Algebraic Pseudorandom Functions with Improved Efficiency from the Augmented Cascade[*]

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Abstract

We construct an algebraic pseudorandom function (PRF) that is more efficient than the classic Naor-Reingold algebraic PRF. Our PRF is the result of adapting the cascade construction, which is the basis of HMAC, to the algebraic settings. To do so we define an augmented cascade and prove it secure when the underlying PRF satisfies a property called parallel security. We then use the augmented cascade to build new algebraic PRFs. The algebraic structure of our PRF leads to an efficient large-domain Verifiable Random Function (VRF) and a large-domain simulatable VRF.

1 Introduction

Pseudorandom functions (PRFs), first defined by Goldreich, Goldwasser, and Micali [GGM86], are a fundamental building block in cryptography and have numerous applications. They are used for encryption, message integrity, signatures, key derivation, user authentication, and many other cryptographic mechanisms. Beyond cryptography, PRFs are used to defend against denial of service attacks [Ber96, CW03] and even to prove lower bounds in learning theory.

In a nutshell, a PRF is indistinguishable from a truly random function. We give precise definitions in the next section. The fastest PRFs are built from block ciphers like AES and security is based on ad-hoc interactive assumptions. In 1996, Naor and Reingold [NR97] presented an elegant PRF whose security can be deduced from the hardness of the Decision Diffie-Hellman problem (DDH) defined in the next section. The Naor-Reingold PRF takes as input an m-bit string $b = b_1 \dots b_m \in \{0,1\}^m$ and a secret key $(h, x_1, \dots, x_m)$ and outputs

$$F_{\text{NR}}\Big(\underbrace{(h, x_1, \dots, x_m)}_{key}, \underbrace{(b_1 \dots b_m)}_{input} \Big) := h^w \quad \text{where} \quad w := \prod_{i=1}^{m} x_i^{b_i} . \tag{1}$$

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We define this PRF more precisely in Section 4.1. Evaluating this PRF amounts to m modular multiplications plus one exponentiation. This PRF was recently generalized by Lewko and Waters [LW09] to work in groups where DDH may be easy, but where a weaker assumption called k-linear may hold. While this has clear security benefits, there is a cost in performance compared to Naor-Reingold.

The algebraic structure of the Naor-Reingold PRF leads to several beautiful applications that are much harder to construct with generic PRFs built from block ciphers. Some examples include Verifiable Random Functions (VRFs) [HW10], oblivious PRFs (used for private keyword search [FIPR05] and secure computation of set-intersection [JL09]), and distributed PRFs [NR97], to name a few. Another algebraic PRF due to Dodis and Yampolskiy [DY05] (based on the signature scheme from [BB04c]) also has many useful applications. However, this PRF is only known to be secure when the domain is small (i.e. polynomial size).

Our results. We describe a new algebraic PRF that has the same domain as Naor-Reingold, but requires fewer multiplications to evaluate and uses shorter private keys. For parameters ℓ and n our PRF takes inputs $(u_1, \ldots, u_n)$ in $\mathbb{Z}_\ell^n$ along with a key $(h, x_1, \ldots, x_n)$ and outputs

$$F\big(\ (h, x_1, \ldots, x_n),\ (u_1 \ldots u_n)\ \big) := h^{1/w} \quad \text{where} \quad w := \prod_{i=1}^{n} (x_i + u_i)\ . \tag{2}$$

For a domain of size 2^m we have $n = m/\log_2 \ell$ and therefore evaluating this PRF requires a factor of $\log_2 \ell$ fewer multiplications than (1) to compute w. Since computing w often takes roughly the same time as the final exponentiation, evaluating this PRF is about twice as fast as evaluating the Naor-Reingold PRF. The secret key is shorter by a factor of $\log_2 \ell$. We prove security of this PRF from the ℓ-DDH assumption defined in the next section. The larger ℓ gets the stronger the assumption becomes and therefore one should keep ℓ small. Setting $\ell = 16$ or 256 for example is a reasonable choice.

Techniques. We prove security of the PRF by developing a PRF composition theorem that generalizes the classic cascade construction of Bellare, Canetti and Krawczyk [BCK96b]. The cascade construction, shown in Figure 1(a), constructs a PRF with a large domain from a PRF with a small domain and is the basis for the NMAC and HMAC PRFs [BCK96a, Bel06].

Unfortunately, the cascade construction is insufficient for our purposes because it requires the output of the underlying PRF to be at least as long as the secret key. We therefore define the augmented cascade, shown in Figure 1(b), which eliminates this requirement by using supplemental secret information in every block. The augmented cascade can be applied directly to PRFs whose output is much smaller than the secret key. Suprisingly, security of the augmented cascade does not follow from security of the underlying PRF. We therefore develop a sufficient condition on the underlying PRF, called parallel security, that implies security of the augmented cascade.

Armed with the augmented cascade theorem, we build our large-domain PRF by plugging the Dodis-Yampolskiy small-domain PRF [DY05] into the augmented cascade. To prove security, we prove that the Dodis-Yampolskiy PRF is parallel secure. As a short aside, we show the power of the augmented cascade theorem by using it to quickly prove security of the Naor-Reingold and Lewko-Waters PRFs.

Verifiable Random Functions. The algebraic structure of the PRF in (2) enables many of the same applications as the Naor-Reingold PRF. In Sections 6.2 and 7 we show how to convert this PRF into an efficient Verifiable Random Function (VRF) with a large domain in groups with a bilinear map. A VRF, as defined in [MRV99], is a PRF that also outputs a proof that it was evaluated correctly. VRFs give signature schemes where every message has a unique signature. They were also used to construct e-cash schemes [BCKL09, ASM07].

Hohenberger and Waters [HW10] recently constructed an elegant VRF with a large domain. Our VRF is a little less efficient, but surprisingly is based on a weaker assumption. Their construction requires an assumption where the problem instance has size $O(mQ)$ where 2^m is the size of the domain and Q is the number of adversarial queries. We only require a problem instance of size $O(m)$. Our security proof makes use of admissible hash functions as in [BB04b]. We also describe a large-domain *simulatable* VRF, as defined in [CL07].

2 Preliminaries

2.1 Pseudorandom Functions

We begin by reviewing the definition of pseudorandom functions [GGM86]. Informally, a pseudorandom function is an efficiently computable function such that no efficient adversary can distinguish the function from a truly random function given only black-box access.

More precisely, a PRF is an efficiently computable function $F : K \times X \to Y$ where K is called the key space, X is called the domain, and Y is called the range. Security for a PRF is defined using two experiments between a challenger and an adversary $\mathcal{A}$. For $b \in \{0, 1\}$ the challenger in Exp_b works as follows.

When $b = 0$ the challenger chooses a random key $k \in K$ and sets $f(\cdot) := F(k, \cdot)$.
When $b = 1$ the challenger chooses a random function $f : X \to Y$.
The adversary (adaptively) sends input queries $x_1, \ldots, x_q$ in X to the challenger and the challenger responds with $f(x_1), \ldots, f(x_q)$. Eventually the adversary outputs a bit $b' \in \{0, 1\}$.

For $b \in \{0, 1\}$ let W_b be the probability that $\mathcal{A}$ outputs 1 in Exp_b.

Definition 1. *A PRF $F : K \times X \to Y$ is secure if for all efficient adversaries $\mathcal{A}$ the quantity*

$$\mathit{PRF}_{\mathit{adv}}[\mathcal{A}, F] := |W_0 - W_1|$$

is negligible.

As usual, one makes the terms "efficient" and "negligible" precise using asymptotic notation by equating efficient with probabilistic polynomial time and equating negligible with functions smaller than all inverse polynomials. Here, we use non-asymptotic language to simplify the notation.

2.2 Complexity assumptions

Notation. In this section and in Section 4.2 it is convenient to use vector notation defined as follows. Let $\mathbb{G}$ be a group of prime order p with generator g.

- for vectors $\bar{g} = (g_1, \ldots, g_n) \in \mathbb{G}^n$ and $\bar{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_p^n$ define $\bar{g}^{\bar{x}} := \left(g_1^{x_1}, \ldots, g_n^{x_n}\right) \in \mathbb{G}^n$. For a scalar $g \in \mathbb{G}$ define $g^{\bar{x}} := (g^{x_1}, \ldots, g^{x_n})$.
- for a matrix $A = (a_{i,j}) \in \mathbb{Z}_p^{n \times m}$ and a vector $\bar{g} \in \mathbb{G}^m$ define

$$A \cdot \bar{g} := \bar{h} \in \mathbb{G}^n \quad \text{where} \quad h_i := \prod_{j=1}^{m} g_j^{a_{i,j}} \quad \text{for } i = 1, \ldots, n.$$

  and for a scalar $g \in \mathbb{G}$ define $g^A := (g^{(a_{i,j})}) \in \mathbb{G}^{n \times m}$.

We also use $[k]$ to denote the set $\{1, \ldots, k\}$.

The k-linear assumption. Let V_k be the linear subspace of $\mathbb{Z}_p^{k+1}$ containing all vectors orthogonal to $(-1, 1, 1, \ldots, 1)$; its dimension is k. A vector $v = (v_0, \ldots, v_k)$ is in V_k if v_0 is the sum of the remaining coordinates. When $k = 1$ a vector $v = (v_0, v_1)$ is in V_1 if and only if $v_0 = v_1$.
For an algorithm $\mathcal{A}$ define

$$\mathsf{LIN}^{(k)}_{\text{adv}}[\mathcal{A}, \mathbb{G}] := \left| \Pr[\mathcal{A}(\bar{g},\ \bar{g}^{\bar{x}}) = 1] - \Pr[\mathcal{A}(\bar{g},\ \bar{g}^{\bar{y}}) = 1] \right|$$

where $\bar{g}$ is uniform in $\mathbb{G}^{k+1}$, $\bar{x}$ is uniform in V_k, and $\bar{y}$ is uniform in $\mathbb{Z}_p^{k+1}$.

Definition 2. *For $k \geq 1$ we say that the k-linear assumption holds for the group $\mathbb{G}$ if for all efficient algorithms $\mathcal{A}$ the advantage $\mathsf{LIN}^{(k)}_{adv}[\mathcal{A}, \mathbb{G}]$ is negligible.*

The 1-linear assumption is identical to the standard Decision Diffie-Hellman (DDH) problem in $\mathbb{G}$ and we write $\mathsf{DDH}_{\text{adv}}[\mathcal{A}, \mathbb{G}]$ to denote $\mathsf{LIN}^{(1)}_{\text{adv}}[\mathcal{A}, \mathbb{G}]$. For $k = 2$ we obtain the decision linear assumption defined in [BBS04]. For larger k we obtain the generalized linear assumption defined in [Sha07, HK07].

It is not difficult to show that if the k-linear assumption holds for $\mathbb{G}$ then so does the ℓ-linear assumption for all $\ell > k$. It is believed that the larger k is the weaker the assumption becomes. In particular, the 2-linear assumption may hold in groups where the 1-linear assumption (a.k.a DDH) is false.

The k-DDH assumption. For $x \in \mathbb{Z}_p$ let $\text{pow}(x, k)$ be the vector $(1, x, x^2, \ldots, x^k) \in \mathbb{Z}_p^{k+1}$. The k-DDH assumption states that $g^{1/x}$ is indistinguishable from a random group element given $g^{\text{pow}(x,k)}$. More precisely, for an algorithm $\mathcal{A}$ define

$$\mathsf{DDH}^{(k)}_{\text{adv}}[\mathcal{A}, \mathbb{G}] := \left| \Pr[\mathcal{A}(g^{\text{pow}(x,k)},\ g^{1/x}) = 1] - \Pr[\mathcal{A}(g^{\text{pow}(x,k)},\ h) = 1] \right|$$

where g, h are uniform in $\mathbb{G}$ and x is uniform in $\mathbb{Z}_p$. When $x = 0$ we define $g^{1/x}$ to be 1 in $\mathbb{G}$.

Definition 3. *For $k \geq 1$ we say that the k-DDH assumption holds for the group $\mathbb{G}$ if for all efficient algorithms $\mathcal{A}$ the advantage $\mathsf{DDH}^{(k)}_{adv}[\mathcal{A}, \mathbb{G}]$ is negligible.*

This assumption was previously used in [BB04a, DY05] where it was called k-DDHI. The 1-DDH assumption implies the standard DDH assumption. Moreover, for $k > 1$ the k-DDH assumption implies the ℓ-DDH assumption for $\ell < k$.

A hierarchy. From the facts stated above we obtain a hierarchy of complexity assumptions from the k-linear and k-DDH assumptions:

$$\ldots \leq k\text{-lin} \leq \ldots \leq 1\text{-lin} \equiv \text{DDH} \leq 1\text{-DDH} \leq \ldots \leq k\text{-DDH} \leq \ldots$$

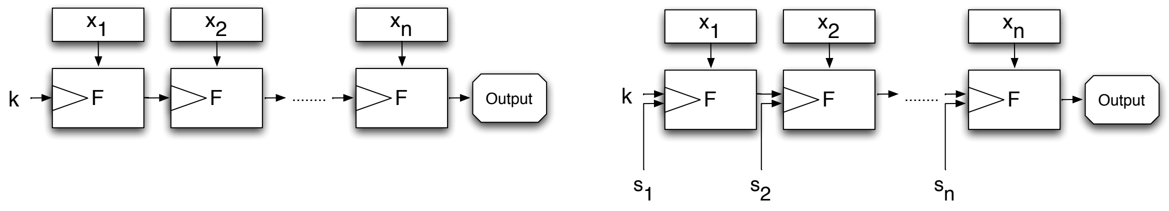
The assumptions becomes stronger as one moves from left to right. In the generic group model this hierarchy can be shown not to collapse [BBG05].

A useful lemma. We will need the following lemma from [BHHO08] (Lemma 1). Let $\mathbb{Z}_p^{n \times m}$ be the set of $n \times m$ matrices over $\mathbb{Z}_p$ and let $\text{Rk}_1(\mathbb{Z}_p^{n \times m})$ be the set of matrices in $\mathbb{Z}_p^{n \times m}$ of rank 1.

Let $\mathbb{G}$ be a group of order p with generator g. Let A_0 be uniform in $\text{RK}_1(\mathbb{Z}_p^{n \times m})$ and A_1 be uniform in $\mathbb{Z}_p^{n \times m}$. For an algorithm $\mathcal{A} : \mathbb{G}^{n \times m} \to \{0, 1\}$ define

$$\text{adv}[\mathcal{A}] := \left| \Pr[\mathcal{A}(g^{(A_0)}) = 1] - \Pr[\mathcal{A}(g^{(A_1)}) = 1] \right|$$

The following lemma shows that when DDH is hard in $\mathbb{G}$, no efficient adversary can distinguish a rank 1 matrix in the exponent from a random matrix in the exponent.



(a) The cascade construction, key $= k$ (b) The augmented cascade, key $= (s_1, \ldots, s_n, k)$

Figure 1: Cascade and augmented cascade

Lemma 1. *For every algorithm $\mathcal{A}$ there exists an algorithm $\mathcal{B}$ with about the same running time as $\mathcal{A}$ so that*

$$adv[\mathcal{A}] \leq \min(m, n) \cdot \mathsf{DDH}_{adv}[\mathcal{B}, \mathbb{G}]$$

2.3 Bilinear maps

We briefly review the necessary facts about bilinear maps and bilinear map groups [Mil04]. Let $\mathbb{G}$ and $\mathbb{G}_{\mathrm{T}}$ be two (multiplicative) cyclic groups of prime order p and let g be a generator of $\mathbb{G}$. A bilinear map is a map $e : \mathbb{G} \times \mathbb{G} \to \mathbb{G}_{\mathrm{T}}$ with the following properties:

1. Bilinear: for all $u, v \in \mathbb{G}$ and $a, b \in \mathbb{Z}$ we have $e(u^a, v^b) = e(u, v)^{ab}$.
2. Non-degenerate: $e(g, g) \neq 1$.

We say that $\mathbb{G}$ is a bilinear group if the group action in $\mathbb{G}$ can be computed efficiently and there exists a group $\mathbb{G}_{\mathrm{T}}$ and an efficiently computable bilinear map $e : \mathbb{G} \times \mathbb{G} \to \mathbb{G}_{\mathrm{T}}$ as above. Note that $e(\cdot, \cdot)$ is symmetric since $e(g^a, g^b) = e(g, g)^{ab} = e(g^b, g^a)$.

The k-BDH assumption. For $x \in \mathbb{Z}_p$, let $g^{\mathrm{pow}(x,k)}$ be the vector $(g, g^x, \ldots, g^{(x^k)})$. The k-BDH assumption states that $e(g, u)^{1/x}$ is indistinguishable from a random group element in $\mathbb{G}_{\mathrm{T}}$ given u and $g^{\mathrm{pow}(x,k)}$. More precisely, for an algorithm $\mathcal{A}$ define

$$\mathsf{BDH}^{(k)}_{\mathrm{adv}}[\mathcal{A}, \mathbb{G}] := \big| \Pr[\mathcal{A}(u, g^{\mathrm{pow}(x,k)}, e(g^{1/x}, u)) = 1] - \Pr[\mathcal{A}(u, g^{\mathrm{pow}(x,k)}, \gamma) = 1] \big|$$

where g, u are uniform in $\mathbb{G}$, x is uniform in $\mathbb{Z}_p$, and γ is uniform in $\mathbb{G}_T$.

Definition 4. *For $k \geq 1$, we say that the k-BDH assumption holds for the group $\mathbb{G}$ if for all efficient algorithms $\mathcal{A}$, the advantage $\mathsf{BDH}^{(k)}_{adv}[\mathcal{A}, \mathbb{G}]$ is negligible.*

3 The Augmented Cascade

3.1 The cascade PRF

The cascade pseudorandom function, defined in [BCK96b], constructs a secure PRF with domain X^n from a secure PRF with domain X. The cascade construction is shown in Figure 1(a). More precisely, let $F : K \times X \to K$ be a secure PRF. We define the cascade of F denoted $F^{*n} : K \times X^n \to K$ as:

```
input: key $k_0 \in K$, and $(x_1, \ldots, x_n) \in X^n$
for $i = 1, \ldots, n$ do:
    $k_i \leftarrow F(k_{i-1}, x_i)$
output $k_n$
```

Here the output range of F must equal the key space K.

Cascade is the basis for the NMAC and HMAC message authentication codes [BCK96b, Bel06]. Cascade is a generalization of the GGM PRF [GGM86], which can be viewed as a method to convert a PRF with a 1-bit domain into a PRF with an n-bit domain. The security of the cascade construction is stated concretely in the following theorem, which is shown in [BCK96b].

Theorem 2. *For every q-query PRF adversary $\mathcal{A}$ attacking F^{*n} there exists a q-query PRF adversary $\mathcal{B}$ attacking F such that*

$$\mathsf{PRF}_{adv}[\mathcal{A}, F^{*n}] \leq nq \cdot \mathsf{PRF}_{adv}[\mathcal{B}, F]$$

where $\mathcal{B}$ runs in about the same time as $\mathcal{A}$.

3.2 Augmented Cascade PRF

The cascade construction works with a PRF F whose output is as long as the PRF key. When constructing algebraic PRFs, the starting point is often a PRF F whose output is shorter than required for cascade. We therefore need to augment the output of F so that its output is a valid key for F. Consider a PRF F operating on the following spaces:

$$F : \underbrace{(S \times K)}_{\text{key}} \times X \to K$$

Notice that the key for F is a pair in (S, K) while the output is in K and therefore not a complete key. In the augmented cascade we append a fresh random string to the output to make it into a valid key.

We define the augmented cascade, denoted $\hat{F}^{*n}$, as a function

$$\hat{F}^{*n} : \underbrace{(S^n \times K)}_{\text{key}} \times X^n \to K$$

The function's domain is X^n and its keys are tuples of the form $(s_1, \ldots, s_n, k) \in S^n \times K$. The augmented cascade is shown in Figure 1(b) and is defined as follows:

```
input:  key $(s_1, \ldots, s_n, k_0) \in S^n \times K$, and
        value $(x_1, \ldots, x_n) \in X^n$
for $i = 1, \ldots, n$ do:
    $k_i \leftarrow F(\ (s_i, k_{i-1}),\ x_i)$
output $k_n$
```

Security. Unfortunately, the augmented cascade can be insecure even if the underlying function F is a secure PRF. For example, F can be a secure PRF even if it ignores the part of the key in K (i.e. F only uses the part of the key in S). In this case, since we ignore k_i (for all i), the last block of the augmented cascade construction is evaluated independently of the first $n-1$ blocks. Thus, the resulting augmented cascade construction $\hat{F}^{*n}$ ignores the first $n-1$ input blocks and hence cannot be a secure PRF. In the next two sections we establish sufficient conditions for security of the augmented cascade.

3.3 Parallel composition security

In Theorem 3 below we will show that the augmented cascade is a secure PRF provided that the underlying function F satisfies a property we call *parallel security*. This property says that F remains a secure PRF when the adversary has access to multiple instances of the function with different but related keys.

For a function $F : (S \times K) \times X \to K$ and an integer $q > 0$ we define q related keys $(s, k_1), \ldots, (s, k_q)$ where $s \in S$ and $k_1, \ldots, k_q \in K$. These keys are related since they all share the same s. We say that the function F is q-parallel secure if the resulting set of q functions is indistinguishable from q random independent functions.
More precisely, let $F^{(q)}$ be the function: $F^{(q)} : (S \times K^q) \times \big(X \times [q]\big) \to K$ defined by

$$F^{(q)}\big(\ \underbrace{(s, k_1, \ldots, k_q)}_{\text{key}}\ ,\ \underbrace{(x, i)}_{\text{input}}\big) := F\big(\ (s, k_i)\ ,\ x\big)$$

Here $i \in [q]$ selects the key (s, k_i) to be used in the function F. Thus, $F^{(q)}$ emulates q functions F whose keys are (s, k_i) for $i = 1, \ldots, q$.

Definition 5. *We say that $F : (S \times K) \times X \to K$ is a q-parallel secure PRF if $F^{(q)}$ is a secure PRF.*

The function F need not be q-parallel secure even if it is secure as a PRF. For example, as above, a secure PRF $F : (S \times K) \times X \to K$ that ignores the part of the key in K (i.e. only uses the S part of the key) is clearly not 2-parallel secure. Even when S is small (e.g. $S = \{0, 1\}$) the function F may not be 2-parallel secure even though F is a secure PRF.

3.4 Security of the augmented cascade

We now prove security of the augmented cascade provided that the underlying PRF is parallel-secure.

Theorem 3. *If F is q-parallel secure then the augmented cascade $\hat{F}^{*n}$ is a secure PRF against q-query adversaries.*
In particular, for every q-query PRF adversary $\mathcal{A}$ attacking $\hat{F}^{*n}$ there is a q-query PRF adversary $\mathcal{B}$ attacking $F^{(q)}$ such that

$$\textsf{PRF}_{\textsf{adv}}[\mathcal{A}, \hat{F}^{*n}] \leq n \cdot \textsf{PRF}_{\textsf{adv}}[\mathcal{B}, F^{(q)}]$$

where $\mathcal{B}$ runs in about the same time as $\mathcal{A}$.

The proof uses a hybrid argument similar to the proof of the original cascade [BCK96b], but is sufficiently different to require its own proof.

Proof of Theorem 3. Given an adversary $\mathcal{A}$ we construct an adversary $\mathcal{B}$ as required. The intuition for the construction of $\mathcal{B}$ comes from the following sequence of $n + 1$ hybrid experiments between a challenger and adversary $\mathcal{A}$. In hybrid i, the challenger replaces the first i stages of the augmented cascade with a truly random function, while the last $n - i$ stages are carried out as in the standard augmented cascade.

More precisely, for $i = 0, \ldots, n$ define the challenger in hybrid experiment $\mathcal{P}_i$ as follows:

setup: the challenger chooses a random function $f : X^i \to K$ and random keys $s_1, \ldots, s_n$ in S.

queries: to respond to a query $(x_1, \ldots, x_n) \in X^n$ from $\mathcal{A}$ do:
let $k_i \leftarrow f(x_1, \ldots, x_i) \in K$

```
for $j = i+1, \ldots, n$ do:
    $k_j \leftarrow F(\ (s_j, k_{j-1}),\ x_j)$
send $k_n$ to $\mathcal{A}$
```

For $i = 0, \ldots, n$, let W_i be the probability that $\mathcal{A}$ outputs 1 in hybrid number i. Observe that in hybrid $\mathcal{P}_0$ the adversary $\mathcal{A}$ interacts with the function $\hat{F}^{*n}$ while in hybrid $\mathcal{P}_n$ the adversary interacts with a random function $f : X^n \to K$. Therefore,

$$\mathsf{PRF}_{\mathsf{adv}}[\mathcal{A}, \hat{F}^{*n}] = |W_n - W_0|$$

It follows by the standard hybrid argument that there exists a $t \in [1, n]$ such that

$$\mathsf{PRF}_{\mathsf{adv}}[\mathcal{A}, \hat{F}^{*n}] \le n \cdot |W_{t-1} - W_t| \tag{3}$$

We construct a q-query PRF adversary $\mathcal{B}$ such that

$$\mathsf{PRF}_{\mathsf{adv}}[\mathcal{B}, F^{(q)}] = |W_{t-1} - W_t| \tag{4}$$

Combining (3) and (4) proves the theorem.

Adversary $\mathcal{B}$ emulates the challenger in hybrid $\mathcal{P}_t$ or $\mathcal{P}_{t+1}$. This requires $\mathcal{B}$ to emulate a random function $f : X^t \to K$. To do so, it is convenient to describe $\mathcal{B}$ using an associative array T that maps elements of X^t to numbers in $\{1, \ldots, q\}$. Initially the array T is empty.

Adversary $\mathcal{B}$ interacts with its $F^{(q)}$ challenger and emulates a $\hat{F}^{*n}$ challenger for $\mathcal{A}$. $\mathcal{B}$ works as follows:

```
setup: $T \leftarrow \emptyset$, ctr $\leftarrow 0$, choose random $s_{t+2}, \ldots, s_n$ in $S$
queries: to respond to a query for $(x_1, \ldots, x_n) \in X^n$ from $\mathcal{A}$ do:
    if $T[x_1 \ldots x_t] = \perp$ (i.e. $x_1 \ldots x_t$ is a new prefix)
        increment ctr by 1 and set $T[x_1 \ldots x_t] :=$ ctr
    let $u \leftarrow T[x_1 \ldots x_t] \in \{1, \ldots, q\}$
    $\mathcal{B}$ queries its $F^{(q)}$ challenger at $(x_{t+1}, u)$ and obtains some $k_{t+1} \in K$
    note: $k_{t+1}$ is either random in $K$ or is equal to $F((s, k^*_u), x_{t+1})$ for some random key $(s, k^*_u)$
            chosen by $\mathcal{B}$'s challenger.
    for $j = t+2, \ldots, n$ do:    (finish the cascade)
        $k_j \leftarrow F(\ (s_j, k_{j-1}),\ x_j)$
    send $k_n$ to $\mathcal{A}$
eventually $\mathcal{A}$ outputs a bit $b' \in \{0,1\}$. $\mathcal{B}$ outputs the same bit and terminates.
```

Since $\mathcal{A}$ makes at most q queries the variable u is always in the range $[1, q]$ and therefore all of $\mathcal{B}$'s queries to its challenger are in the proper range.

When $\mathcal{B}$'s challenger emulates a random function then $\mathcal{B}$ emulates a $\mathcal{P}_{t+1}$ challenger to adversary $\mathcal{A}$. When $\mathcal{B}$'s challenger emulates $F^{(q)}$ then $\mathcal{B}$ emulates a $\mathcal{P}_t$ challenger to adversary $\mathcal{A}$. Therefore (4) holds which completes the proof of the theorem. $\square$

4 Existing Algebraic PRFs

We briefly review two existing algebraic PRFs in the literature and explain how their security neatly follows from the security of the augmented cascade construction.

4.1 The Naor-Reingold PRF

We start with the Naor-Reingold PRF [NR97]. Let $\mathbb{G}$ be a group of order p and let $f : (\mathbb{Z}_p \times \mathbb{G}) \times \{0,1\} \to \mathbb{G}$ be the function

$$f\big((x,h),b\big) := h^{(x^b)} = \begin{cases} h & \text{if } b=0 \\ h^x & \text{if } b=1 \end{cases} \tag{5}$$

Plugging f into the augmented cascade we obtain the following PRF whose domain is $\{0,1\}^n$ and range is $\mathbb{G}$:

$$F_{\text{NR}} := \hat{f}^{*n}\big(\underbrace{(x_1,\ldots,x_n,h)}_{\text{key}},\ \underbrace{(b_1\ldots b_n)}_{\text{input}}\big) = h^{\left(x_1^{b_1}\cdots x_n^{b_n}\right)}$$

To show that F_{NR} is a secure PRF it suffices to show that f is parallel secure. Naor and Reingold do so implicitly in their proof. We state this in the following lemma.

Lemma 4. *If the DDH assumption holds for the group $\mathbb{G}$ then the function f defined in (5) is q-parallel secure for all q polynomial in the security parameter.*

Proof. To prove that f is q-parallel secure we need to show that $f^{(q)}$ is a secure PRF. The function $f^{(q)}$ has domain $\{0,1\} \times [q]$ which is a set of size $2q$. Hence, it suffices to show that enumerating the $2q$ outputs of $f^{(q)}$ gives a secure pseudorandom generator. In particular, all we need to show is that

$$G(x,h_1,\ldots,h_q) := (h_1,\ h_1^x,\ldots,h_q,\ h_q^x)$$

is a secure PRG, assuming DDH holds in $\mathbb{G}$. This is a direct application of the random self reduction of DDH [NR97]. For completeness, we briefly review the reduction.

Let $\mathcal{A}$ be an algorithm that distinguishes the output of G on a random seed from a random tuple in G^{2q}. We build an algorithm $\mathcal{B}$ that breaks DDH in $\mathbb{G}$. Given a tuple (g,h,u,v) as input, algorithm $\mathcal{B}$ chooses random $a_1,\ldots,a_q$ and $b_1,\ldots,b_q$ in $\mathbb{Z}_p$ and computes

$$\big(g^{a_1}u^{b_1},\ h^{a_1}v^{b_1},\ \ldots,\ g^{a_q}u^{b_q},\ h^{a_q}v^{b_q}\big) \in \mathbb{G}^{2q} \tag{6}$$

Naor and Reingold show that if (g,h,u,v) is a DDH tuple then (6) is distributed as the output of G on a random seed. If (g,h,u,v) is a random tuple then (6) is random in $\mathbb{G}^{2q}$. Algorithm $\mathcal{B}$ runs $\mathcal{A}$ on the tuple (6) and outputs whatever $\mathcal{A}$ outputs. Then $\mathsf{DDH}_{\text{adv}}[\mathcal{B},\mathbb{G}] = \mathsf{PRF}_{\text{adv}}[\mathcal{A}, f^{(q)}]$ as required. The running time overhead of $\mathcal{B}$ is polynomial in q. $\square$

Combining Theorem 3 with Lemma 4 proves that the function F_{NR} is a secure PRF whenever DDH holds in $\mathbb{G}$.

4.2 The Lewko-Waters PRF

Lewko and Waters construct a PRF from the k-linear assumption [LW09]. While their PRF is not as efficient as the PRF of Naor and Reingold, their construction can remain secure in groups where DDH is false.

Let $\mathbb{G}$ be a group of order p. Let $k > 0$ be a parameter and define $f : (\mathbb{Z}_p^{k\times k} \times \mathbb{G}^k) \times \{0,1\} \to \mathbb{G}^k$ as the function

$$f\big((A,h),b\big) := A^b \cdot h = \begin{cases} h & \text{if } b=0 \\ A\cdot h & \text{if } b=1 \end{cases} \tag{7}$$

Recall that the notation $A \cdot h$ is defined in Section 2.2. Plugging f into the augmented cascade we obtain the following PRF whose domain is $\{0,1\}^n$:

$$F_{\text{LW}} := \hat{f}^{*n}\big((A_1,\ldots,A_n,h),\ (b_1\ldots b_n)\big) = (A_1^{b_1}\cdots A_n^{b_n})\cdot h\ \in \mathbb{G}^k$$

To show that F_{LW} is a secure PRF it suffices to show that f is parallel secure. Lewko-Waters do so implicitly in their proof. We state this in the following lemma.

Lemma 5. *If the k-linear assumption holds for the group $\mathbb{G}$, then the function f with parameter k defined in (7) is q-parallel secure for all q polynomial in the security parameter.*

Proof sketch. As in the proof of Lemma 4, it suffices to show that

$$G(A,h_1,\ldots,h_q) := (h_1,\ A\cdot h_1,\ldots,h_n,\ A\cdot h_n)$$

is a secure pseudorandom generator, assuming k-linear holds in $\mathbb{G}$. To prove this, one first shows that this G is a secure PRG when A is a random row vector in $\mathbb{Z}_p^k$. This uses the random self reduction of the k-linear problem described in [LW09]. Then one extends this to a $k\times k$ matrix using a hybrid argument over the k rows of the matrix A. Both ingredients are given in the Lewko-Waters proof of security. $\square$

Combining Theorem 3 with Lemma 5 proves that the function F_{LW} with parameter k is a secure PRF whenever the k-linear assumption holds in $\mathbb{G}$.

5 A New Algebraic PRF

Our starting point is a secure PRF due to Dodis and Yampolskiy [DY05] with a domain of size ℓ for some small ℓ. The PRF is proven secure under the ℓ-DDH assumption. Recall that we use $[\ell]$ to denote the set $\{1,\ldots,\ell\}$ and consider the PRF $f : (\mathbb{Z}_p \times \mathbb{G}) \times [\ell] \to \mathbb{G}$ defined as follows:

$$f(\ \underbrace{(s,h)}_{\text{key}},\ x\ ) := h^{1/(s+x)} \tag{8}$$

As before we define $h^{1/0} = 1$. Dodis and Yampolskiy prove the following theorem.

Theorem 6 ([DY05])**.** *Suppose the ℓ-DDH assumption holds in $\mathbb{G}$. Then f is a secure PRF provided the domain size ℓ is polynomial in the security parameter.*

In particular, for every PRF adversary $\mathcal{A}$ there is an ℓ-DDH algorithm B such that

$$\textsf{PRF}_{adv}[\mathcal{A},f] = \textsf{DDH}^{(\ell)}_{adv}[\mathcal{B},\mathbb{G}] \quad \text{and} \quad time(\mathcal{B}) = time(\mathcal{A}) + O(\ell\cdot T)$$

where T is the maximum time for exponentiation in $\mathbb{G}$.

Plugging f into the augmented cascade we obtain a PRF whose domain $[\ell]^n$ has exponential size. The resulting PRF is defined as follows:

$$F := \hat{f}^{*n}(\underbrace{(s_1,\ldots,s_n,h)}_{\text{key}},\ \underbrace{(x_1,\ldots,x_n)}_{\text{input}}) := h^{\left[1/\prod_{i=1}^{n}(s_i+x_i)\right]} \tag{9}$$

As discussed in the introduction, this PRF is more efficient than the Naor-Reingold PRF since it processes $\log_2 \ell$ bits per block rather than just one bit per block. The cost of this increased efficiency is reliance on a stronger assumption, namely ℓ-DDH.

Theorem 7. *The PRF defined in (9) is secure assuming the ℓ-DDH assumption holds in $\mathbb{G}$.*

To prove the theorem it suffices to show that f defined in (8) is parallel secure; namely that $f^{(q)}$ is a secure PRF for all polynomial q. We state this in the following lemma.

Lemma 8. *If the function f defined in (8) is a secure PRF and the DDH assumption holds in $\mathbb{G}$ then f is q-parallel secure for all q polynomial in the security parameter.*

In particular, for every PRF adversary $\mathcal{A}$ there are adversaries $\mathcal{B}_1$ and $\mathcal{B}_2$, whose running time is about the same as $\mathcal{A}$'s up to a polynomial factor, such that

$$\mathsf{PRF}_{adv}[\mathcal{A}, f^{(q)}] \le \mathsf{PRF}_{adv}[\mathcal{B}_1, f] + q \cdot \mathsf{DDH}_{adv}[\mathcal{B}_2, \mathbb{G}]$$

Note that the DDH assumption is implied by the k-DDH assumption and hence the DDH assumption used in Lemma 8 does not add an assumption beyond the one already used to prove that the underlying f is a secure PRF.

Proof of Lemma 8. Our goal is to show that $f^{(q)}$ is a secure PRF. We present the proof as a sequence of three games between a challenger and a PRF adversary $\mathcal{A}$ that attacks $f^{(q)}$. For $i = 0, 1, 2$, let W_i be the probability that $\mathcal{A}$ outputs 1 at the end of Game i.

Game 0. The challenger in this game behaves as a standard challenger presenting the adversary with an oracle for the pseudorandom function $f^{(q)}$ with a random key $(s, h_1, \ldots, h_q)$.

Game 1. The challenger in this game chooses a random function $u : [\ell] \to \mathbb{G}$. It also chooses random $r_1, \ldots, r_q$ in $\mathbb{Z}_p$. Now, given a query $(x, i) \in [\ell] \times [q]$ from the adversary, the challenger responds with $u(x)^{r_i}$.

We show that Games 0 and 1 are indistinguishable, assuming f is a secure PRF. In particular, there is a PRF adversary $\mathcal{B}_1$, whose running time is about the same as $\mathcal{A}$'s, such that

$$|W_0 - W_1| = \mathsf{PRF}_{\text{adv}}[\mathcal{B}_1, f] \tag{10}$$

Adversary $\mathcal{B}_1$ interacts with a PRF challenger for f and plays the role of an $f^{(q)}$ PRF challenger for $\mathcal{A}$. Adversary $\mathcal{B}_1$ works as follows:

```
choose random $r_1, \ldots, r_q$ in $\mathbb{Z}_p$.

given a query $(x, i) \in [\ell] \times [q]$ from $\mathcal{A}$ do:
    issue a query to $\mathcal{B}_1$'s challenger with input $x$ and
        obtain $y$ in response.
    respond on $\mathcal{A}$ with $y^{r_i}$.

finally, output whatever $\mathcal{A}$ outputs.
```

When $\mathcal{B}_1$'s challenger emulates an oracle for the function f with random key (s, h) it responds to query x with $y = h^{1/(s+x)}$. For $i = 1, \ldots, q$ define $h_i := h^{r_i}$. Then $\mathcal{B}_1$'s response to $\mathcal{A}$'s query for (x, i) is simply $h_i^{1/(s+x)}$ which is precisely $f^{(q)}(\ (s, h_1, \ldots, h_q),\ (x, i)\ )$. Hence, in this case $\mathcal{B}_1$ emulates a Game 0 challenger for $\mathcal{A}$.

When $\mathcal{B}_1$'s challenger emulates a random function $u : [\ell] \to \mathbb{G}$ then $\mathcal{B}_1$'s response to $\mathcal{A}$'s query for (x, i) is simply $u(x)^{r_i}$ which is precisely how a Game 1 challenger would respond. These two arguments prove (10), as required.

Game 2. The challenger presents the adversary with an oracle for a random function $w : [\ell] \times [q] \to \mathbb{G}$.

We use Lemma 1 to argue that Games 1 and 2 are indistinguishable assuming the DDH assumption holds in $\mathbb{G}$. In particular, there is a DDH algorithm $\mathcal{B}_2$ such that

$$|W_1 - W_2| \leq q \cdot \mathsf{DDH}_{\mathrm{adv}}[\mathcal{B}_2, \mathbb{G}] \tag{11}$$

Let $(x_1, i_1), \ldots, (x_q, i_q)$ be $\mathcal{A}$'s queries to its challenger. Recall that in Game 1 the challenger responds to $\mathcal{A}$'s queries using a random function $u : X \to G$ and random $r_1, \ldots, r_q \in \mathbb{Z}_p$. Let $A \in \mathbb{Z}_p^{q \times q}$ be the matrix $A := (r_i s_j)_{1 \leq i,j \leq q}$. Clearly A has rank 1.

In Game 1 the adversary is given q entries in the matrix $g^A \in \mathbb{G}^{q \times q}$. In Game 2 the adversary is given q random values in $\mathbb{G}$ which we treat as q entries in a random $q \times q$ matrix in $\mathbb{G}$. By Lemma 1 there is an algorithm $\mathcal{B}_2$ that satisfies (11), as required.

Summary. Combining (10) and (11) shows that

$$\mathsf{PRF}_{\mathrm{adv}}[\mathcal{A}, f^{(q)}] = |W_0 - W_2| \leq |W_0 - W_1| + |W_1 - W_2| \leq \mathsf{PRF}_{\mathrm{adv}}[\mathcal{B}_1, f] + q \cdot \mathsf{DDH}_{\mathrm{adv}}[\mathcal{B}_2, \mathbb{G}]$$

which completes the proof of the theorem. $\square$

The proof of Theorem 7 follows by combining Theorem 3 with Lemma 8, which shows that the function F with parameter ℓ is a secure PRF whenever the ℓ-DDH assumption holds.

6 Verifiable Random Functions

Verifable Random Functions, introduced by Micali, Rabin, and Vadhan [MRV99] are PRFs where the party holding the secret key can produce a non-interactive proof that the PRF was evaluated correctly. The proof should not interfere with the pseudorandom properties of the PRF.

We give two VRF constructions from the augmented cascade. In this section, for a parameter ℓ, we use the Dodis-Yampolskiy small-domain VRF to construct VRFs for a domain of size ℓ^n for *constant* n. Security is based on the $n\ell$-BDH assumption in bilinear groups. In comparison, the core Dodis-Yampolskiy construction requires the ℓ^n-BDH assumption for a VRF on a domain of size ℓ^n.

In Section 7 we construct a VRF for a domain of size 2^m for arbitrary m from the $O(m)$-BDH assumption. Our construction makes use of admissible hash functions introduced in [BB04b].

Hohenberger and Waters [HW10] recently constructed an elegant large domain VRF from the Naor-Reingold PRF for domain of size 2^m for arbitrary m. Security against a Q-query adversary relies on the $O(mQ)$-BDHE assumption, where t-BDHE is an assumption of the same flavor as the t-BDH assumption. While the efficiency of our VRF is worse than that of Hohenberger and Waters, the required complexity assumption is weaker: $O(m)$ vs. $O(mQ)$. The proof techiques for the two constructions are quite different. Hohenberger and Waters use the pile-up approach of Waters [Wat05] while we use admissible hash functions [BB04b].

Other VRFs include Abdalla et al. [ACF09] who give a construction using the m-wBDH assumption in blinear groups for a domain of size 2^m. The construction is limited to polynomial size domains since security degrades exponentially in m. Early VRFs outputting one bit were given by Lysyanskaya [Lys02] and Dodis [Dod03] based on stronger assumptions.

6.1 Definition of VRFs

A VRF is an efficiently computable function $F : K \times X \to Y$ equipped with three algorithms:

- $\mathsf{Gen}(1^\lambda)$ outputs a pair of keys $(\mathsf{pk}, \mathsf{sk})$ for a security parameter λ.
- $\mathsf{Prove}(\mathsf{sk}, x)$ computes $\big(F(\mathsf{sk}, x), \pi(\mathsf{sk}, x)\big)$, where $\pi(\mathsf{sk}, x)$ is a proof of correctness.
- $\mathsf{Verify}(\mathsf{pk}, x, y, \pi)$ verifies that $y = F(\mathsf{sk}, x)$ using the proof π.

Security for a VRF is defined using two experiments between a challenger and an adversary $\mathcal{A}$. For $b \in \{0, 1\}$, the challenger in Exp_b works as follows:

The challenger chooses a random key $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^\lambda)$.
The adversary (adaptively) sends queries $x_1, \ldots, x_q$ in X to the challenger who responds with $F(\mathsf{sk}, x_i)$ and proof $\pi(\mathsf{sk}, x_i)$ for $i = 1, \ldots, q$.
The adversary issues a special challenge query x^*. If $b = 0$, the challenger responds with $F(\mathsf{sk}, x^*)$. If $b = 1$, the challenger chooses random $y \in Y$ and responds with y.
Eventually the adversary outputs a bit $b' \in \{0, 1\}$.

For $b \in \{0, 1\}$ let W_b be the probability that $\mathcal{A}$ outputs 1 in Exp_b. Define $\mathsf{VRF}_{\mathrm{adv}}[\mathcal{A}, F] := |W_0 - W_1|$.

Definition 6. *A VRF is said to be secure if it satisfies the following properties.*

1. **Pseudorandomness:** *For every efficient adversary $\mathcal{A}$, $\mathsf{VRF}_{adv}[\mathcal{A}, F]$ is a negligible function.*
2. **Verifiability:** *For* $(y, \pi) \leftarrow \mathsf{Prove}(\mathsf{sk}, x)$*,* $\mathsf{Verify}(\mathsf{pk}, x, y, \pi) = 1$.
3. **Uniqueness:** *no values of* $(\mathsf{pk}, x, y_1, y_2, \pi_1, \pi_2)$ *satisfy*
$\mathsf{Verify}_{\mathsf{pk}}(x, y_1, \pi_1) = \mathsf{Verify}_{\mathsf{pk}}(x, y_2, \pi_2) = 1$ *for* $y_1 \neq y_2$.

6.2 Building a VRF using augmented cascade

We construct a secure VRF with domain of size ℓ^n using the $n\ell$-BDH assumption and the augmented cascade. Evaluating the VRF takes n multiplications and one exponentiation. Our VRF is built from the augmented cascade using the Dodis-Yampolskiy VRF as the underlying function. The Dodis-Yampolskiy VRF uses pairings and outputs elements in $\mathbb{G}_\mathrm{T}$ while its key uses elemets in $\mathbb{G}$. We therefore need to slightly tweak the augmented cascade to compensate for the difference between $\mathbb{G}_\mathrm{T}$ and $\mathbb{G}$, but this is easily done. The resulting VRF is defined as follows:

Algorithm $\mathsf{Gen}(1^\lambda)$: Fix a group $\mathbb{G}$ of prime order p with a bilinear pairing. Choose random generators $g, u \in \mathbb{G}$ and random values $s_1, s_2, \ldots, s_n \in \mathbb{Z}_p$. Set $t_i := g^{s_i}$ for $i = 1, \ldots, n$ and output the keys

$$\mathsf{pk} = (g, u, t_1, \ldots, t_n), \quad \mathsf{sk} = (g, u, s_1, \ldots, s_n).$$

Function $F : (\mathbb{G}^2 \times \mathbb{Z}_p^n) \times [\ell]^n \to \mathbb{G}_{\mathrm{T}}$. On input sk and $\mathbf{x} = (x_1, \ldots, x_n) \in [\ell]^n$ output

$$F(\mathsf{sk}, \mathbf{x}) := e(\ g^{[1/\prod_{i=1}^{n}(x_i+s_i)]},\ u)\ .$$

Algorithm $\mathsf{Prove}(\mathsf{sk}, \mathbf{x})$: This algorithm outputs $F(\mathsf{sk}, \mathbf{x})$ along with a proof π as follows: for $i = 1$ to n, compute $\pi_i = g^{[1/\prod_{j=1}^{i}(x_j+s_j)]} \in \mathbb{G}$ and output the proof $\pi := (\pi_1, \pi_2, \ldots, \pi_n) \in \mathbb{G}^n$.

Algorithm $\mathsf{Verify}(\mathsf{pk}, \mathbf{x}, y, \pi)$: First verify that the proof π contains legal encodings of elements in $\mathbb{G}$. Next, check that

$$e(\pi_i,\ g^{x_i}\, t_i) = e(\pi_{i-1}, g) \quad \text{for } i = 1, \ldots, n,$$

where $\pi_0 := g$. Finally, check that $e(\pi_n, u) = y$, where y is the output of the VRF. Verify returns 1 iff all the checks are satisfied.

6.3 Proof of VRF security

We prove security for a polynomial size domain. For a domain of size ℓ^n we use the $n\ell$-BDH assumption.

Theorem 9. *Suppose the $n\ell$-BDH assumption holds in a bilinear group $\mathbb{G}$ (of order p). If ℓ^n is polynomial in the security parameter then the VRF defined in section 6.2 is secure. In particular, for every VRF adversary $\mathcal{A}$ there is an $n\ell$-DDH algorithm $\mathcal{B}$, whose running time is about the same as $\mathcal{A}$'s up to a polynomial factor, such that*

$$\mathit{VRF}_{adv}[\mathcal{A}, F] \le \ell^n \cdot \mathit{BDH}_{adv}^{(n\ell)}[\mathcal{B}, \mathbb{G}].$$

Proof. Verifiability of the VRF is straightforward. *Uniqueness* follows from the group structure: for any input there is only one group element in $\mathbb{G}$ that is a valid output, and moreover, it is not possible (even for an unbounded adversary) to devise a valid proof for another element. It remains to prove *pseudorandomness*.

Intuition. Given a $n\ell$-BDH instance, enumerate all possible inputs for each of the n input blocks. Consider an $n \times \ell$ matrix where the $(i, j)^{\text{th}}$ entry corresponds to the monomial $(x + j - b_i^*)$, for randomly chosen b_i^*. Any query from the adversary can be seen as a path through the matrix that visits exactly one cell in each column (corresponding to each input block). The simulator chooses n random cells called "mines," one mine per column, and constructs a public key that lets it answer any query from the adversary that does not visit *all* n mines. If the adversar's challenge query hits each and every mine, which happens with probability ℓ^{-n}, then the simulator can use the adversary to solve the given $n\ell$-BDH instance. We formalize this intuition below and in Section 7 extend this idea, with error-correcting codes, to construct a VRF with a large domain.

Let $\mathcal{A}$ be a VRF adversary attacking F. We construct the following algorithm $\mathcal{B}$ that breaks the $n\ell$-BDH assumption in $\mathbb{G}$ with advantage $\mathsf{VRF}_{\text{adv}}[\mathcal{A}, F]/\ell^n$.

Input: Algorithm $\mathcal{B}$ is given a tuple $(g, u, g^x, \ldots, g^{x^{n\ell}}, y) \in (\mathbb{G})^{n\ell+2} \times \mathbb{G}_{\mathrm{T}}$ and is to determine if y is $e(g, u)^{1/x}$ or drawn randomly from $\mathbb{G}_{\mathrm{T}}$.

Key generation: In this step, the algorithm $\mathcal{B}$ chooses a random $\mathbf{b}^* = (b_1^*, \dots, b_n^*) \overset{\mathrm{R}}{\leftarrow} [\ell]^n$ and constructs an instance of the VRF as follows: First, $\mathcal{B}$ constructs the polynomials

$$p_i(z) = \prod_{a \in [\ell]} (z + a - b_i^*) \qquad \text{and} \qquad p(z) = z^{-1} \prod_{i=1}^{n} p_i(z)$$

in $\mathbb{Z}_p[z]$. Since the product $\prod_{i=1}^{n} p_i(z)$ is over *all* possible inputs $a \in [\ell]$ the product is divisible by z^n and therefore $p(z)$ is in $\mathbb{Z}_p[z]$. Write

$$p(z) = \sum_{j=0}^{n\ell-1} c_j \cdot z^j$$

for some c_j in $\mathbb{Z}_p$. $\mathcal{B}$ then constructs the generator h as:

$$h = g^{p(x)} = \prod_{j=0}^{n\ell-1} \left(g^{(x^j)^{c_j}} \right)$$

Next, $\mathcal{B}$ constructs the public key values

$$t_i = h^{(x-b_i^*)} = g^{p(x)(x-b_i^*)} = \prod_{j=1}^{n\ell} \left(\left(g^{(x^j)} \right)^{c_{j-1} - b_i^* c_j} \right)$$

for $i = 1, \dots, n$ and sends $\mathcal{A}$ the public key $\mathsf{pk} = (h, u, t_1, \dots, t_n)$. The secret key values $s_1, \dots, s_n$ corresponding to this public key (and are unknown to $\mathcal{B}$) are:

$$s_i = x - b_i^* \quad \text{for } i = 1, \dots, n.$$

Responding to Oracle Queries: Without loss of generality, assume $\mathcal{A}$ never repeats a query. Consider the i^{th} query $(1 \le i < \ell^n)$ on message $\mathbf{b^{(i)}} = (b_1^{(i)}, \dots, b_n^{(i)}) \in [\ell]^n$. Every j such that $b_j^{(i)} = b_i^*$ contributes a factor z to the denominator in the definition of F. As long as $\mathbf{b}^{(i)} \neq \mathbf{b}^*$, $\mathcal{B}$ successfully answers the query, as follows. Construct n polynomials, $p^{(i,1)}$ through $p^{(i,n)}$ as

$$p^{(i,j)}(z) = \frac{p(z)}{(z + b_1^{(i)} - b_1^*) \cdots (z + b_j^{(i)} - b_j^*)} = \sum_{k=0}^{n\ell-j-1} d_{j,k} z^k \quad \in \mathbb{Z}_p[z] \tag{12}$$

for some constants $d_{j,k} \in \mathbb{Z}_p$, $j = 1, \dots, n$. Note $p^{(i,j)}(z)$ is a polynomial in $\mathbb{Z}_p[z]$ since $p(z)$ is divisible by the denominator in (12) unless the denominator is z^n which only happens when $\mathbf{b^{(i)}} = \mathbf{b}^*$.

To compute proofs, $\mathcal{B}$ computes:

$$\pi_j = \prod_{k=0}^{n\ell-j-1} \left(g^{(x^j)} \right)^{d_{j,k}} = h^{1/\left[\prod_{k=1}^{j} (x + b_k^{(i)} - b_k^*)\right]}$$

for $j = 1, \dots, n$ and sets $\pi = (\pi_1, \dots, \pi_n)$. Finally $\mathcal{B}$ computes the function on the query,

$$F(\mathsf{sk}, \mathbf{b^{(i)}}) = e(\pi_n, u) = e(h^{1/\left[\prod_{k=1}^{n} (s_k + b_k^{(i)})\right]}, u)$$

and responds on $\mathcal{A}$'s query with $F(\mathsf{sk}, \mathbf{b}^{(\mathbf{i})})$ and π. In the unlikely event that $\mathbf{b}^{(\mathbf{i})} = \mathbf{b}^*$ algorithm $\mathcal{B}$ aborts the simulation and outputs $\perp$.

Challenge: Eventually, $\mathcal{A}$ outputs a message $\hat{\mathbf{b}}$ on which it wants to be challenged. If $\hat{\mathbf{b}} \neq \mathbf{b}^*$, then $\mathcal{B}$ aborts and outputs $\perp$. If $\hat{\mathbf{b}} = \mathbf{b}^*$, $\mathcal{B}$ proceeds as follows.

Since $\mathcal{A}$ is a VRF adversary, it can distinguish between $e(h,u)^{1/[\prod_{i=1}^{n}(s_i+b_i^*)]} = e(h,u)^{1/x^n}$ and a random element in $\mathbb{G}_{\mathrm{T}}$ with $\mathsf{VRF}_{\mathrm{adv}}[\mathcal{A}, F]$. Now, recall that $p(z)$ is divisible by z^{n-1} but not by z^n. Thus, there are scalars $\rho \neq 0$ and $\rho_0, \ldots, \rho_{n\ell-n-1}$ in $\mathbb{Z}_p$ such that:

$$s(z) = \frac{p(z)}{z^n} - \frac{\rho}{z} = \sum_{j=0}^{n\ell-n-1} \rho_j z^j \ .$$

Now, using the challenge y, algorithm $\mathcal{B}$ computes:

$$y^* = y^{\rho} \cdot \prod_{j=0}^{n\ell-n-1} e\left((g^{(x^j)})^{\rho_j}, u\right)$$

If $y \overset{R}{\leftarrow} \mathbb{G}_{\mathrm{T}}$, then y^* is random in $\mathbb{G}_{\mathrm{T}}$. However, if $y = e(g,u)^{1/x}$ then y^* is $F(\mathsf{sk}, \mathbf{b}^*) = e(h,u)^{1/x^n}$. Therefore, $\mathcal{B}$ responds to $\mathcal{A}$'s challenge query with y^*.

Guess: Algorithm $\mathcal{A}$ makes more queries, if required, to which $\mathcal{B}$ responds as before, and then finally outputs a guess bit $b' \in \{0,1\}$. $\mathcal{B}$ outputs b' as its guess.

Success probability. The running time of $\mathcal{B}$ is dominated by responding to oracle queries. Each query requires one exponentiation and polynomially many multiplications. $\mathcal{A}$ queries $\mathcal{B}$ at most a polynomial many number of times therefore $\mathcal{B}$'s running time is roughly a polynmial factor off the running time of $\mathcal{A}$.

Correctness of the algorithm $\mathcal{B}$ follows in a fairly straightforward manner. If the challenge query is $\mathbf{b}^*$, $\mathcal{B}$ distinguishes the $n\ell$-BDH challenge with the same advantage the adversary $\mathcal{A}$ has against the VRF. This occurs with a probability of ℓ^{-n} because $\mathbf{b}^*$ is chosen uniformly at random by $\mathcal{B}$. Therefore

$$\mathsf{BDH}_{\mathrm{adv}}^{(n\ell)}[\mathcal{B}, \mathbb{G}] \geq \frac{\mathsf{VRF}_{\mathrm{adv}}[\mathcal{A}, F]}{\ell^n}$$

which completes the proof of the theorem. □

7 VRFs with large input domains

In this section, we show how to construct a secure VRF with an input domain of $\{0,1\}^m$ for arbitrary m. Security depends on the $O(m)$-BDH assumption. Evaluating the VRF requires about $48m$ multiplications and one exponentiation. Our construction uses error correcting codes with a large minimum distance.

Definition 7. *A $(m,n,d)_\ell$-error correcting code is an injective function $H : \{0,1\}^m \to [\ell]^n$ such that for every two words $c_1, c_2 \in \{0,1\}^n$ we have:*

$$\mathrm{HDist}\left(H(c_1), H(c_2)\right) \geq d,$$

where $\mathrm{HDist}(\cdot,\cdot)$ denotes the hamming distance between the codewords (the number of co-ordinates where the two codewords differ). We say that an error-correcting code is efficient *if the function H is efficiently computable.*

We need the following lemma from [BS98].

Lemma 10. *For any positive integers ℓ, m, let $n = 8m\ell$. Then there exists a $(m, n, d)_\ell$-error correcting code with $d > n(1 - 2/\ell)$.*

In particular, for the construction of the VRF it suffices to take $\ell = 6$. Therefore,

Corollary 11. *For any positive integer m, there exists a $(m, n, d)_6$-error correcting code with $d > 2n/3$ and $n = 48m$.*

We will need the function H defining the error-correcting code to be efficiently computable. We can make Corollary 11 efficient using an appropriate pseudo-random function or by using an explicit low rate error-correcting code from [ABN+92].

7.1 VRF construction

In this section, we construct a class of verifiable random functions parametrized by the output alphabet size $\ell \geq 6$. For $\ell = 6$ we obtain (using Corollary 11) an efficient VRF on domain $\{0,1\}^m$ requiring $48m$ multiplications per function evaluation and the $O(m)$-BDH assumption for security.

Algorithm $\mathsf{Gen}(1^\lambda)$: Fix a group $\mathbb{G}$ of prime order p with a bilinear pairing. Select random generators $g, u \in \mathbb{G}$, random values $s_1, s_2, \ldots, s_n \in \mathbb{Z}_p$ and set $t_i = g^{s_i}$. Choose an efficient error correcting code $H : \{0,1\}^m \to [\ell]^n$ with a minimum distance $d > n(1 - 1/e)$; (for $\ell = 6$ taking $n = 48m$ is sufficient by Corollary 11). Output the keys:

$$\mathsf{pk} = (g, u, t_1, \ldots, t_n), \quad \mathsf{sk} = (g, u, s_1, \ldots, s_n).$$

Function $F : (\mathbb{G}^2 \times \mathbb{Z}_p^n) \times \{0,1\}^m \to \mathbb{G}_\mathrm{T}$. On input sk and a bit string $x = (x_1, x_2, \ldots, x_m)$ in $\{0,1\}^m$ output:

$$F(\mathsf{sk}, x) := e\left(g^{\left[1/\prod_{i=1}^{n}(H(x)_i + s_i)\right]},\ u\right)$$

where $H(x)_i$ refers to the i^{th} coordinate of $H(x)$.

Algorithm $\mathsf{Prove}(\mathsf{sk}, x)$: Upon input sk and x output $F(\mathsf{sk}, x)$ along with a proof π as follows. For $i = 1$ to n, compute $\pi_i = g^{\left[1/\prod_{j=1}^{i}(H(x)_j + s_j)\right]}$. Output the proof:

$$\pi := (\pi_1, \pi_2, \ldots, \pi_n) \quad \in \mathbb{G}^n .$$

Algorithm $\mathsf{Verify}(\mathsf{pk}, x, y, \pi)$: First verify that π contains legal encodings of elements in $\mathbb{G}$. Next, check that:

$$e(\pi_i,\ g^{H(x)_i} \cdot t_i) = e(\pi_{i-1}, g),$$

where $\pi_0 := g$. Finally, check that $e(\pi_n, u) = y$, where y is the output of the VRF. Verify returns 1 iff all the checks are true.

Theorem 12. *If $H : \{0,1\}^m \to [\ell]^n$ is a $(m, n, d)_\ell$-error correcting code with $d > n(1 - 1/e)$ and $\ell = o(\sqrt{n}/\ln n)$, the VRF constructed in section 7.1 is secure under the $n\ell$-BDH assumption.*

In particular, for every VRF adversary $\mathcal{A}$ there is an $n\ell$-DDH algorithm $\mathcal{B}$ such that

$$\mathit{VRF}_{adv}[\mathcal{A}, F] \leq O(n) \cdot \mathit{BDH}_{adv}^{(n\ell)}[\mathcal{B}, \mathbb{G}].$$

Proof Outline. Recall the intuition behind the proof of Theorem 9. Given an $n\ell$-BDH instance, construct an $n \times \ell$ matrix where the $(i,j)^{\text{th}}$ entry corresponds to the monomial $(x + j - b_i^*)$, for randomly chosen b_i^*. In the proof of Theorem 9 the public parameters were constructed to enable the simulator to answer any query that does not visit all n mines. An adversary whose special query visits all n mines can be used to solve the $n\ell$-BDH instance. Since the fraction of challenge queries that do not cause the adversary to abort is exponentially small in n (i.e. ℓ^{-n}), this proof technique works only for small n.

In this proof, for a parameter k, we choose a random integer $r \in [k, n]$ and modify the simulated public parameters to enable the simulator to use challenge queries that visit at least r (rather than all n) mines. k is chosen so that the challenger fails on a $(1/n)$ fraction of all possible inputs with high probability, which can then be used to break the $n\ell$-BDH assumption. This requires k to be $O(\log Q)$. To mitigate an adversary forcing the simulator to abort by constructing highly correlated queries, a code with large minimum distance is used which lower bounds the success probability to at least $\Omega(1/n)$ which is sufficient for the simulation.

Proof. Along the lines of the previous proof, given an adversary $\mathcal{A}$ that distinguishes the VRF from a random function with non-negligible probability, we construct an algorithm $\mathcal{B}$ that solves the $n\ell$-BDH instance.

Input: Algorithm $\mathcal{B}$ given a tuple $(g, g^x, \ldots, g^{x^{n\ell}}, y) \in (\mathbb{G})^{n\ell+1} \times \mathbb{G}_{\mathrm{T}}$ is to determine if y is $e(g,u)^{1/x}$ or y is random in $\mathbb{G}_{\mathrm{T}}$.

Key generation: $\mathcal{B}$ sets a parameter $k = O(\log Q)$ and picks a number r uniformly at random from $\{k+1, \ldots, n\}$ and constructs the VRF as follows. $\mathcal{B}$ chooses $\mathbf{b}^* = (b_1^*, \ldots, b_n^*)$ uniformly at random from $[\ell]^n$. The public and private keys for algorithm $\mathcal{A}$ are generated as follows: $\mathcal{B}$ enumerates all possible inputs in time ℓ to construct

$$p_i(z) = \prod_{a \in [\ell]} (z + a - b_i^*) \quad \text{and} \quad p(z) = z^{-(n-r+1)} \prod_{i=1}^{n} p_i(z) = \sum_{j=0}^{n\ell-(n-r+1)} c_j \cdot z^j,$$

for some coefficients c_j in $\mathbb{Z}_p$. The product $\prod p_i(z)$ is over *all* possible inputs $a \in [\ell]$ which implies z^n is a factor. Therefore, it can be divided by z^{n-r+1}. The construction implies that $\mathcal{B}$ cannot answer any query that matches $\mathbf{b}^*$ on more than r values. $\mathcal{B}$ then computes the generator h:

$$h = g^{p(x)} = \prod_{j=0}^{n\ell-(n-r+1)} \left(g^{(x^j)^{c_j}} \right).$$

The secret keys s_i (unknown to $\mathcal{B}$) are $x - b_i^*$ for $i = 1, \ldots, n$. The public key component t_i corresponding to the secret key component s_i is:

$$t_i = h^{(x-b_i^*)} = g^{p(x)(x-b_i^*)} = \prod_{j=1}^{n\ell-n+r} \left(\left(g^{(x^j)} \right)^{c_{j-1} - b_i^* c_j} \right).$$

$\mathcal{B}$ computes t_i for $i = 1, \ldots, n$ and sends the public key $\mathsf{pk} = (h, u, t_1, \ldots, t_n)$ to $\mathcal{A}$.

Responding to Oracle Queries: $\mathcal{B}$ responds to queries in a manner almost identical to the one in the proof of Theorem 9. Let q_i be the i^{th} query from $\mathcal{A}$. $\mathcal{B}$ first computes $\mathbf{b}^{(\mathbf{i})} = H(q_i)$. If $\mathrm{HDist}(\mathbf{b}^{(\mathbf{i})}, \mathbf{b}^*) \leq (n-r)$ (i.e., they agree on at least r coordinates), as before, the query cannot be answered because the polynomial $p(z)$ is not divisible by the denominator in (12). In this case $\mathcal{B}$ outputs $\perp$ and aborts. Otherwise, $\mathcal{B}$ evaluates the function and responds to $\mathcal{A}$ in a manner identical to the one in the proof of Theorem 9.

Challenge: Eventually $\mathcal{A}$ outputs a message $\hat{b}$ on which it wants to be challenged.

If $\mathrm{HDist}(H(\hat{b}), \mathbf{b}^*) \neq (n-r)$, $\mathcal{B}$ outputs $\perp$ and aborts. Otherwise, let w denote the product $(s_i + H(\hat{b})_i)$ over the $n-r$ coordinates that $H(\hat{b})$ and $\mathbf{b}^*$ differ. Since $\mathcal{A}$ is a VRF adversary, it can distinguish between $e(h,u)^{1/[\prod_{i=1}^{n}(s_i+H(\hat{b})_i)]} = e(h,u)^{1/(x^r \cdot w)}$ and a random element in $\mathbb{G}_{\mathrm{T}}$ with $\mathsf{VRF}_{\mathrm{adv}}[\mathcal{A}, F]$. Now, recall that $p(z)$ is divisible by $z^{r-1} \cdot w$ but not by $z^r \cdot w$. Thus, there are scalars $\rho \neq 0$ and $\rho_0, \ldots, \rho_{n\ell-2n+r-1}$ in $\mathbb{Z}_p$ such that:

$$s(z) := \frac{p(z)}{z^r \cdot w} - \frac{\rho}{z} = \sum_{j=0}^{n\ell-2n+r-1} \rho_j z^j \quad \in \mathbb{Z}_p[z].$$

Now, using the challenge y, algorithm $\mathcal{B}$ computes:

$$y^* = y^{\rho} \cdot \prod_{j=0}^{n\ell-2n+r-1} e\left((g^{(x^j)})^{\rho_j}, u\right).$$

Now, note that if $y \xleftarrow{R} \mathbb{G}_{\mathrm{T}}$, then y^* is distributed randomly in $\mathbb{G}_{\mathrm{T}}$. However, if $y = e(g,u)^{1/x}$, then $y^* = e(g,u)^{p(z)/(x^r \cdot w)} = e(h,u)^{1/(x^r \cdot w)} = F(\mathsf{sk}, \hat{b})$. Therefore, $\mathcal{B}$ gives $\mathcal{A}$ the value y^* as computed above.

Guess: At the end of the interaction, algorithm $\mathcal{A}$ makes a guess and outputs a bit $b' \in \{0,1\}$ which $\mathcal{B}$ also returns.

Success Probability. If $\mathcal{B}$ does not abort, correctness follows in a straightfoward manner. $\mathcal{B}$ successfully simulates the challenge query only if $\mathrm{HDist}(H(\hat{b}), \mathbf{b}^*) = n - r$. Since $\mathcal{B}$ guesses r uniformly at random from $[k, n]$ this happens with probability at least $1/n$. Given a correct guess for r, we estimate the probability that $\mathcal{A}$ can force $\mathcal{B}$ to output $\perp$.

Let A_i be the event that the i^{th} query issued by the adversary cannot be answered by the simulator and $(A_i \,|\, r)$ be the event A_i given a particular value r. The $(Q+1)^{\mathrm{th}}$ query is the challenge query. The probability of the simulator successfully simulating the adversary without aborting is:

$$\Pr[\text{Success}] = \Pr\left[\bar{A_1} \wedge \ldots \wedge \bar{A_Q} \wedge A_{Q+1}\right].$$

We can bound this using the inclusion-exclusion principle:

$$\Pr\left[\bar{A_1} \wedge \ldots \wedge \bar{A_Q} \wedge A_{Q+1}\right] \geq \Pr\left[A_{Q+1}\right] - \sum_{i=1}^{Q} \Pr\left[A_i \wedge A_{Q+1}\right] \tag{13}$$

One can easily verify that:

$$\Pr\left[A_{Q+1} \,|\, r\right] = \binom{n}{r} \cdot \frac{(\ell-1)^{n-r}}{\ell^n}.$$

Recall that H has minimum distance $d > n(1 - 1/e)$. For the rest of the analysis, let the minimum distance of the code d be $n(1-\alpha)$, for some $\alpha < 1/e$. Both A_i and A_{Q+1} can occur simultaneously iff there are exactly r mines in the α fraction of coordinates on which they coincide, and the rest of the mines were placed randomly in the remaining coordinates. Thus, we get:

$$\Pr\left[A_i \wedge A_{Q+1} \,|\, r\right] \leq \binom{n\alpha}{r} \frac{(\ell-1)^{n-r}}{\ell^n}.$$

Therefore, plugging in the value of $\Pr[A_{Q+1}]$, we get:

$$\begin{aligned}\Pr[\text{Success} \mid r] &\geq \binom{n}{r}\frac{(\ell-1)^{n-r}}{\ell^n} - Q\binom{n\alpha}{r}\frac{(\ell-1)^{n-r}}{\ell^n} \\ &\geq \binom{n}{r}\frac{(\ell-1)^{n-r}}{\ell^n}\left(1 - Q\binom{n\alpha}{r}\cdot\binom{n}{r}^{-1}\right) \qquad (14)\end{aligned}$$

Now, using the fact that binomials satisfy $\left(\frac{n}{r}\right)^r \leq \binom{n}{r} \leq \left(\frac{ne}{r}\right)^r$ and the fact that $k = O(\log Q)$ and therefore $r \geq k \geq \log 2Q/(\log(1/e\alpha))$ we obtain that

$$Q\binom{n\alpha}{r}\binom{n}{r}^{-1} \leq Q(\alpha e)^r \leq 1/2$$

Plugging this bound into (14) we get:

$$\Pr[\text{Success} \mid r] \geq \binom{n}{r}\frac{(\ell-1)^{n-r}}{\ell^n}\left(1 - \frac{1}{2}\right).$$

Since $\mathcal{B}$ guesses a particular r with probability more than $1/n$, defining β such that $\beta \cdot n = k$ and summing over all possible values of r we get

$$\begin{aligned}\Pr[\text{Success}] = \sum_{r\geq\beta\cdot n}\left\{\frac{1}{n}\cdot\Pr[\text{Success} \mid r]\right\} &\geq \frac{1}{2n}\left\{\left(1-\frac{1}{\ell}\right)^n \sum_{r\geq\beta n}\binom{n}{r}(\ell-1)^{-r}\right\} \\ &= \frac{1}{2n}\left\{\sum_{r\geq\beta n}\binom{n}{r}\left(\frac{1}{\ell}\right)^r\left(\frac{\ell-1}{\ell}\right)^{n-r}\right\} \qquad (15)\end{aligned}$$

The quantity inside the paranthesis in (15) is the tail of a Binomial distribution. Since $\mathcal{A}$ runs in polynomial time, Q is $\mathrm{poly}(n)$ from which it follows β is $O(\log n/n)$ which allows us to get a lower bound using a Chernoff inequality with a large deviation. We get:

$$\left\{\sum_{r\geq\beta n}\binom{n}{r}\left(\frac{1}{\ell}\right)^r\left(\frac{\ell-1}{\ell}\right)^{n-r}\right\} \geq 1 - \left\{\exp\left(-n\frac{(\ell-1)}{8\ell^3}\right)\right\}.$$

Substituting this bound in equation (15), we obtain:

$$\Pr[\text{Success}] \geq \frac{1}{2n}\cdot\left\{1 - \exp\left(-n\frac{(\ell-1)}{8\ell^3}\right)\right\} = \frac{1}{2n} - \text{negl}(n) \qquad (16)$$

where the last equality follows since $\ell = o(\sqrt{n}/\ln n)$. Thus, if algorithm $\mathcal{A}$ breaks the VRF, algorithm $\mathcal{B}$ breaks the $n\ell$-BDH assumption with probability at least $\Omega(1/n)$. Therefore,

$$\mathsf{VRF}_{\text{adv}}[\mathcal{A}, F] \leq O(n)\cdot\mathsf{BDH}^{(n\ell)}_{\text{adv}}[\mathcal{B}, \mathbb{G}],$$

which concludes the proof of the theorem.

□

7.2 Simulatable VRFs

Chase and Lysyanskaya [CL07] introduced *simulatable* VRFs (sVRF), which they used to convert single-theorem non-interactive zero knowledge (NIZK) to many-theorem NIZK. Their simulatable VRF, secure under the k-BDH assumption and the subgroup decision assumption (SDA), has a polynomial size domain. We briefly outline how the augmented cascade gives a large-domain sVRF using the same assumptions.

Stated informally, Chase and Lysyanskaya show that by modifying the proof π of the Dodis-Yampolskiy VRF, there exist algorithms $(\mathsf{SimG}, \mathsf{SimSample}, \mathsf{SimProve})$ (analogous to $(\mathsf{Gen}, F, \mathsf{Prove})$ in the definition of VRFs) and a way to simulate parameters $\mathsf{SimParam}$ with the following properties:

1. SimSample, using the parameters output by SimParams, produces a random distribution (that is indistinguishable from the distribution of the outputs of F, since F is a sVRF).
2. SimProve is able to simulate proofs for these random outputs that are indistinguishable from proofs produced by Prove, and any adversary that is able to distinguish between the simulated proofs and real proofs can be used to break SDA.

The augmented cascade theorem generalizes to sVRFs and can be used to construct large-domain sVRFs from small-domain ones, provided the underlying sVRF has parallel-security. The simulatability of the sVRF makes it possible to push the hybrid proof of the augmented cascade (Theorem 3) to the settings of sVRFs. We note that this was not possible for VRFs since the simulator cannot provide proofs in the hybrid experiments. Now, plugging the Chase-Lysyanskaya sVRF into this augmented cascade, we obtain a large-domain sVRF.

8 Conclusions

We presented a generalization of the cascade construction called the augmented cascade. We used the augmented cascade to construct large-domain PRFs from small-domain algebraic PRFs. The augmented cascade provides a unified framework for analysing the constructions of Naor-Reingold and Lewko-Waters. We used the augmented cascade to extend the Dodis-Yampolskiy PRF to a PRF on large domains, resulting in the most efficient algebraic PRF to date.

The new large-domain PRF can be converted into a large-domain VRF in a bilinear group and proven secure based on the m-BDH assumption for some parameter m that depends on the domain size. For small domains the resulting VRF uses a weaker assumption than its Dodis-Yampolskiy origin. We obtain an efficient large domain VRF using error correcting codes. The algebraic structure of these constructions will likely find many applications, as was the case for the Naor-Reingold PRF. As an example, we briefly noted a simulatable-VRF for large domains.

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