#E(\mathbb{F}_p) \mid \#E'(\mathbb{F}_{p^2})$.*

Proof. Remark 2.13 in [35] shows that the order of the desired twist E' over $\mathbb{F}_{p^2}$ is $n' = \#E'(\mathbb{F}_{p^2}) = n(2p-n)$. Since n and p are both odd, also n' is odd. If $\xi \in \mathbb{F}_{p^2}$ is neither a square nor a cube, the correct twist is either given by $y^2 = x^3 + b/\xi$ or $y^2 = x^3 + b/\xi^5$. Since $p \equiv 1 \pmod{3}$ and $b = N(\xi)$ the value b/ξ^5 is a cube by Lemma 3. This means that the curve given by $y^2 = x^3 + b/\xi^5$ has a point of order 2, hence the order of this particular twist is even. Therefore, $E' : y^2 = x^3 + b/\xi$ is the twist one seeks. Notice that $b/\xi = \bar{\xi}$. $\qquad\square$

3 The proposed family of curves

We propose to use BN curves of form $E_{b,\ell} : y^2 = x^3 + b$ where either $b = c^4 + d^6$ or $b = c^6 + 4d^4$ for some $c, d \in \mathbb{N} \setminus \{0\}$ with the following properties: the BN prime p satisfies $p \equiv 3 \pmod{4}$ (and possibly also $p \equiv 4 \pmod{9}$); the Hamming weights of the (signed) binary representations of either the BN parameter u or the loop order $\omega = 6u + 2$ of the optimal ate pairing (or both) are minimal for each bit length $\ell := \lceil \lg p \rceil$; and b is as small as possible (preferably with low Hamming weight). We slightly abuse notation and identify the integer b with a field element in $\mathbb{F}_p$. The corresponding $\xi \in \mathbb{F}_{p^2}$ such that $b = N(\xi)$ is $\xi = c^2 + d^3 i$ or $\xi = c^3 + 2d^2 i$, respectively. Notice that the choice of b is consistent with both Lemma 2 and Theorem 1, and it is also compatible with [39, Algorithm 3.5] in the sense that one can certainly look for c and d such that the resulting b has the form prescribed by that algorithm.

The rationale for our proposal is summarized as follows.

Pairing efficiency

First and foremost, pairing computation must be as efficient as possible, since this is the most expensive operation in any pairing-based protocol. Low-weight ω minimizes the cost of the Miller loop in optimal pairings, while low-weight u minimizes the cost of the final exponentiation [42]. Small values of b favor faster pairing computation [14], especially if b has low Hamming weight, which is clearly possible with the prescribed form we suggest (e.g. if c and d are small powers of 2). One of the best situations, though not the only one, arises when $b = 2$ and $\xi = 1 + i$. In this case, multiplications by b are most efficient on all platforms (not only on those where a dedicated multiplication by a small constant is readily available, but also those where it has to be emulated with simpler operations like shifts or additions). Furthermore, the computation of conjugates, which involves multiplications by ξ, incurs the least overhead.

Overall efficiency

All operations involved in pairing-based protocols must be as efficient as possible. Works like [37] and [8] only consider pairing computation speed as a metric, disregarding operations like generating random points or hashing to the pairing groups $\mathbb{G}_1$ and $\mathbb{G}_2$ which are essential to most cryptographic schemes based on pairings. For BN curves, this means there must be a very efficient method to compute square roots in $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$. This is least expensive when $p \equiv 3$ (mod 4) and $p^2 \equiv 9$ (mod 16), since the Cippolla-Lehmer method simplifies to one square detection and one exponentiation for square roots in $\mathbb{F}_p$, namely, $\sqrt{a} = a^{(p+1)/4}$, and the KCYL [28] method applies to the computation of square roots in $\mathbb{F}_{p^2}$, taking one square detection and 1.5 exponentiations. The case $p^2 \equiv 17$ (mod 32) is almost as efficient, taking one square detection and 2 exponentiations to compute roots in $\mathbb{F}_{p^2}$ with the method of [34]. In certain scenarios (e.g. when threefold pairing compression is desired) one might wish to require $p \equiv 4$ (mod 9) as well, since this facilitates the computation of cube roots with methods similar to those for computing square roots [5, Section 3.1].

Uniform finite field arithmetic

Arithmetic in all finite fields involved must be efficient. Operations in $\mathbb{G}_1$ and $\mathbb{G}_2$ already need efficient arithmetic in $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$. Further processing of pairing values as for example explicit or implicit exponentiation needs efficient algorithms for $\mathbb{F}_{p^{12}}$ itself, or in some cases for the subfields $\mathbb{F}_{p^6}$ or $\mathbb{F}_{p^4}$. Efficient subfield arithmetic is needed if pairing compression techniques are adopted (by factors of 2 and 3, respectively). Also, potential support for efficient conversions between different representations has to be planned for the sake of interoperability.

Generator simplicity

Obvious generators that do not involve any extra processing or storage are clearly desirable. A curve equation of form $E : y^2 = x^3 + (c^4 + d^6)$ admits the obvious solution $G = (-d^2, c^2)$, while one of form $E : y^2 = x^3 + (c^6 + 4d^4)$ admits the solution $G = (-c^2, 2d^2)$. By Theorem 1 the sextic twists of form $E' : y^2 = x^3 + (c^2 - d^3 i)$ and $E' : y^2 = x^3 + (c^3 - 2d^2 i)$, respectively, always contain a subgroup of the same order n as E, and the curve equations for E' admit the obvious solution $G' = (-di, c)$ and $G' = (-c, d(1 - i))$, respectively. Then the point $[h]G'$, where $h = 2p - n$, only fails to be a generator of $E'(\mathbb{F}_{p^2})[n]$ with negligibly low probability $O(1/h)$. The cofactor multiplication can be carried out very efficiently [41, Section 6].

Suitable field sizes

An obvious bottleneck is $\mathbb{F}_{p^2}$ arithmetic, since it is the foundation of all operations in $\mathbb{G}_2$, $\mathbb{G}_T$, and pairing computation. Choosing p slightly smaller than a multiple of the platform word size (say, more than two bits but less than three

bits) is interesting because it enables not only postponing modular reductions in critical operations like $\mathbb{F}_{p^2}$ multiplication or squaring, but also simplifying the actual reduction when it is finally applied, as pointed out in [8, Section 5.2].

4 Sample curves

In Table 1, we provide practical curves of the proposed family for fields of bit length $\ell := 32m - 2$ where $5 \leqslant m \leqslant 20$, thus ranging between 80-bit and 320-bit security levels. All of them have the form $E_{c^4+1,\ell} : y^2 = x^3 + (c^4 + 1)$ over $\mathbb{F}_p$ with $p = p(u)$, prime order $n = n(u)$, and admit a twist of correct order given by $E' : y^2 = x^3 + (c^2 - i)$ over $\mathbb{F}_{p^2}$. Also, c is always a power of 2. These parameters were obtained from a script in Magma [11] that searches primes with prescribed properties.

We note that the values for u, c, and b uniquely determine all needed parameters, i.e. the primes p and n, the curve equations for E and its twist as well as the generator points. Namely, field extensions $\mathbb{F}_{p^{2r}}$ can be represented directly as $\mathbb{F}_{p^2}[z]/(z^r - c^2 - i)$ for $r = 2, 3, 6$, or via towers as indicated in Example 1. And the pairing groups are $\mathbb{G}_1 = \langle G \rangle$ for $G = (-1, c^2)$, and $\mathbb{G}_2 = \langle H' \rangle$ for $H' = [h](-i, c)$ with $h = 2p - n$, respectively. The low weight of u enables very efficient multiplication by the cofactor h [41, Section 6].

The peculiar choice $\ell = 32m - 2$ deserves some attention, since it is smaller (albeit not by much) than a multiple of typical word sizes (more precisely, a multiple of 8 bits) and hence leads to security levels that are very slightly lower than usual. This was done so that, adopting Montgomery arithmetic in the base field, all values listed here enable all modular reductions involved in an $\mathbb{F}_{p^2}$ multiplication or squaring to be postponed and carried out only once at the very end of that operation, in a very simple and efficient manner as suggested by [8, Section 5.2]. The value $\lfloor 2^{32m}/p \rfloor$ indicates how many modular reductions can be postponed if $\mathbb{F}_p$ elements are held in $32m$-bit variables. With the suggested choice of $\ell = 32m - 2$, $\lfloor 2^{32m}/p \rfloor = 7$ for all examples on Table 1 except for the entry at $\ell = 254$, where it is 6 ($\mathbb{F}_{p^2}$ multiplication and squaring do not need this value to be larger than 5).

Square roots in $\mathbb{F}_{p^2}$ can be efficiently computed with the suggested method, either KCYL [28] or Müller [34].

Example 2. The 254-bit curve corresponding to the parameter $u = -(2^{62} + 2^{55} + 1)$ is given by $E_{2,254} : y^2 = x^3 + 2$ with $G = (-1, 1)$, $E' : y^2 = x^3 + (1 - i)$, $G' = [h](-i, 1)$. □

Example 3. All the examples on Table 1 are of the first form of implementation-friendly curves we suggest. As an example of the second form, to be used in scenarios where efficient cube root computation is desired, one could adopt the 254-bit curve (not listed on Table 1) with $u = -(2^{62} - 2^{49} - 2^2 + 1)$ and $E_{5,254} : y^2 = x^3 + 5$, $G = (-1, 2)$, $E' : y^2 = x^3 + (1 - 2i)$, $G' = [h](-1, 1 - i)$. One can check by direct inspection that $p \equiv 4 \pmod 9$ for this curve. □

The particular curve of Example 2 has been apparently first suggested in [38, Section 4.2], and curves with $c = 1$ (and hence $b = 2$), which make up the majority of Table 1, have been singled out in [43], albeit without the benefit of a unified view of the curve equation, its correct twist, and the finite fields involved as pointed out in Section 3.

Table 1. Sample curves $E_{b,\ell} : y^2 = x^3 + b$ with $b = c^4 + 1$ over $\mathbb{F}_p$, where $p = p(u)$, bitsize of p is $l = 32m - 2$, and suggested method for square root computation in $\mathbb{F}_{p^2}$

m	ℓ	u	$\mathsf{wt}(6u + 2)$	c	b	$\sqrt{\mathbb{F}_{p^2}}$
5	158	$-(2^{38} + 2^{28} + 1)$	5	2	17	KCYL
6	190	$-(2^{46} + 2^{23} + 2^{22} + 1)$	5	8	4097	KCYL
7	222	$2^{54} - 2^{44} + 1$	5	4	257	Müller
8	254	$-(2^{62} + 2^{55} + 1)$	5	1	2	KCYL
9	286	$-(2^{70} + 2^{58} + 2^{38} + 1)$	7	1	2	KCYL
10	318	$2^{78} + 2^{62} + 2^1 + 1$	6	1	2	KCYL
11	350	$-(2^{86} - 2^{69} + 2^{28} + 1)$	7	1	2	KCYL
12	382	$-(2^{94} + 2^{76} + 2^{72} + 1)$	7	1	2	KCYL
13	414	$-(2^{102} + 2^{84} - 2^{55} + 1)$	7	1	2	KCYL
14	446	$2^{110} + 2^{36} + 1$	5	4	257	Müller
15	478	$-(2^{118} - 2^{55} - 2^{19} + 1)$	7	1	2	KCYL
16	510	$-(2^{126} + 2^{53} - 2^{50} + 1)$	6	4	257	KCYL
17	542	$-(2^{134} + 2^{114} + 2^{30} + 1)$	7	1	2	KCYL
18	574	$-(2^{142} + 2^{120} - 2^{99} + 1)$	7	1	2	KCYL
19	606	$-(2^{150} - 2^{95} + 2^8 + 1)$	7	1	2	KCYL
20	638	$2^{158} - 2^{128} - 2^{68} + 1$	7	4	257	Müller

Efficiency

It is instructive to compare the relative efficiency of the proposed family with available results in the literature. Curves at roughly the same security level as $E_{2,254}$ of Example 2 appeared in [37, 8, 2]. The results are summarized on Table 2. Following [8], we denote by $\tilde{m}$ the number of $\mathbb{F}_{p^2}$ multiplications, and by $\tilde{s}$ the corresponding number of squarings, needed to compute one (optimal ate) pairing, within the Miller loop (ML), the final exponentiation (FE), and the total count (TC). We also provide the number m of equivalent $\mathbb{F}_p$ multiplications incurred, with $\tilde{m} \approx 3m$ and $\tilde{s} \approx 2m$. We only provide operation counts for on-the-fly arithmetic.

The figures in Table 2 refer to the following implementation decisions:

- Joint point-and-line computations within the Miller loop as suggested by Costello *et al.* [14, Section 5] (see also [13, Section 4]);
- Tailored multiplication in $\mathbb{F}_{p^{12}}$ to accumulate line function values, which are known to be rather sparse for even embedding degrees [14, Section 3];

Table 2. Experimental comparison of optimal ate pairing performance for 254 bits

source	$\tilde{m}$	$\tilde{s}$	m	Uses our BN Subclass?
Naehrig et al. [37]	2022 (ML)	590 (ML)	7246 (ML)	
	673 (FE)	1719 (FE)	5457 (FE)	
	2695 (TC)	2309 (TC)	**12703** (TC)	No
Beuchat et al. [8]	1954 (ML)	568 (ML)	6998 (ML)	
	443 (FE)	1719 (FE)	4767 (FE)	
	2397 (TC)	2287 (TC)	**11765** (TC)	No
This work	1256 (ML)	1209 (ML)	6186 (ML)	
	1162 (FE)	66 (FE)	3618 (FE)	
	2418 (TC)	1275 (TC)	**9804** (TC)	Yes

- Improved computation of the hard part of the final exponentiation as proposed by Scott *et al.* [42];
- Squaring in $\mathbb{F}_{p^{12}}$ via the Chung-Hasan SQR_3 algorithm [12], which positively affects both the joint point-and-line computations and the easy part of the final exponentiation after the Miller loop;
- Improved Granger-Scott squaring [22] and Karabina compressed squaring [26] in the cyclotomic subgroup of $G_{\phi_6(q)}$, which positively affect the hard part of the final exponentiation and also contributes for efficient post-processing (e.g. further exponentiation) of pairing values as needed in several protocols;
- Careful scheduling of the product of conjugates during the computation of the inverse in $\mathbb{F}_{p^{12}}$ (see A) to keep most of the operations in subfields;
- (Minor optimization) Simplification of the final line functions that occur in the optimal ate pairing [37, Section 3.2]. This includes omitting the third line function and multiplying together the sparse values of the two remaining ones.
- (Minor optimization) When the BN parameter u is negative, replacement of the extra inversion incurred by a conjugation after final exponentiation [2].

We have implemented the optimal ate pairing algorithm as described above on the curve $E_{2,254}$ given in Table 1 in Java. The implementation is a pure Java software not using any assembly or other processor specific optimization. It is therefore clear that it does not achieve the absolute speed of the implementations we compare to in Table 2. Instead it is a more flexible implementation, fairly independent of platform restrictions.

After a preliminary version of this paper was completed, [2] pointed out that [26] offers further optimization opportunities for exponentiating elements in the cyclotomic subgroup with sparse exponents when using the BN subclass proposed herein. However, due to distinct implementation choices motivated by a rather different research focus, overall [2] does not attain the full level of optimization achievable by following the above list of recommended techniques.

While we do not claim that these figures are optimal, one can see in this example that our proposal achieves an about 17% better pairing computation

speed than the best previous results without taking advantage of our subfamily [8], with the added bonus of simpler, enhanced arithmetic (including faster square root extraction) in the finite fields $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ underlying groups $\mathbb{G}_1$ and $\mathbb{G}_2$ as needed by most pairing-friendly protocols. Automatically, this also holds for arithmetic in $\mathbb{G}_T$ enabled by the tower-friendly nature of the suggested finite-field representation.

While most of the listed techniques are applicable to different parameter choices as well as to other pairing-friendly elliptic curve families, our proposal was chosen so as to favor the best cases of each technique, and hence to maximize their potential for efficient processing.

5   Conclusion

We have presented a subclass of Barreto-Naehrig curves that generically favors efficient implementation while retaining a very simple description. Our proposal targets not only pairing computation speed but also the efficiency of all typical arithmetic operations needed to instantiate pairing-based cryptographic protocols, and focuses on offering optimization opportunities on the broadest possible landscape of platforms rather than narrowing down to any particular one.

As a highlight for future research, we point out that one problem still open regarding efficient implementation of pairing-based cryptography is that of deterministic, highly efficient hashing onto the groups $\mathbb{G}_1$ and $\mathbb{G}_2$. Although our proposal partially addresses this problem by supporting the fastest known arithmetic algorithms for these groups (particularly square root extraction), more advanced hashing techniques like that of Icart [25] are currently not applicable to any BN curve. Finding a secure hashing method of that kind for those groups or describing a subclass of BN curves where such a method is available is of great importance for many pairing-based protocols.

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A Scheduling the product of conjugates for $\mathbb{F}_{p^{12}}$ inversion

We now describe a simple technique to save operations when inverting an element $\gamma \in \mathbb{F}_{q^6}$ by keeping most of the involved operations in subfields. We follow the notation of Example 1 and Section 4.

One can invert $\gamma \in \mathbb{F}_{q^6} \setminus \{0\}$ by computing (see e.g. [29, Section 3.1])

$$\gamma^{-1} = \gamma^{v-1} \cdot \gamma^{-v},$$

where $v := 1+q+q^2+q^3+q^4+q^5$. Defining the quantities $\lambda := \gamma^{1+q^3} \in \mathbb{F}_{q^3}$, $\mu := \lambda^q \cdot \lambda^{q^2} \in \mathbb{F}_{q^3}$, $\varepsilon := \gamma^v \in \mathbb{F}_q$, and $\eta := \mu \cdot \varepsilon^{-1}$, we can write $\varepsilon = \gamma^{1+q+q^2+q^3+q^4+q^5} = \lambda \cdot \mu$ and $\gamma^{v-1} = \gamma^{q+q^2+q^3+q^4+q^5} = \gamma^{q^3} \cdot \gamma^{q(1+q^3)} \cdot \gamma^{q^2(1+q^3)} = \gamma^{q^3} \cdot \lambda^q \cdot \lambda^{q^2} = \gamma^{q^3} \cdot \mu$, whereby $\gamma^{-1} = \gamma^{q^3} \cdot (\mu \cdot \varepsilon^{-1}) = \gamma^{q^3} \cdot \eta$.

Writing $\gamma = \alpha + \beta w$ for $\alpha, \beta \in \mathbb{F}_{q^3}$, one sees that $\lambda = (\alpha + \beta w)(\alpha - \beta w) = \alpha^2 - \beta^2 \xi$, where the $\mathbb{F}_{q^3}$ squarings can be performed via the Chung-Hasan SQR$_3$ method for $\mathbb{F}_{q^3}$ over $\mathbb{F}_q$, incurring an overall cost of $2\tilde{m}+8\tilde{s}$, while $\mu = (\lambda \cdot \lambda^q)^q$ can be computed at a cost $3\tilde{m}+3\tilde{s}$, apart from conjugation. Performing the product $\varepsilon = \lambda \cdot \mu$ requires only $3\tilde{m}$, since this value is known to lie in $\mathbb{F}_q$. Computing $\eta = \mu \cdot \varepsilon^{-1}$ then incurs one $\mathbb{F}_q$ inversion and one multiplication between an element from $\mathbb{F}_{q^3}$ and another from $\mathbb{F}_q$ (which takes $3\tilde{m}$). Finally we are faced with the multiplication $\gamma^{-1} = \gamma^{q^3} \cdot \eta$ between an element from $\mathbb{F}_{q^6}$ and another from $\mathbb{F}_{q^3}$, which takes $2 \cdot 6\tilde{m}$, apart from conjugation.

Therefore, apart from conjugations and one $\mathbb{F}_q$ inversion, the overall cost is $23\tilde{m} + 11\tilde{s} \approx 91m$, which compares well against the corresponding $288m$ cost

incurred by naively computing γ^{-1} as a product of conjugates in sequential order.