Computational Soundness about Formal Encryption in Presence of Secret Shares and Key Cycles *

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Abstract. The computational soundness of formal encryption is researched extensively after the work by Abadi and Rogaway. A recent work by Abadi and Warinschi extends this work to a scenario in which secret sharing is used. A more recent work by Micciancio extends this work to deal the formal encryption in presence of key cycles by using of co-induction definition of the adversarial knowledge. In this paper, we prove a computational soundness theorem of formal encryption which is in presence of both key cycles and secret shares.

1 Introduction

There are two main approaches to security analysis. One is based on formal model and another is baseed on computational model. In formal model[1][2][3][4],

- messages are considered as formal expressions;
- encryption operation is only an abstract function;
- security is modeled by formal formulas;
- and analysis of security is done by formal reasoning.

In computational model[5][6][7],

- messages are considered as bit-strings;
- the encryption operation of message is a concrete arithmetic;
- security is defined by computational bounded adversary successfully attacking in negligible probability;
- and analysis of security is done by reduction.

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Each of the methods has its advantage and disadvantage. Generally, the former is simple but cannot guarantee the computational soundness. The later does exactly the opposite. From 1980's, these two methods developed according to their own directions independently. Till the beginning of this century, in their seminal work[8], Abadi and Rogaway give a method to bridge the gap between these two approaches, and build the computational soundness of formal security analysis. Intuitively, in security analysis, the computational soundness means that if two formal expressions are equivalence in formal model, then their computational interpretations are undistinguished in computational model. During the last ten years, computational soundness has gained a lot of attention[8][9][10][11][12][13][14][15] [16][17]and works in this area are still in full swing.

Our analysis aims to ensure the computational soundness about formal encryption in presence of secret shares and key cycles.

Secret Shares. In a secret sharing scheme, a key may be separated into several secret shares, and only those who can get the specific shares can get this key, otherwise, nothing can be got about this key. The secret sharing scheme is proposed in [18], and since then, it is used extensively in cryptography. Moreover, it can be used in other security applications. In [19], Miklau and Suciu implement access control policies in data publishing by using the encryption scheme and secret sharing scheme. Using of secret sharing scheme makes the deploying of access control policies more flexibly. What we care about is that whether a formal treatment of secret sharing can keep its computational soundness.

Key cycles. Key cycles is firstly referred in [5], and then be noted since the work [8]. Non-strictly speaking, key cycles means that a key encrypts itself directly or indirectly. At the first glance, it seems that such a problem doesn't deserve so much attention due to the few occurrences of key cycles in a well-defined protocol. However, this is not always the case. For example, a backup system may store the key on disk and then encrypt the entire disk with this key. Another example comes from the situation where the key cycles is needed 'by design'[20] in a system for non-transferable anonymous credentials. Moreover, key cycles take a significant part in resolving the problem of computational soundness. Generally, in formal model, key cycles is allowed according to the definition of the expression[8] if there is no further restriction. While in computational model, the occurrence of key cycle is eliminated according to the standard notion of security for encryption [5]. This is the reason why the key cycles gain so much attention when the computational soundness is referred.

Related Work. In [8], Abadi and Rogaway give the definition of key cycles and then prove the computational soundness of security under formal setting in absence of key cycles. A natural problem is whether a formal encryption with key cycles is computational sound. Recent years, this problem is studied in many works[8][21][13][22][17]. In [21], Laud addresses the problem of reconciling symbolic and computational analysis in the presence of key cycles by strengthening the symbolic adversary[21], that is, weakening the symbolic encryption. Specifically, Laud uses the similar approach in [8] except giving adversary the power to break the encryption with key cycles by adding some additional rules. In [13][22], instead of using restricted or revised formal model, Adão et al. deal with the key cycles by strengthening the computational notion. Specifically, Adão et al. adopt another security notion, i.e., Key-Dependent Message(KDM) security [23] in which the messages are chosen depending on the keys of the encryption scheme itself. Intuitively, different from the standard security notions(CPA or CCA), KDM security implies the security of key cycles and thus is closer to its formal counterpart. However, it's not easy to construct a KDM secure encryption scheme. Compare to the previous definitions of security, [13] shows that KDM security is "orthogonal" to the standard security. That is, KDM security neither implies nor is implied by chosen-ciphertext security (CCA-2). More and more works are focusing on constructing the KDM secure scheme[23][24][25], but most of them are given in the random-oracle model[23], or by a relaxed notion of KDM security[24], or under the restricted adversary[25]. [26] shows that it is impossible to prove KDM security against a family of query functions that contains exponentially hard pseudo-random functions if the proof makes only a black-box use of the query function and the adversary attacking the scheme. In this paper, we do not consider the KDM security. Rather, our work is under CPA security.

In all the approaches mentioned above, when modeling the power of adversary to obtain keys, an inductive method is used. Very recently, different from the inductive method, Micciancio [17] give a general approach to deal with the key cycles in which the powder of the adversary to get keys is modeled by co-induction. The generalization of this approach makes it possible to deal a larger class of cryptographic expressions, e.g., the expressions with pseudo-random keys [27]. Alternatively, in this paper, we will extend this approach to cryptographic expressions that use of secret sharing schemes. Abadi and Warinschi [16] have given an approach to bridge the gap between formal and computational views in presence of secret shares, but the key cycles in it is not allowed. In this paper, we will prove the computational soundness of formal encryption in presence of both key cycles and secret shares. Our extension to [17] is just like the extension in [16] to [8].

Organization The rest of the paper is organized as follows. Section 2 presents syntax of the formal message, patterns, and the notion of equivalence between messages. In section 3, the computational model is defined, and computational semantics of formal message is given. Then, in section 4, the main result of this paper, theorem of computational soundness is proved. Finally, we conclude in Section 5 and discuss the further work.

2 Formal model

In this section we provide the basic notions for our work in formal setting. We do this by summarizing the main definitions and results in previous papers[8][21][16][22][17] with some changes. Such changes are necessary because we take both the key shares and key cycles into consideration.

2.1 Messages

In a formal message, anything is modeled by symbols. We use **Data** and **Keys** to denote the symbols sets of data, and keys respectively. Often, $d, d_1, d_2, \cdots$ range over **Data**, and $k, k_1, k_2, \cdots$ range over **Keys**.

Definition 1 (Shares). *Assume a key can be divided into n secret shares, and k^j denotes the jth secret share of key k. Given a key $k \in \mathbf{Keys}$, we define $\mathbf{s}(k)$ as follows*[1]*:*

$$\mathbf{s}(k) = \{k^j \mid j \in [1, n]\}$$

Given a set $\mathbf{K} \subseteq \mathbf{Keys}$, when we write $\mathbf{s}(\mathbf{K})$, we mean that $\mathbf{s}(\mathbf{K}) = \bigcup_{k \in \mathbf{K}} \mathbf{s}(k)$. Furthermore, we can define the set of secret shares as $\mathbf{Shares} = \mathbf{s}(\mathbf{Keys})$.

For example, if $\mathbf{Keys} = \{k_1, k_2, k_3\}$ and $n = 2$, by dividing each key into two secret shares, we have $\mathbf{Shares} = \{k_1^1, k_1^2, k_2^1, k_2^2, k_3^1, k_3^2\}$.

Generally, the numbers of shares for each key, say n, is an integer constant. When a key is divided into n shares[2], we assume that, only all these shares allow to recovery of this key, and one can get nothing about this key with its p shares where $p < n$.

Based on **Data**, **Keys** and **Shares**, we can define the set of messages.

Definition 2 (Message). *The set of messages is denoted by **Msg** and can be defined by Backus Naur form as follows:*

$$\mathbf{Msg} ::= \mathbf{Data} \mid \mathbf{Keys} \mid \mathbf{Shares} \mid (\mathbf{Msg}, \mathbf{Msg}) \mid \{\!|\mathbf{Msg}|\!\}_{\mathbf{Keys}}$$

Informally, (m_1, m_2) represents the concatenation of m_1 and m_2, and $\{\!|m|\!\}_k$ represents the encryption of m under k.

Obviously, in a message, some parts may occur in forms of cleartext, and other parts may occur in forms of ciphertext. Without the decryption key, the parts in forms of ciphertext show nothing but its structure at most. To reflect this fact, we need to extend the set of messages **Msg** to the set of extended messages **MSG** by introducing some specific symbols.

Definition 3 (Extended message). *The set of extended messages, written as **MSG**, is defined under $\mathbf{Data} \cup \{\square\}, \mathbf{Keys} \cup \{\Diamond\}$, and $\mathbf{Shares} \cup \{\Diamond^j\}$ with the similar syntax of **Msg**:*

$$\begin{aligned}\mathbf{MSG} ::= &\mathbf{Data} \cup \{\square\} \mid \mathbf{Keys} \cup \{\Diamond\} \mid \mathbf{Shares} \cup \{\Diamond^j\} \\ &\mid (\mathbf{MSG}, \mathbf{MSG}) \mid \{\!|\mathbf{MSG}|\!\}_{\mathbf{Keys} \cup \{\Diamond\}}\end{aligned}$$

Intuitively, $\square, \Diamond$ and $\Diamond^j$ denote the unknown data, keys and secret shares respectively.

From definitions of **Msg** and **MSG**, we can see that **Msg** is in fact included in **MSG**, and thus, in most time, when we refer to *message*, we means a member of **MSG**, and use $m, m', m'', \cdots, m_1, m_2, \cdots$ to range over **MSG**.

Similar to [17], we accept the following notational conventions:

[1] By using $j \in [1, n]$, we mean $1 \leq j \leq n$.

[2] It is assumed that each key is shared only once.

- $(m_1, m_2, \cdots, m_n) \triangleq (m_1, (m_2, \cdots, m_n))$;
- $\{\!|(m_1, m_2)|\!\}_k \triangleq \{\!|m_1, m_2|\!\}_k$;
- $\{\!|m|\!\}_\Diamond \triangleq \{\!|m|\!\}$.

Moreover, to simplify our presentation, we will use the symbols of the first order logic in the following definition. For example, we use $\wedge$ for *and*, $\vee$ for *or*, $\neg$ for *negation*, $\exists$ for *exists*, and $\forall$ for *for all*.

Definition 4 (Sub-message). *Let $m, m' \in$ **MSG**. We say message m' is a sub-message of m, written as $m' \preccurlyeq m$, if one of the following holds:*

1. $m' = m$;
2. $m = (m_1, m_2) \wedge (m' \preccurlyeq m_1 \vee m' \preccurlyeq m_2)$;
3. $m = \{\!|m''|\!\}_k \wedge m' \preccurlyeq m''$.

Definition 5 (Occurrence). *Let $x \in$ **Keys**$\cup$**Shares** and $m \in$ **MSG**. x occurs in m, written as $x \lessdot m$, if one of the following holds:*

1. $x = m$;
2. $m = (m_1, m_2) \wedge (x \lessdot m_1 \vee x \lessdot m_2)$;
3. $m = \{\!|m'|\!\}_k \wedge (x = k \vee x \lessdot m')$.

With Definition 5, we can define a function **keys** : **MSG** $\to$ **Keys**, which returns the set of the keys occurring in a message or whose shares occurring in this message. More formally, given $m \in$ **MSG**, we have

$$\mathbf{keys}(m) = \{k | (k \in \mathbf{Keys}) \wedge ((k \lessdot m) \vee \exists j \in [1, n].(k^j \lessdot m))\}.$$

Definition 6 (Encryption relation). *Let $m \in$ **MSG**$, k_1, k_2 \in$ **keys**(m), we say k_1 encrypts k_2 in m, written as $k_1 \sqsubset_m k_2$, if there exists a message m' such that*

$$\{\!|m'|\!\}_{k_1} \preccurlyeq m \wedge k_2 \in \mathbf{keys}(m').$$

Example 1. Let $m = \{\!|k_1, k_2^1|\!\}_{k_3}$, we have

- $\{\!|k_1, k_2^1|\!\}_{k_3} \preccurlyeq m, (k_1, k_2^1) \preccurlyeq m, k_1 \preccurlyeq m, k_2^1 \preccurlyeq m$;
- $k_1 \lessdot m, k_2^1 \lessdot m, k_3 \lessdot m$;
- $k_3 \sqsubset_m k_1, k_3 \sqsubset_m k_2$.

Definition 7 (Key cycle[3]). *Let $m \in$ **MSG**, we can construct a directed graph $G = (N, E)$ in which $N = \{k | k \in \mathbf{keys}(m)\}$ is the set of the nodes, and $E = \{k_1 k_2 | k_1 \in N \wedge k_2 \in N \wedge k_1 \sqsubset_m k_2\}$ is the set of the edges. We say there exists a key cycle in the message m, if and only if there exists a cycle in the graph G.*

From the definitions above, we can see that the secret shares are considered in messages. Moreover, the rest of our work does not eliminate the key cycles from the messages. Both of which make our work different from previous ones.

[3] There are many different definitions of key cycles in the literatures, in which [8] is the most general one. The definition here is similar to the definition in [8] except that secret share is considered. Such a general definition is used to emphasize that any form of key cycle is allowed.

2.2 Patterns

Since a message may contain some sub-messages in form of ciphertext, a message will show different views given different keys. When given no further information other than the the message itself, the view of the message can be uniquely determined. Informally speaking, this view is just the pattern of the message.

Owing to the presence of the secret shares, the keys related to the message become more complicated. So, before formally defining the pattern, we need to give several functions.

– $\mathbf{sbk}(m) : \mathbf{MSG} \to \mathbf{Keys}$, return the keys which are the sub-messages of m, or whose shares are the sub-messages of m:

$$\mathbf{sbk}(m) = \{k | (k \in \mathbf{Keys}) \wedge ((k \preccurlyeq m) \vee \exists j \in [1, n].(k^j \preccurlyeq m))\}$$

– $\mathbf{rck}(m) : \mathbf{MSG} \to \mathbf{Keys}$, return the keys which can possibly be recovered from m. Specifically, it return the keys which are the sub-messages of m, or all of whose shares are sub-messages of m:

$$\mathbf{rck}(m) = \{k | (k \in \mathbf{Keys} \wedge k \preccurlyeq m) \vee \forall j \in [1, n].(k^j \preccurlyeq m)\}$$

– $\mathbf{psk}(m) : \mathbf{MSG} \to \mathbf{Keys}$, return the set of keys which do not occur directly as the sub-message of m, but whose secret shares partially occur in m. It can be simply defined by $\mathbf{sbk}(m)$ and $\mathbf{rck}(m)$:

$$\mathbf{psk}(m) = \mathbf{sbk}(m) \setminus \mathbf{rck}(m)$$

– $\mathbf{eok}(m) : \mathbf{MSG} \to \mathbf{Keys}$, return the set of keys which only occur in m as the encryption keys:

$$\mathbf{eok}(m) = \mathbf{keys}(m) \setminus \mathbf{sbk}(m)$$

By the definition above, we have the more intuitive properties as follows:

$$\mathbf{sbk}(m) \cup \mathbf{eok}(m) = \mathbf{keys}(m); \tag{1}$$
$$\mathbf{sbk}(m) \cap \mathbf{eok}(m) = \emptyset; \tag{2}$$
$$\mathbf{rck}(m) \cup \mathbf{psk}(m) = \mathbf{sbk}(m); \tag{3}$$
$$\mathbf{rck}(m) \cap \mathbf{psk}(m) = \emptyset. \tag{4}$$

Example 2. This example is given to illustrate various functions about keys[4]. Let $m = (\{\!|k_1, k_2^1|\!\}_{k_1}, \{\!|k_3, \{\!|\{\!|k_4|\!\}_{k_5}|\!\}_{k_4}|\!\}_{k_2}, \{\!|k_2^2|\!\}_{k_3}, \{\!|k_4^2|\!\}_{k_6}, k_5^1, k_6^1, \{\!|k_4^1|\!\}_{k_7})$, we have

$$\mathbf{keys}(m) = \{k_1, k_2, k_3, k_4, k_5, k_6, k_7\} \qquad \mathbf{sbk}(m) = \{k_1, k_2, k_3, k_4, k_5, k_6\}$$
$$\mathbf{rck}(m) = \{k_1, k_2, k_3, k_4\} \qquad \mathbf{psk}(m) = \{k_5, k_6\}$$
$$\mathbf{eok}(m) = \{k_7\}$$

struct : $\mathbf{MSG} \rightarrow \mathbf{MSG}$	$\mathbf{struct}(d) = \Box;$ $\mathbf{struct}(k) = \Diamond;$ $\mathbf{struct}(k^j) = \Diamond^j;$ $\mathbf{struct}((m_1, m_2)) = (\mathbf{struct}(m_1), \mathbf{struct}(m_2));$ $\mathbf{struct}(\{\|m\|\}_k) = \{\|\mathbf{struct}(m)\|\}.$
p : $\mathbf{MSG} \times \mathbf{Keys} \rightarrow \mathbf{MSG}$	$\mathbf{p}(d, \mathbf{K}) = d;$ $\mathbf{p}(k, \mathbf{K}) = k;$ $\mathbf{p}(k^j, \mathbf{K}) = k^j$, (for $j \in \{1..n\}$); $\mathbf{p}((m_1, m_2), \mathbf{K}) = (\mathbf{p}(m_1, \mathbf{K}), \mathbf{p}(m_2, \mathbf{K}));$ $\mathbf{p}(\{\|m\|\}_k, \mathbf{K}) = \begin{cases} \{\|\mathbf{p}(m)\|\}_k & (\text{ if } k \in \mathbf{K}); \\ \{\|\mathbf{struct}(m)\|\}_k & (\text{otherwise.}). \end{cases}$

Fig. 1. Rules defining the function **p**, and auxiliary function **struct**

To define the patterns of the messages, we need the functions of **p** and an auxiliary function **struct**, which are defined in Fig. 1.

The function **p** and **rck** satisfy the following fundamental properties:

$$\mathbf{p}(m, \mathbf{keys}(m)) = m \tag{5}$$

$$\mathbf{p}(\mathbf{p}(m, \mathbf{K}), \mathbf{K}') = \mathbf{p}(m, \mathbf{K} \cap \mathbf{K}') \tag{6}$$

$$\mathbf{rck}(\mathbf{p}(m, \mathbf{K})) \subseteq \mathbf{rck}(m) \tag{7}$$

These three properties are similar to the properties of **p** and **r** in [17]. Moreover, about **p**, we have the following proposition:

Proposition 1. *If* $\mathbf{K}' \cap \mathbf{keys}(m) = \emptyset$*, then* $\mathbf{p}(m, \mathbf{K} \cup \mathbf{K}') = \mathbf{p}(m, \mathbf{K})$.

Proof. Given $k \in \mathbf{keys}(m)$, since $\mathbf{K}' \cap \mathbf{keys}(m) = \emptyset$, we have $k \notin \mathbf{K}'$. So, if $k \in \mathbf{K} \cup \mathbf{K}'$, then $k \in \mathbf{K}$. On the other hand, if $k \notin \mathbf{K} \cup \mathbf{K}'$, then $k \notin \mathbf{K}$. From the definition of **p**, what we can get from m by the help of $\mathbf{K}$ is just what we can get from m by the help of $\mathbf{K} \cup \mathbf{K}'$.

Intuitively, this proposition means that, given a message m and a key set $\mathbf{K}$, additional key which is unrelated to m can not provide additional information about m.

Definition 8 (Key getting function). *Given a message m, we define a function $\mathcal{F}_m : \wp(\mathbf{Keys}) \rightarrow \wp(\mathbf{Keys})$. Precisely, given a set $\mathbf{K} \subseteq \mathbf{Keys}$, we have*

$$\mathcal{F}_m(\mathbf{K}) = \mathbf{rck}(\mathbf{p}(m, \mathbf{K})) \tag{8}$$

Proposition 2. *The function $\mathcal{F}_m : \wp(\mathbf{Keys}) \rightarrow \wp(\mathbf{Keys})$ is monotone.*

[4] To keep continuity, the message used in this example will also be used in the followed examples.

Proof. Assume $K_1 \in \wp(\mathbf{Keys}), \mathbf{K}_2 \in \wp(\mathbf{Keys})$, and $\mathbf{K}_1 \subseteq \mathbf{K}_2$, we will show that $\mathcal{F}_m(\mathbf{K}_1) \subseteq \mathcal{F}_m(\mathbf{K}_2)$.

By equation (8), we have $\mathcal{F}_m(\mathbf{K}_1) = \mathbf{rck}(\mathbf{p}(m, \mathbf{K}_1))$, and $\mathcal{F}_m(\mathbf{K}_2) = \mathbf{rck}(\mathbf{p}(m, \mathbf{K}_2))$. So, to show $\mathcal{F}_m(\mathbf{K}_1) \subseteq \mathcal{F}_m(\mathbf{K}_2)$, we only need to prove that $\mathbf{rck}(\mathbf{p}(m, \mathbf{K}_1)) \subseteq \mathbf{rck}(\mathbf{p}(m, \mathbf{K}_2))$:

$$\begin{aligned}
&\mathbf{rck}(\mathbf{p}(m, \mathbf{K}_1)) && \\
&= \mathbf{rck}(\mathbf{p}(m, \mathbf{K}_2 \cap \mathbf{K}_1)) && \text{by assumption } \mathbf{K}_1 \subseteq \mathbf{K}_2 \\
&= \mathbf{rck}(\mathbf{p}(\mathbf{p}(m, \mathbf{K}_2), \mathbf{K}_1)) && \text{by (6)} \\
&\subseteq \mathbf{rck}(\mathbf{p}(m, \mathbf{K}_2)) && \text{by (7)}
\end{aligned}$$

Intuitively, given message m and a key set $\mathbf{K}$, $\mathcal{F}_m(\mathbf{K})$ computes the set of keys which occur as the sub-message of $\mathbf{p}(m, \mathbf{K})$, or whose secret shares fully occur in $\mathbf{p}(m, \mathbf{K})$.

The monotonicity of the function $\mathcal{F}_m$ makes it possible to define the greatest fix-point of $\mathcal{F}_m$.

Definition 9 (The greatest fix point of $\mathcal{F}_m$). *The greatest fix-point of $\mathcal{F}_m$, written $\mathbf{FIX}(\mathcal{F}_m)$, is defined as follows:*

$$\mathbf{FIX}(\mathcal{F}_m) = \bigcap_{i=0}^{\ell} \mathcal{F}_m^i(\mathbf{keys}(m)) \tag{9}$$

where $\ell = |\mathbf{keys}(m)|$.

Obviously, by the definition of greatest fix-point and the monotonicity of $\mathcal{F}_m$, we have

$$\mathbf{FIX}(\mathcal{F}_m) = \mathcal{F}_m^{\ell}(\mathbf{keys}(m)) \tag{10}$$

Definition 10 (Pattern of the message). *The pattern of the message m, written as* $\mathbf{pattern}(m)$, *is define as*

$$\mathbf{pattern}(m) = \mathbf{p}(m, \mathbf{FIX}(\mathcal{F}_m)) \tag{11}$$

Example 3. Let m be the the same in Example 2, and the full share numbles of a key, say n, is assumed to be 2:

$$m = (\{\!|k_1, k_2^1|\!\}_{k_1}, \{\!|k_3, \{\!|\{\!|k_4|\!\}_{k_5}|\!\}_{k_4}|\!\}_{k_2}, \{\!|k_2^2|\!\}_{k_3}, \{\!|k_4^2|\!\}_{k_6}, k_5^1, k_6^1, \{\!|k_4^1|\!\}_{k_7})$$

Starting from the set $\mathbf{K}_0 = \mathbf{keys}(m)$, the greatest fix point of $\mathcal{F}_m$ can be computed recursively as follows:

$$\begin{aligned}
\mathbf{K}_0 &= \{k_1, k_2, k_3, k_4, k_5, k_6, k_7\} \\
\mathbf{p}(m, \mathbf{K}_0) &= (\{\!|k_1, k_2^1|\!\}_{k_1}, \{\!|k_3, \{\!|\{\!|k_4|\!\}_{k_5}|\!\}_{k_4}|\!\}_{k_2}, \{\!|k_2^2|\!\}_{k_3}, \{\!|k_4^2|\!\}_{k_6}, k_5^1, k_6^1, \{\!|k_4^1|\!\}_{k_7}) \\
\mathbf{K}_1 &= \mathcal{F}_m(\mathbf{K}_0) = \mathbf{rck}(\mathbf{p}(m, \mathbf{K}_0)) = \{k_1, k_2, k_3, k_4\} \\
\mathbf{p}(m, \mathbf{K}_1) &= (\{\!|k_1, k_2^1|\!\}_{k_1}, \{\!|k_3, \{\!|\{\!|\Diamond|\!\}_{k_5}|\!\}_{k_4}|\!\}_{k_2}, \{\!|k_2^2|\!\}_{k_3}, \{\!|\Diamond^2|\!\}_{k_6}, k_5^1, k_6^1, \{\!|\Diamond^1|\!\}_{k_7})
\end{aligned}$$

$$\mathbf{K}_2 = \mathcal{F}_m(\mathbf{K}_1) = \mathbf{rck}(\mathbf{p}(m, \mathbf{K}_1)) = \{k_1, k_2, k_3\}$$
$$\mathbf{p}(m, \mathbf{K}_2) = (\{\!|k_1, k_2^1|\!\}_{k_1}, \{\!|k_3, \{\!|\{\!|\Diamond|\!\}|\!\}_{k_4}|\!\}_{k_2}, \{\!|k_2^2|\!\}_{k_3}, \{\!|\Diamond^2|\!\}_{k_6}, k_5^1, k_6^1, \{\!|\Diamond^1|\!\}_{k_7})$$
$$\mathbf{K}_3 = \mathcal{F}_m(\mathbf{K}_2) = \mathbf{rck}(\mathbf{p}(m, \mathbf{K}_2)) = \{k_1, k_2, k_3\}$$

Then, we have

$$\mathbf{FIX}(\mathcal{F}_m) = \{k_1, k_2, k_3\}$$
$$\mathbf{pattern}(m) = \mathbf{p}(m, \mathbf{FIX}(\mathcal{F}_m))$$
$$= (\{\!|k_1, k_2^1|\!\}_{k_1}, \{\!|k_3, \{\!|\{\!|\Diamond|\!\}|\!\}_{k_4}|\!\}_{k_2}, \{\!|k_2^2|\!\}_{k_3}, \{\!|\Diamond^2|\!\}_{k_6}, k_5^1, k_6^1, \{\!|\Diamond^1|\!\}_{k_7})$$

2.3 Equivalence

As usual, the keys in a formal message is considered as bound names(like in spi calculus[4]), so, they can be renamed without effecting the essential meaning of the formal message. However, since the secret shares of the keys is considered in the formal model, we must redefine the renaming to messages.

Definition 11 (Renaming). *There are three types of renaming: K-renaming(Keys renaming), KS-renaming(Keys and shares renaming) and S-renaming(Shares only renaming). KS-renaming and S-renaming are all defined based on K-renaming.*

1. *Let $\mathbf{K} \subseteq \mathbf{Keys}$. A K-renaming on $\mathbf{K}$ is a bijection on $\mathbf{K}$, often written as $\sigma[\mathbf{K}]$ or $\theta[\mathbf{K}]$.*

2. *KS-renaming is defined by extending the K-renaming. Let $\mathbf{K}, \mathbf{K}' \subseteq \mathbf{Keys}$, $\mathbf{K} \subseteq \mathbf{K}'$, and $\sigma[\mathbf{K}']$ be a K-renaming. A KS-renaming on $\mathbf{K} \cup \mathbf{s}(\mathbf{K})$, written as $\bar{\sigma}[\mathbf{K} \cup \mathbf{s}(\mathbf{K})]$ is defined as follows:*
$$\bar{\sigma}(k) = \sigma(k) \qquad (k \in \mathbf{K})$$
$$\bar{\sigma}(k^j) = \sigma(k)^j \qquad (k^j \in \mathbf{s}(\mathbf{K}))$$

3. *S-renaming is also defined based on the K-renaming. Let $\mathbf{K}, \mathbf{K}' \subseteq \mathbf{Keys}$, $\mathbf{K} \subseteq \mathbf{K}'$ $\sigma[\mathbf{K}']$ be K-renaming. An S-renaming on $\mathbf{s}(\mathbf{K})$, written as $\hat{\sigma}[\mathbf{s}(\mathbf{K})]$ is defined as follows:*
$$\hat{\sigma}(k^j) = \sigma(k)^j \qquad (k \in \mathbf{K})$$

As a conventional notation, we have
$$\sigma(\mathbf{K}) \triangleq \{k' | k \in \mathbf{K} \wedge \sigma(k) = k'\}.$$

Similar notations can be used on $\bar{\sigma}$ and $\hat{\sigma}$. When there's no confusion according to the context, we often write $\sigma[\mathbf{K}], \bar{\sigma}[\mathbf{K} \cup \mathbf{s}(\mathbf{K})]$ and $\hat{\sigma}[\mathbf{s}(\mathbf{K})]$ as σ, $\bar{\sigma}$ and $\hat{\sigma}$ respectively for short.

Let $m \in \mathbf{MSG}$, $\bar{\sigma}[\mathbf{K} \cup \mathbf{s}(\mathbf{K})]$ be a KS-renaming. We use $m\bar{\sigma}$ as applying $\bar{\sigma}$ to message m. That is, rename all the key $k_i \in \mathbf{K}$ and its secret shares k_i^j occurring in m with $\bar{\sigma}(k_i)$ and $\bar{\sigma}(k_i^j)$ respectively.

Similarly, let $m \in \mathbf{MSG}$, $\hat{\sigma}[\mathbf{s}(\mathbf{K})]$ be an S-renaming on $\mathbf{s}(\mathbf{K})$. We use $m\hat{\sigma}$ as applying $\hat{\sigma}$ to message m. That is, rename all secret shares $k^j \in \mathbf{s}(\mathbf{K})$ with $\hat{\sigma}(k^j)$ without renaming of k itself.

Note 1. In this paper, when applying an S-renaming $\hat{\sigma}$ based on K-renaming $\sigma[\mathbf{K}]$ to message m, we always assume that,

$$\sigma(\mathbf{K}) \cap \mathbf{keys}(m) = \emptyset. \tag{12}$$

Intuitively, (12) is used to assure that a secret share in m is renamed to a fresh symbol. For example, in $m\bar{\sigma}$, $\bar{\sigma}(k^j)$ is a share of $\bar{\sigma}(k)$ if k^j is a share of k in m, while in $m\hat{\sigma}$ where σ meet (12), such relation is broken.

Now, it suffices to define the equivalence of the messages.

Definition 12 (Equivalence of the message). *Given $m, m' \in \mathbf{MSG}$, Message m' is said to equivalent to m, written as $m' \cong m$, if and only if, there exists a KS-renaming $\bar{\sigma}$ based on K-renaming $\sigma[\mathbf{keys}(m)]$, or, additionally an S-renaming $\hat{\theta}$ based on K-renaming $\theta[\mathbf{psk}(m\bar{\sigma})]$, such that one of the following holds:*

1. $\mathbf{pattern}(m') = \mathbf{pattern}(m)\bar{\sigma}$
2. $\mathbf{pattern}(m') = (\mathbf{pattern}(m)\bar{\sigma})\hat{\theta}$

This definition of equivalence differs from the equivalence in [16] in that the S-renaming is considered. So, for example, $(\{|k_2|\}_{k_1}, k_1^1)$ and $(\{|k_2|\}_{k_1}, k_3^1)$ are equivalent according to Definition 12, but not equivalent in [16].

Example 4. In this example, we will illustrate the three types of renaming and its applying in message equivalence.

Recall Example 3, we have

$$m = \left(\{\!|k_1, k_2^1|\!\}_{k_1}, \{\!|k_3, \{\!|\{\!|k_4|\!\}_{k_5}|\!\}_{k_4}|\!\}_{k_2}, \{\!|k_2^2|\!\}_{k_3}, \{\!|k_4^2|\!\}_{k_6}, k_5^1, k_6^1, \{\!|k_4^1|\!\}_{k_7}\right)$$

Let $\mathbf{K} = \mathbf{keys}(\mathbf{pattern}(m))$, $\mathbf{K}' = \mathbf{psk}(\mathbf{pattern}(m)\bar{\sigma})$. We then define a KS-renaming $\bar{\sigma}$ based on a K-renaming $\sigma[\mathbf{K}]$, and an S-renaming $\hat{\theta}$ based on a K-renaming $\theta[\mathbf{K}']$, which are showed in Fig. 2.

From Definition 12 and Fig. 2, if one of the following two condition holds,

$$\mathbf{pattern}(m') = (\{\!|k_7, k_6^1|\!\}_{k_7}, \{\!|k_5, \{\!|\{\!|\Diamond|\!\}|\!\}_{k_4}|\!\}_{k_6}, \{\!|k_6^2|\!\}_{k_5}, \{\!|\Diamond^2|\!\}_{k_2}, k_3^1, k_2^1, \{\!|\Diamond^1|\!\}_{k_1})$$
$$\mathbf{pattern}(m') = (\{\!|k_7, k_6^1|\!\}_{k_7}, \{\!|k_5, \{\!|\{\!|\Diamond|\!\}|\!\}_{k_4}|\!\}_{k_6}, \{\!|k_6^2|\!\}_{k_5}, \{\!|\Diamond^2|\!\}_{k_2}, k_{3'}^1, k_{2'}^1, \{\!|\Diamond^1|\!\}_{k_1})$$

we have $m' \cong m$.

3 Computational model

In computational model, the message is just a bit-string which belongs to $\{0,1\}^*$.

Definition 13 (Indistinguishability). *Let $D = \{D_\eta\}_{\eta \in \mathbb{N}}$ be an ensemble, i.e., a collection of distributions over strings. We say two ensembles D and D' are indistinguishable, written as $D \approx D'$, if for every probabilistic polynomial-time adversary $\mathcal{A}$, there exists a negligible function* **negl**, *such that*

$$\mathbf{Pr}[x \leftarrow D_\eta : \mathcal{A}(1^\eta, x) = 1] - \mathbf{Pr}[x \leftarrow D'_\eta : \mathcal{A}(1^\eta, x) = 1] = \mathbf{negl}(\eta)$$

where $x \leftarrow D_\eta$ means that x is sampled from the distribution D_η.

$\mathbf{pattern}(m)$	$(\{\!\vert k_1, k_2^1\vert\!\}_{k_1}, \{\!\vert k_3, \{\!\vert\{\!\vert\Diamond\vert\!\}\vert\!\}_{k_4}\vert\!\}_{k_2}, \{\!\vert k_2^2\vert\!\}_{k_3}, \{\!\vert\Diamond^2\vert\!\}_{k_6}, k_5^1, k_6^1, \{\!\vert\Diamond^1\vert\!\}_{k_7})$
$\begin{array}{c}\mathbf{K}\\ \downarrow\\ \sigma(\mathbf{K})\end{array}$	$\begin{array}{ccccccc}k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7\\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow\\ k_7 & k_6 & k_5 & k_4 & k_3 & k_2 & k_1\end{array}$
$\begin{array}{c}\mathbf{K}\cup\mathbf{s}(\mathbf{K})\\ \downarrow\\ \bar{\sigma}(\mathbf{K}\cup\mathbf{s}(\mathbf{K}))\end{array}$	$\begin{array}{cccccccccccccccc}k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_1^1 & k_1^2 & \cdots & k_5^1 & k_5^2 & k_6^1 & k_6^2 & k_7^1 & k_7^2\\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow\\ k_7 & k_6 & k_5 & k_4 & k_3 & k_2 & k_1 & k_7^1 & k_7^2 & \cdots & k_3^1 & k_3^2 & k_2^1 & k_2^2 & k_1^1 & k_1^2\end{array}$
$\mathbf{pattern}(m)\bar{\sigma}$	$(\{\!\vert k_7, k_6^1\vert\!\}_{k_7}, \{\!\vert k_5, \{\!\vert\{\!\vert\Diamond\vert\!\}\vert\!\}_{k_4}\vert\!\}_{k_6}, \{\!\vert k_6^2\vert\!\}_{k_5}, \{\!\vert\Diamond^2\vert\!\}_{k_2}, k_3^1, k_2^1, \{\!\vert\Diamond^1\vert\!\}_{k_1})$
$\begin{array}{c}\mathbf{K}'\\ \downarrow\\ \theta(\mathbf{K}')\end{array}$	$\begin{array}{cc}k_3 & k_2\\ \downarrow & \downarrow\\ k_{3'} & k_{2'}\end{array}$
$\begin{array}{c}\mathbf{s}(\mathbf{K}')\\ \downarrow\\ \hat{\theta}(\mathbf{s}(\mathbf{K}'))\end{array}$	$\begin{array}{cccc}k_3^1 & k_3^2 & k_2^1 & k_2^2\\ \downarrow & \downarrow & \downarrow & \downarrow\\ k_{3'}^1 & k_{3'}^2 & k_{2'}^1 & k_{2'}^2\end{array}$
$(\mathbf{pattern}(m)\bar{\sigma})\hat{\theta}$	$(\{\!\vert k_7, k_6^1\vert\!\}_{k_7}, \{\!\vert k_5, \{\!\vert\{\!\vert\Diamond\vert\!\}\vert\!\}_{k_4}\vert\!\}_{k_6}, \{\!\vert k_6^2\vert\!\}_{k_5}, \{\!\vert\Diamond^2\vert\!\}_{k_2}, k_{3'}^1, k_{2'}^1, \{\!\vert\Diamond^1\vert\!\}_{k_1})$

Fig. 2. An example for KS-renaming and an S-renaming

A typical property of indistinguishability is that it is transitive [21][5], i.e.,

$$\text{if } D \approx D' \text{ and } D' \approx D'', \text{ then } D \approx D'' \tag{13}$$

Definition 14 (Private-key encryption scheme). *A private-key encryption scheme is a tuple of algorithms* $\mathbf{\Pi} = (\mathbf{Gen}, \mathbf{Enc}, \mathbf{Dec})$ *such that:*

1. *The key-generation algorithm* **Gen** *takes as input the security parameter* 1^η *and outputs a key* k*. This process can be written as* $k \leftarrow \mathbf{Gen}(1^\eta)$*.*
2. *The encryption algorithm* **Enc** *takes as input a key* k *and a message* $m \in \{0,1\}^*$*, and output a ciphertext* c*. This process can be written as* $c \leftarrow \mathbf{Enc}_k(m)$*.*
3. *The decryption algorithm* **Dec** *takes as input a key* k *and a ciphertext* c*, and outputs a message* m*. This process is often written as* $m := \mathbf{Dec}_k(c)$*.*

It is required that $\mathbf{Dec}_k(\mathbf{Enc}_k(m)) = m$*.*

We will use a standard notion of security for encryption: indistinguishability against chosen plaintext attacks(CPA).

Definition 15 (CPA security). *For any probabilistic polynomial time adversaries* $\mathcal{A}$ *and polynomial* **poly***, let* $\mathbf{\Pi} = (\mathbf{Gen}, \mathbf{Enc}, \mathbf{Dec})$ *is an encryption scheme,* $n = \mathbf{poly}(\eta)$*,* $k_1, \cdots, k_n$ *are the keys generated by* **Gen***,* b *is a random bit chosen uniformly from* $\{0,1\}$*,* $O_b(i,m)$*is an encryption oracle that outputs*

[5] Strictly speaking, such transitivity can only be applied polynomial times about security parameter η. We adopt this property only to simplify the proof of some theorems.

$\mathbf{Enc}_{k_i}(m)$ *if* $b = 1$*, or* $\mathbf{Enc}_{k_i}(0^{|m|})$ *if* $b = 0$*. The encryption scheme* $\mathbf{\Pi}$ *is indistinguishable under chosen plaintext attack(or is* **CPA***-secure) if there exists a negligible function* **negl** *such that*

$$\mathbf{Pr}[\mathcal{A}^{O_1}(1^\eta) = 1] - \mathbf{Pr}[\mathcal{A}^{O_0}(1^\eta) = 1] = \mathbf{negl}(\eta)$$

This definition is equivalent to the definition of **IND-CPA** in which only one encryption oracle is given[17].

Definition 16 (Secret sharing scheme). *An* n*-out-of-*n *secret sharing scheme for sharing keys of a encryption scheme* $\mathbf{\Pi}$ *is a tuple of algorithms* $\mathbf{\Lambda} = (\mathbf{Crt}, \mathbf{Com})$ *such that:*

1. *The share creation algorithm* **Crt** *takes as input a key* k *and the security parameter* 1^η *and outputs* n *shares of* $k : k^1, k^2, \cdots, k^n$*. This process can be written as* $k^1, k^2, \cdots, k^n \leftarrow \mathbf{Crt}(k, 1^\eta)$*.*
2. *The share combination algorithm* **Com** *takes as input* n *shares* $k^1, k^2, \cdots, k^n$ *and output a key* k*. This process can be written as* $k := \mathbf{Com}(k^1, k^2, \cdots, k^n)$*.*

It is required that $\mathbf{Com}(\mathbf{Crt}(k, 1^\eta)) = k$*.*

Definition 17 (Security of secret sharing). *For any probabilistic polynomial time adversaries* $\mathcal{A}$ *and polynomial* **poly***, let* $\mathbf{\Pi} = (\mathbf{Gen}, \mathbf{Enc}, \mathbf{Dec})$ *be an encryption scheme,* $\mathbf{\Lambda} = (\mathbf{Crt}, \mathbf{Com})$ *be an secret sharing scheme,* $n = \mathbf{poly}(\eta)$*,* $\mathbf{sh}(k)$ *be the set of* n *secret shares of key* k *generated by* **Crt***, and* $\mathbf{sh}(k)|_S$ *be the restriction of* $\mathbf{sh}(k)$ *to the secret shares whose indexes are in* $S \subseteq \{1, 2, \cdots, n\}$*. The secret sharing scheme* $\mathbf{\Lambda}$ *is secure if for any* $S \subset \{1, 2, \cdots, n\}$*, there exists a negligible function* **negl** *such that*

$$\begin{aligned} &\mathbf{Pr}\left[k_0, k_1 \leftarrow \mathbf{Gen}(1^\eta), \mathbf{sh}(k_0) \leftarrow \mathbf{Crt}(k_0, 1^\eta) : \mathcal{A}(k_0, k_1, \mathbf{sh}(k_0)|_S) = 1\right] - \\ &\mathbf{Pr}\left[k_0, k_1 \leftarrow \mathbf{Gen}(1^\eta), \mathbf{sh}(k_1) \leftarrow \mathbf{Crt}(k_1, 1^\eta) : \mathcal{A}(k_0, k_1, \mathbf{sh}(k_1)|_S) = 1\right] \\ = &\,\mathbf{negl}(\eta) \end{aligned}$$

Definition 18 (Computational model). *A computational model is a 4-tuple* $\mathbf{M} = (\mathbf{\Pi}, \mathbf{\Lambda}, \omega, \gamma)$*, in which*

- $\mathbf{\Pi}$ *is an encryption scheme.*
- $\mathbf{\Lambda}$ *is a secret sharing scheme.*
- $\omega : \mathbf{Data} \to \{0, 1\}^*$ *is an interpretation function to evaluate each symbol in* **Data** *to a bit-string.*
- $\gamma : \{0, 1\}^* \times \{0, 1\}^* \to \{0, 1\}^*$ *is a function to connect two bit-strings to a single bit-string. It can be viewed as the computational counterpart of message concatenation in formal model.*

Definition 19 (Computational interpretation of messages). *Given a computational model* $\mathbf{M} = (\mathbf{\Pi}, \mathbf{\Lambda}, \omega, \gamma)$ *and a formal message* m*, we can get the computational interpretation of* m*, that is, associate a collection of distributions (i.e., ensemble) over a bit-string* $[\![m]\!]_\mathbf{M} = \{[\![m]\!]_{\mathbf{M}(\eta)}\}$ *to the formal message* m*. Assume* $\ell = |\mathbf{keys}(m)|$ *and the number of shares for each key is* n*, we can get* $[\![m]\!]_\mathbf{M}$ *by the following steps:*

1. *Initialization. Construct an ℓ vector κ to save the interpretation of keys, and an $\ell \times n$ array ς to save the interpretation of shares. Then, evaluate $\kappa[i](1 \leq i \leq \ell)$ and $\varsigma[i,j](1 \leq i \leq \ell, 1 \leq j \leq n)$ by following procedure:*

$$\textit{for } i = 1 \textit{ to } \ell \textit{ do}$$
$$\left\{ \begin{array}{l} \kappa[i] \leftarrow \mathbf{Gen}(1^\eta); \\ \textit{for } \; j = 1 \textit{ to } n \textit{ do} \\ \quad \varsigma[i,j] \leftarrow \mathbf{Crt}(k_i, 1^\eta). \end{array} \right\}$$

2. *Interpretation. Interpretation of the message m can be done recursively as follows:*
   - $[\![d]\!]_{\mathbf{M}} = \omega(d)$, *for* $d \in \mathbf{Data}$.
   - $[\![k_i]\!]_{\mathbf{M}} = \kappa[i]$, *for* $k_i \in \mathbf{Keys}$ *and* $1 \leq i \leq \ell$.
   - $[\![k_i^j]\!]_{\mathbf{M}} = \varsigma[i,j]$, *for* $k_i^j \in \mathbf{Shares}$ *and* $1 \leq j \leq n$.
   - $[\![(m_1, m_2)]\!]_{\mathbf{M}} = \gamma\left([\![m_1]\!]_{\mathbf{M}}, [\![m_2]\!]_{\mathbf{M}}\right)$.
   - $[\![\{\!|m|\!\}_{k_i}]\!]_{\mathbf{M}} = \mathbf{Enc}_{[\![k_i]\!]_{\mathbf{M}}}[\![m]\!]_{\mathbf{M}}$.
   - $[\![\mathbf{struct}(m)]\!]_{\mathbf{M}} = 0^{|[\![m]\!]_{\mathbf{M}}|}$, *where* $|[\![m]\!]_{\mathbf{M}}|$ *denotes the length of* $[\![m]\!]_{\mathbf{M}}$.

4 Computational soundness

Intuitively, Computational soundness means that, if two messages are equivalent in the formal model, their interpretation in computational model will be indistinguishable.

Before proving our main result of computational soundness, we need some lemmas. To clarify the proof, we use Fig.3 to list the invoking structure of these lemmas and the propositions in proving the computational soundness theorem, where $a \rightarrow b$ means that a is invoked in proving b.

$$\begin{array}{ccccccccc}
 & & & & \text{Proposition 1} & & & & \text{Proposition 2} \\
 & & & & \downarrow & & & & \downarrow \\
\text{Lemma 1} & & \text{Lemma 2} & \rightarrow & \text{Lemma 3} & \rightarrow & \text{Lemma 4} & \rightarrow & \text{Lemma 5} \\
\downarrow & & \downarrow & & & & & & \downarrow \\
\hline
 & & & & \downarrow & & & & \\
 & & & & \text{Computational soundness theorem} & & & &
\end{array}$$

Fig. 3. The invoking structure in proving the computational soundness theorem

Lemma 1. *Let $m \in \mathbf{MSG}$, $\bar{\sigma}$ be an KS-renaming based on K-renaming $\sigma[keys(m)]$. Given a computational model $\mathbf{M}$, it holds that*

$$[\![m]\!]_{\mathbf{M}} \approx [\![m\bar{\sigma}]\!]_{\mathbf{M}}$$

Proof. According to Definition of KS-renaming in Definition 11, $m\bar{\sigma}$ is got from m by consistently renaming its keys and key shares according to $\bar{\sigma}$, but the distribution associated with a message is decided only by their meaning, not by the symbols used in the message, so, this lemma holds.

In fact, Lemma 1 is the same as Lemma 8 in [16]. Here, KS-renaming is the consistent renaming[6] in [16].

The following lemma is similar to Lemma 1 except that S-renaming is used. However, Lemma 1 cannot be naturally applied on S-renaming, simply because S-renaming is actually not a consistent renaming.

Lemma 2. *Let $m \in \mathbf{MSG}$, $\hat{\theta}$ be an S-renaming based on K-renaming $\theta[\mathbf{psk}(m)]$. Given a computational model $\mathbf{M} = (\mathbf{\Pi}, \mathbf{\Lambda}, \omega, \gamma)$, if $\mathbf{\Pi}$ is a* **CPA** *secure encryption scheme and $\mathbf{\Lambda}$ is a secure secret sharing scheme, then, it holds that*

$$[\![m]\!]_{\mathbf{M}} \approx [\![m\hat{\theta}]\!]_{\mathbf{M}}$$

Proof. Assume $|\mathbf{psk}(m)| = \rho$ is polynomially bounded in the length of message m, and thus $\mathbf{psk}(m) = \{k_{a_1}, k_{a_2}, \cdots, k_{a_\rho}\}$. Let $m_0 = m$, and $m_i = m_{i-1}\hat{\theta}[\{k_{a_i}\}]$ where $1 \leq i \leq \rho$. we have $m_\rho = m\hat{\theta}[\mathbf{psk}(m)]$, i.e.,$m_\rho = m\hat{\theta}$. By using the hybrid argument, to show $[\![m]\!]_{\mathbf{M}} \approx [\![m\hat{\theta}]\!]_{\mathbf{M}}$, we only need to show $[\![m_{i-1}]\!]_{\mathbf{M}} \approx [\![m_i]\!]_{\mathbf{M}}$, where $1 \leq i \leq \rho$.

Let's evaluate message m_{i-1} and m_i according to Definition 19. Intuitively, the only difference between m_{i-1} and m_i is that, in m_i, the secret shares of k_{a_i} is replaced by the secret shares of a new key $k_{a'_i}$. So, we can use Definition 19 to get computational interpretations of each symbols in m_{i-1}, and complete evaluating message m_{i-1}. To evaluate message m_i, we use the same computational interpretation to m_i except the secret shares of k_{a_i}. To give the computational interpretation of shares of k_{a_i}, we firstly generate a new key by **Gen** of $\mathbf{\Pi}$; then create n secret shares of this key by **Crt** of $\mathbf{\Lambda}$, and save them in $\varsigma[i,1]$ to $\varsigma[i,n]$ respectively. By doing such, we get $[\![m_{i-1}]\!]_{\mathbf{M}}$ and $[\![m_i]\!]_{\mathbf{M}}$.

Let $\mathcal{D}_1$ be a probabilistic polynomial-time distinguisher, and set

$$\begin{aligned}\varepsilon_1(\eta) \triangleq & \mathbf{Pr}\left[v_1 \leftarrow [\![m_{i-1}]\!]_{\mathbf{M}(\eta)} : \mathcal{D}_1(v, 1^\eta) = 1\right] - \\ & \mathbf{Pr}\left[v_1 \leftarrow [\![m_i]\!]_{\mathbf{M}(\eta)} : \mathcal{D}_1(v, 1^\eta) = 1\right].\end{aligned}$$

Now, assume for contradiction that $\mathcal{D}_1$ distinguishes $[\![m_{i-1}]\!]_{\mathbf{M}}$ from $[\![m_i]\!]_{\mathbf{M}}$ with non-negligible probability, i.e., $\varepsilon_1(\eta)$ is non-negligible. Then we construct an adversary $\mathcal{A}_1$ to break the security of sharing scheme $\mathbf{\Lambda}$ by the help of distinguisher $\mathcal{D}_1$.

Let n be the numbers of shares created by $\mathbf{\Lambda}$, $S_i = \{j | k^j_{a_i} \lessdot m\}$ be the set of indexes j such that the key share $k^j_{a_i}$ occurs in m. Since $k_{a_i} \in \mathbf{psk}(m)$, the shares of k_{a_i} only partially occur in m, that is $|S_i| < n$. From the definition of m_{i-1}, we know that the shares of k_{a_i} occur in m is exactly the shares of k_{a_i} occur in m_{i-1}. So, the share numbers of k_{a_i} occur in m_{i-1} is also $|S_i|$.

Adversary 1 ($\mathcal{A}_1$) *The adversary is given two keys $\hat{k}_0, \hat{k}_1 \leftarrow$* **Gen**$(1^\eta)$ *and a set of shares*[7] $\{\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_p\}$ *either sampled from* **Crt**$(\hat{k}_0, 1^\eta)|_{S_{i-1}}$, *or sampled from* **Crt**$(\hat{k}_1, 1^\eta)|_{S_{i-1}}$, *where $p = |S_i| < n$.*

[6] Informally speaking, consistent renaming means that, when k_i occurred in m is renamed to $k_{i'}$, the share of k_i, say k^j_i, is renamed to $k^j_{i'}$ accordingly.

[7] Here, we use $\hat{k}$ or $\hat{s}$ instead of k or s to distinguish the bit-string keys or shares from the formal symbols of keys or shares.

1. *$\mathcal{A}_1$ evaluate m_{i-1} to get a value v_1:*
   (a) *Let $|\mathbf{keys}(m_{i-1})| = \ell$. Construct an ℓ vector κ and an $(\ell \times n)$ array ς;*
   (b) *$\kappa[j] (j \neq i)$ is initialized by sampling from $\mathbf{Gen}(1^\eta)$;*
   (c) *$\varsigma[j,1], \varsigma[j,2], \cdots, \varsigma[j,n] (j \neq i)$ are initialized by sampling from $\mathbf{Crt}(\kappa[j], 1^\eta)$;*
   (d) *m_{i-1} is evaluated to value v_1 according to Definition 19 except k_{a_i} and the shares of k_{a_i}. More precisely, k_{a_i} is interpret by $\hat{k_0}$, and the shares of k_{a_i} is interpreted by $\{\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_p\}$.*
2. *$\mathcal{A}_1$ runs $\mathcal{D}_1(v_1, 1^\eta)$, and outputs whatever $\mathcal{D}_1(v_1, 1^\eta)$ outputs.*

Note that both $\hat{k}_0$ and $\hat{k}_1$ are generated by **Gen**. So, if $\{\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_p\}$ sampled from $\mathbf{Crt}(\hat{k}_0, 1^\eta)|_{S_i}$, then v_1 just sampled from $[\![m_{i-1}]\!]_{\mathbf{M}}$. If $\{\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_p\}$ sampled from $\mathbf{Crt}(\hat{k}_1, 1^\eta)|_{S_i}$, then k_{a_i} is interpreted by $\hat{k_0}$, while the shares of k_{a_i} are interpreted by the shares of $\hat{k_1}$. By the definition of m_i, in this situation, v_1 is just sampled from $[\![m_i]\!]_{\mathbf{M}}$. Considering that $\mathcal{A}_1$ outputs whatever $\mathcal{D}_1(v_1, 1^\eta)$ outputs, we have

$$
\begin{aligned}
&\mathbf{Pr}\left[\hat{k}_0, \hat{k}_1 \leftarrow \mathbf{Gen}(1^\eta), \mathbf{sh}(\hat{k}_0) \leftarrow \mathbf{Crt}(\hat{k}_0, 1^\eta) : \mathcal{A}_1(\hat{k}_0, \hat{k}_1, \mathbf{sh}(\hat{k}_0)|_{S_i}) = 1\right] - \\
&\mathbf{Pr}\left[\hat{k}_0, \hat{k}_1 \leftarrow \mathbf{Gen}(1^\eta), \mathbf{sh}(\hat{k}_1) \leftarrow \mathbf{Crt}(\hat{k}_1, 1^\eta) : \mathcal{A}_1(\hat{k}_0, \hat{k}_1, \mathbf{sh}(\hat{k}_1)|_{S_i}) = 1\right] \\
= &\mathbf{Pr}\left[v_1 \leftarrow [\![m_{i-1}]\!]_{\mathbf{M}(\eta)} : \mathcal{D}_1(v_1, 1^\eta) = 1\right] - \mathbf{Pr}\left[v_1 \leftarrow [\![m_i]\!]_{\mathbf{M}(\eta)} : \mathcal{D}_1(v_1, 1^\eta) = 1\right] \\
= &\varepsilon_1(\eta)
\end{aligned}
$$

This shows that $\mathcal{A}_1$ can break $\mathbf{\Lambda}$ with non-negligible probability, which is in contradiction with the security of $\mathbf{\Lambda}$, and thus Lemma 2 holds.

Example 5. Recall message m in Example 2, we have $\mathbf{psk}(m) = \{k_5, k_6\}$, Fig. 4 shows an S-renaming and the messages m_0, m_1, and m_2 constructed according to the approach in proof of Lemma 2. Given a computational model $\mathbf{M}$, from Lemma 2, we know that $[\![m]\!]_{\mathbf{M}} \approx [\![m\hat{\theta}]\!]_{\mathbf{M}}$.

$\mathbf{s}(\mathbf{psk}(m))$ $\downarrow$ $\hat{\theta}(\mathbf{s}(\mathbf{psk}(m)))$	$k_5^1 \; k_5^2 \; k_6^1 \; k_6^2$ $\downarrow \; \downarrow \; \downarrow \; \downarrow$ $k_{5'}^1 \; k_{5'}^2 \; k_{6'}^1 \; k_{6'}^2$
$m_0 = m$ $m_1 = m_0\hat{\theta}[\mathbf{s}(k_5)]$ $m_2 = m_1\hat{\theta}[\mathbf{s}(k_6)] = m\hat{\theta}$	$(\{\!\vert k_1, k_2^1 \vert\!\}_{k_1}, \{\!\vert k_3, \{\!\vert \{\!\vert k_4 \vert\!\}_{k_5} \vert\!\}_{k_4} \vert\!\}_{k_2}, \{\!\vert k_2^2 \vert\!\}_{k_3}, \{\!\vert k_4^2 \vert\!\}_{k_6}, k_5^1, k_6^1, \{\!\vert k_4^1 \vert\!\}_{k_7})$ $(\{\!\vert k_1, k_2^1 \vert\!\}_{k_1}, \{\!\vert k_3, \{\!\vert \{\!\vert k_4 \vert\!\}_{k_5} \vert\!\}_{k_4} \vert\!\}_{k_2}, \{\!\vert k_2^2 \vert\!\}_{k_3}, \{\!\vert k_4^2 \vert\!\}_{k_6}, k_{5'}^1, k_6^1, \{\!\vert k_4^1 \vert\!\}_{k_7})$ $(\{\!\vert k_1, k_2^1 \vert\!\}_{k_1}, \{\!\vert k_3, \{\!\vert \{\!\vert k_4 \vert\!\}_{k_5} \vert\!\}_{k_4} \vert\!\}_{k_2}, \{\!\vert k_2^2 \vert\!\}_{k_3}, \{\!\vert k_4^2 \vert\!\}_{k_6}, k_{5'}^1, k_{6'}^1, \{\!\vert k_4^1 \vert\!\}_{k_7})$

Fig. 4. An example for applying S-renaming in proof of Lemma 2.

Lemma 3. *Let $m \in \mathbf{MSG}$. Given a K-renaming $\theta[\mathbf{psk}(m)]$, and thus an S-renaming $\hat{\theta}[\mathbf{s}(\mathbf{psk}(m))]$, we have*

$$[\![\mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta}))]\!]_{\mathbf{M}} \approx [\![\mathbf{p}(m, \mathbf{rck}(m))]\!]_{\mathbf{M}}$$

Proof. If we can show

$$\mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta})) = \mathbf{p}(m, \mathbf{rck}(m))\hat{\theta} \tag{14}$$

then, by Lemma 2, we can directly show that Lemma 3 holds. We then show (14) by the following two steps:

$$\mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta})) = \mathbf{p}(m\hat{\theta}, \mathbf{rck}(m)) \tag{15}$$

$$\mathbf{p}(m\hat{\theta}, \mathbf{rck}(m)) = \mathbf{p}(m, \mathbf{rck}(m))\hat{\theta} \tag{16}$$

Proof of (15). From the definition of $\mathbf{sbk}$ and S-renaming, we have $\mathbf{sbk}(m\hat{\theta}) = \mathbf{rck}(m) \cup \theta(\mathbf{psk}(m))$. Considering that $\theta(\mathbf{psk}(m)) \cap \mathbf{keys}(m) = \emptyset$ by (12), and $\mathbf{rck}(m) \subseteq \mathbf{keys}(m)$ by (3) and (1), we have $\theta(\mathbf{psk}(m)) \cap \mathbf{rck}(m) = \emptyset$. Together with Proposition 1, we get that

$$\begin{aligned}\mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta})) &= \mathbf{p}(m\hat{\theta}, \mathbf{rck}(m) \cup \theta(\mathbf{psk}(m))) \\ &= \mathbf{p}(m\hat{\theta}, \mathbf{rck}(m))\end{aligned}$$

Proof of (16). From (4), we know that $\mathbf{psk}(m) \cap \mathbf{rck}(m) = \emptyset$. So, $\mathbf{p}(m\hat{\theta}, \mathbf{rck}(m))$ is only different from $\mathbf{p}(m, \mathbf{rck}(m))$ in that the shares of keys in $\mathbf{psk}(m)$ is renamed according to $\hat{\theta}$. Therefore, by using the same $\hat{\theta}$ on $\mathbf{p}(m, \mathbf{rck}(m))$, we can get $\mathbf{p}(m\hat{\theta}, \mathbf{rck}(m))$.

Example 6. Continue the Example 5, we have

$$\begin{aligned}
m &= \left(\{\!|k_1, k_2^1|\!\}_{k_1}, \{\!|k_3, \{\!|\{\!|k_4|\!\}_{k_5}|\!\}_{k_4}|\!\}_{k_2}, \{\!|k_2^2|\!\}_{k_3}, \{\!|k_4^2|\!\}_{k_6}, k_5^1, k_6^1, \{\!|k_4^1|\!\}_{k_7}\right) \\
\mathbf{rck}(m) &= \{k_1, k_2, k_3, k_4\} \\
\mathbf{p}(m, \mathbf{rck}(m)) &= (\{\!|k_1, k_2^1|\!\}_{k_1}, \{\!|k_3, \{\!|\{\!|\Diamond|\!\}_{k_5}|\!\}_{k_4}|\!\}_{k_2}, \{\!|k_2^2|\!\}_{k_3}, \{\!|\Diamond^2|\!\}_{k_6}, k_5^1, k_6^1, \{\!|\Diamond^1|\!\}_{k_7}) \\
\mathbf{p}(m, \mathbf{rck}(m))\hat{\theta} &= \left(\{\!|k_1, k_2^1|\!\}_{k_1}, \{\!|k_3, \{\!|\{\!|\Diamond|\!\}_{k_5}|\!\}_{k_4}|\!\}_{k_2}, \{\!|k_2^2|\!\}_{k_3}, \{\!|\Diamond^2|\!\}_{k_6}, k_{5'}^1, k_{6'}^1, \{\!|\Diamond^1|\!\}_{k_7}\right) \\
m\hat{\theta} &= \left(\{\!|k_1, k_2^1|\!\}_{k_1}, \{\!|k_3, \{\!|\{\!|k_4|\!\}_{k_5}|\!\}_{k_4}|\!\}_{k_2}, \{\!|k_2^2|\!\}_{k_3}, \{\!|k_4^2|\!\}_{k_6}, k_{5'}^1, k_{6'}^1, \{\!|k_4^1|\!\}_{k_7}\right) \\
\mathbf{sbk}(m\hat{\theta}) &= \{k_1, k_2, k_3, k_4, k_{5'}, k_{6'}\} \\
\mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta})) &= \left(\{\!|k_1, k_2^1|\!\}_{k_1}, \{\!|k_3, \{\!|\{\!|\Diamond|\!\}_{k_5}|\!\}_{k_4}|\!\}_{k_2}, \{\!|k_2^2|\!\}_{k_3}, \{\!|\Diamond^2|\!\}_{k_6}, k_{5'}^1, k_{6'}^1, \{\!|\Diamond^1|\!\}_{k_7}\right) \\
\mathbf{p}(m\hat{\theta}, \mathbf{rck}(m)) &= \left(\{\!|k_1, k_2^1|\!\}_{k_1}, \{\!|k_3, \{\!|\{\!|\Diamond|\!\}_{k_5}|\!\}_{k_4}|\!\}_{k_2}, \{\!|k_2^2|\!\}_{k_3}, \{\!|\Diamond^2|\!\}_{k_6}, k_{5'}^1, k_{6'}^1, \{\!|\Diamond^1|\!\}_{k_7}\right)
\end{aligned}$$

Obviously, $\mathbf{p}(m, \mathbf{rck}(m))\hat{\theta} = \mathbf{p}(m\hat{\theta}, \mathbf{rck}(m))$. Then, from Lemma 2, we get

$$[\![\mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta}))]\!]_{\mathbf{M}} \approx [\![\mathbf{p}(m, \mathbf{rck}(m))]\!]_{\mathbf{M}}$$

as expected.

Lemma 4. *Given a formal messages m, and a computational model $\mathbf{M} = (\mathbf{\Pi}, \mathbf{\Lambda}, \omega, \gamma)$, if $\mathbf{\Pi}$ is a* **CPA** *secure encryption scheme and $\mathbf{\Lambda}$ is a secure secret sharing scheme, then, it holds that*

$$[\![m]\!]_{\mathbf{M}} \approx [\![\mathbf{p}(m, \mathbf{rck}(m))]\!]_{\mathbf{M}}$$

Proof. Assume $\hat{\theta}$ be an S-renaming based on K-renaming $\theta[\mathbf{psk}(m)]$. We have

$$\llbracket m \rrbracket_{\mathbf{M}} \approx \llbracket m\hat{\theta} \rrbracket_{\mathbf{M}} \qquad \text{by Lemma 2}$$
$$\llbracket \mathbf{p}(m, \mathbf{rck}(m)) \rrbracket_{\mathbf{M}} \approx \llbracket \mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta})) \rrbracket_{\mathbf{M}} \qquad \text{by Lemma 3}$$

Therefore, to prove $\llbracket m \rrbracket_{\mathbf{M}} \approx \llbracket \mathbf{p}(m, \mathbf{rck}(m)) \rrbracket_{\mathbf{M}}$, we only need to show

$$\llbracket m\hat{\theta} \rrbracket_{\mathbf{M}} \approx \llbracket \mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta})) \rrbracket_{\mathbf{M}}$$

Let's evaluate message $m\hat{\theta}$ and $\mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta}))$ according to Definition 19.

Intuitively, the difference between m and $m\hat{\theta}$ is that, in $m\hat{\theta}$, the secret shares of k in $\mathbf{psk}(m)$ is replaced by the secret shares of a new key. So, we can evaluate $m\hat{\theta}$ by generating $|\mathbf{psk}m|$ more keys and their secret shares.

The difference between $m\hat{\theta}$ and $\mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta}))$ is that, all the sub-messages of $m\hat{\theta}$ in form of $\{m'\}_{k_i}$,where $k_i \in \mathbf{eok}(m\hat{\theta}) = \mathbf{keys}(m\hat{\theta}) \setminus \mathbf{sbk}(m\hat{\theta})$, is replaced by $\{\mathbf{struct}(m')\}_{k_i}$. So, according to Definition 19, we can evaluate $\mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta}))$ by using the same computational interpretation of $m\hat{\theta}$ except the sub-message in form of $\{m'\}_{k_i}$ where $k_i \in \mathbf{eok}(m\hat{\theta})$. The computational interpretation of $\{\!|m'|\!\}_{k_i}$ is simply interpreted by $0^{|\llbracket m' \rrbracket_{\mathbf{M}}|}$.

By doing such, we get $\llbracket m\hat{\theta} \rrbracket_{\mathbf{M}}$ and $\llbracket \mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta})) \rrbracket_{\mathbf{M}}$.

Let $\mathcal{D}_2$ be a probabilistic polynomial-time distinguisher, and set

$$\begin{aligned}\varepsilon_2(\eta) \triangleq & \mathbf{Pr}\left[v_2 \leftarrow \llbracket m\hat{\theta} \rrbracket_{\mathbf{M}(\eta)} : \mathcal{D}_2(v_2, 1^\eta) = 1\right] - \\ & \mathbf{Pr}\left[v_2 \leftarrow \llbracket \mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta})) \rrbracket_{\mathbf{M}(\eta)} : \mathcal{D}_2(v_2, 1^\eta) = 1\right].\end{aligned}$$

Assume for contradiction that there is a distinguisher $\mathcal{D}_2$ which can distinguish $\llbracket m\hat{\theta} \rrbracket_{\mathbf{M}}$ from $\llbracket \mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta})) \rrbracket_{\mathbf{M}}$ with non-negligible probability. We then construct an adversary $\mathcal{A}_2$ to break the encryption scheme $\mathbf{\Pi}$.

Adversary 2 ($\mathcal{A}_2$) *The adversary is given the security parameter 1^η and an encryption oracle $O_b(\cdot,\cdot)$ about $\mathbf{eok}(m\hat{\theta})$. Given a query (i, m') where $k_i \in \mathbf{eok}(m\hat{\theta})$, $O_b(\cdot,\cdot)$ outputs $\mathbf{Enc}_{k_i}(m')$ if $b = 1$, or $\mathbf{Enc}_{k_i}(0^{|m'|})$ if $b = 0$.*

1. *$\mathcal{A}_2$ evaluate $m\hat{\theta}$ to get a value v_2:*
   - *(a) Let $|\mathbf{sbk}(m\hat{\theta})| = \ell$. Construct an ℓ vector κ and an $(\ell \times n)$ array ς;*
   - *(b) $\kappa[j](k_j \in \mathbf{sbk}(m\hat{\theta}))$ is initialized by sampling from $\mathbf{Gen}(1^\eta)$;*
   - *(c) $\varsigma[j,1], \varsigma[j,2], \cdots, \varsigma[j,n](k_j \in \mathbf{sbk}(m\hat{\theta}))$ are initialized by sampling from $\mathbf{Crt}(\kappa[j], 1^\eta)$;*
   - *(d) $m\hat{\theta}$ is evaluated to value v_2 according to Definition 19 except keys in $\mathbf{eok}(m\hat{\theta}) = \mathbf{keys}(m\hat{\theta}) \setminus \mathbf{sbk}(m\hat{\theta})$[8]. More precisely, a message in form of $\{m'\}_{k_i}$, where $k_i \in \mathbf{eok}(m\hat{\theta})$, is evaluated by submitting (i, m') to $O_b(\cdot,\cdot)$.*
2. *$\mathcal{A}_2$ runs $\mathcal{D}_2(v_2, 1^\eta)$, and outputs whatever $\mathcal{D}_2(v_2, 1^\eta)$ outputs.*

[8] Since $\mathbf{eok}(m\hat{\theta}) \cap \mathbf{sbk}(m\hat{\theta}) = \emptyset$, the shares of keys in $\mathbf{eok}(m\hat{\theta})$ never occur in $m\hat{\theta}$.

Since we deal with the messages in presence of key cycles and secret shares, one may wonder that if it's always feasible for the adversary $\mathcal{A}_2$ to construct a query submitted to oracle. After all, for any $m \in \mathbf{MSG}$, it seems that such a query may contain some keys or secret shares that $\mathcal{A}_2$ doesn't know. In fact, by using $m\hat{\theta}$ instead of m itself, considering that $\mathcal{A}_2$ knows all the keys in $\mathbf{sbk}(m\hat{\theta})$ and their shares, it is definitely feasible for $\mathcal{A}_2$ to construct such query.

Moreover, if $b = 1$, we can see that v_2 is just sampled from $[\![m\hat{\theta}]\!]_{\mathbf{M}}$, and if $b = 0$, v_2 is just sampled from $[\![\mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta}))]\!]_{\mathbf{M}}$. Considering that $\mathcal{A}_2$ outputs whatever $\mathcal{D}_2(v_2, 1^\eta)$ outputs, we have

$$\begin{aligned}
&\mathbf{Pr}[{\mathcal{A}_2}^{O_1}(1^\eta) = 1] - \mathbf{Pr}[{\mathcal{A}_2}^{O_0}(1^\eta) = 1] \\
=&\mathbf{Pr}\left[v_2 \leftarrow [\![m\hat{\theta}]\!]_{\mathbf{M}(\eta)} : \mathcal{D}_2(v_2, 1^\eta) = 1\right] - \\
&\mathbf{Pr}\left[v_2 \leftarrow [\![\mathbf{p}(m\hat{\theta}, \mathbf{sbk}(m\hat{\theta}))]\!]_{\mathbf{M}(\eta)} : \mathcal{D}_2(v_2, 1^\eta) = 1\right] \\
=&\varepsilon_2(\eta)
\end{aligned}$$

This shows that $\mathcal{A}_2$ can break $\mathbf{\Pi}$ with non-negligible probability, which is in contradiction with the **CPA** security of $\mathbf{\Pi}$. Therefore, $\varepsilon_2(\eta)$ is negligible, and this completes the lemma.

Lemma 5. *Given a formal messages m, and a computational model $\mathbf{M} = (\mathbf{\Pi}, \mathbf{\Lambda}, \omega, \gamma)$, if $\mathbf{\Pi}$ is a* **CPA** *secure encryption scheme and $\mathbf{\Lambda}$ is a secure secret sharing scheme, then, it holds that*

$$[\![m]\!]_{\mathbf{M}} \approx [\![\mathbf{pattern}(m)]\!]_{\mathbf{M}}$$

Proof. Let $\ell = |\mathbf{Keys}|$ is polynomially bounded in the security parameter η, from (5) and Definition 10, we have

$$\begin{aligned}
&[\![m]\!]_{\mathbf{M}} && [\![\mathbf{pattern}(m)]\!]_{\mathbf{M}} \\
=&[\![\mathbf{p}(m, \mathbf{keys}(m))]\!]_{\mathbf{M}} &=&[\![\mathbf{p}(m, \mathbf{FIX}(\mathcal{F}_m))]\!]_{\mathbf{M}} \\
=&[\![\mathbf{p}(m, \mathcal{F}_m^0(\mathbf{keys}(m)))]\!]_{\mathbf{M}} &=&[\![\mathbf{p}(m, \mathcal{F}_m^\ell(\mathbf{keys}(m)))]\!]_{\mathbf{M}}.
\end{aligned}$$

If we can show

$$[\![\mathbf{p}(m, \mathcal{F}_m^i(\mathbf{Keys}))]\!]_{\mathbf{M}} \approx [\![\mathbf{p}(m, \mathcal{F}_m^{i+1}(\mathbf{Keys}))]\!]_{\mathbf{M}}$$

where $0 \le i \le \ell - 1$, then, by the transitivity (13), we can show $[\![\mathbf{p}(m, \mathcal{F}_m^0(\mathbf{Keys}))]\!]_{\mathbf{M}}$ is distinguishable from $[\![\mathbf{p}(m, \mathcal{F}_m^\ell(\mathbf{keys}(m)))]\!]_{\mathbf{M}}$, i.e.,

$$[\![m]\!]_{\mathbf{M}} \approx [\![\mathbf{pattern}(m)]\!]_{\mathbf{M}}.$$

Let $\mathbf{K} = \mathcal{F}_m^i(\mathbf{keys}(m))$, $m' = p(m, \mathbf{K})$. We have

$$\mathbf{p}(m, \mathcal{F}_m^i(\mathbf{keys}(m))) = p(m, \mathbf{K}) = m' \tag{17}$$

$$\mathbf{p}(m, \mathcal{F}_m^{i+1}(\mathbf{keys}(m))) = p(m, \mathcal{F}_m(\mathbf{K})) \tag{18}$$

Moreover, because $\mathbf{keys}(m)$ is the set of all keys occurring in m, we can get that $\mathcal{F}_m(\mathbf{keys}(m)) \subseteq \mathbf{keys}(m)$. According to proposition **??**, $\mathcal{F}_m$ is monotone. So, we have $\mathcal{F}_m^{i+1}(\mathbf{keys}(m)) \subseteq \mathcal{F}_m^{i}(\mathbf{keys}(m))$, particularly, $\mathcal{F}_m(\mathbf{K}) \subseteq \mathbf{K}$, and thus

$$\mathcal{F}_m(\mathbf{K}) \cap \mathbf{K} = \mathcal{F}_m(\mathbf{K}) \tag{19}$$

Then, we have

$$\begin{aligned}
\mathbf{p}(m', \mathbf{rck}(m')) &= \mathbf{p}(\mathbf{p}(m, \mathbf{K}), \mathcal{F}_m(\mathbf{K})) && \text{by (8)} \\
&= \mathbf{p}(m, \mathbf{K} \cap \mathcal{F}_m(\mathbf{K})) && \text{by (6)} \\
&= \mathbf{p}(m, \mathcal{F}_m(\mathbf{K})) && \text{by (19)}
\end{aligned}$$

From lemma 4, we know that $[\![\mathbf{p}(m', \mathbf{rck}(m'))]\!]_{\mathbf{M}} \approx [\![m']\!]_{\mathbf{M}}$, and thus $[\![m']\!]_{\mathbf{M}} \approx [\![\mathbf{p}(m, \mathcal{F}_m(M))]\!]_{\mathbf{M}}$. Then, with (17) and (18), we get

$$[\![\mathbf{p}(m, \mathcal{F}_m^{i}(\mathbf{keys}(m)))]\!]_{\mathbf{M}} \approx [\![\mathbf{p}(m, \mathcal{F}_m^{i+1}(\mathbf{keys}(m)))]\!]_{\mathbf{M}}$$

and thus Lemma 5 holds.

Now, it's time for us to prove our main result, i.e., the computational soundness theorem.

Theorem 1. *Given two formal messages m, m', and a computational model $\mathbf{M} = (\mathbf{\Pi}, \mathbf{\Lambda}, \omega, \gamma)$, in which $\mathbf{\Pi}$ is an* **CPA** *secure encryption scheme and $\mathbf{\Lambda}$ is a secure secret sharing scheme, if $m \cong m'$, then, $[\![m]\!]_{\mathbf{M}} \approx [\![m']\!]_{\mathbf{M}}$.*

Proof. Since $m \cong m'$, from Definition 12, we know that there exists a KS-renaming $\bar{\sigma}$ based on K-renaming $\sigma[\mathbf{keys}(m)]$, or, additionally an S-renaming $\hat{\theta}$ based on K-renaming $\theta[\mathbf{psk}(m\bar{\sigma})]$, such that one of the following holds:

$$\mathbf{pattern}(m) = \mathbf{pattern}(m')\bar{\sigma} \tag{20}$$

$$\mathbf{pattern}(m) = (\mathbf{pattern}(m')\bar{\sigma})\hat{\theta} \tag{21}$$

From (20) and Lemma 1, we can get $[\![\mathbf{pattern}(m)]\!]_{\mathbf{M}} \approx [\![\mathbf{pattern}(m')]\!]_{\mathbf{M}}$. From (21), Lemma 2, and Lemma 1, we can also get $[\![\mathbf{pattern}(m)]\!]_{\mathbf{M}} \approx [\![\mathbf{pattern}(m')]\!]_{\mathbf{M}}$. So, we can conclude that, if $m \cong m'$, then

$$[\![\mathbf{pattern}(m)]\!]_{\mathbf{M}} \approx [\![\mathbf{pattern}(m')]\!]_{\mathbf{M}}. \tag{22}$$

Moreover, from Lemma 5, we have

$$[\![m]\!]_{\mathbf{M}} \approx [\![\mathbf{pattern}(m)]\!]_{\mathbf{M}}$$
$$[\![m']\!]_{\mathbf{M}} \approx [\![\mathbf{pattern}(m')]\!]_{\mathbf{M}}$$

Together with (13) and (22), we get

$$[\![m]\!]_{\mathbf{M}} \approx [\![m']\!]_{\mathbf{M}}$$

This completes our proof.

5 Conclusion

We proved the computational soundness of formal encryption in presence of secret shares and key cycles. Our work is inspired by [16] and [17], and gives a more general result. To extend the result of [16] to consider key cycles, we model the adversary's knowledge by co-induction which is proposed in [17]. Presence of secret shares and key cycles makes the cryptographic setting more complicated and need more consideration. For example, when both keys and shares are occurred in a key cycle, we must reconsider what keys can be recover from it and what cannot. Moreover, by using CPA secure encryption scheme in computational model, we must deal the conflict between definition of CPA and the key cycles, especially the secret shares are involved. These problems also need more considerations in the proof of computational soundness. All these make our work different from the previous.

In further research, one can take the work in this paper to the setting of the asymmetric cryptography. Still another work can be down is to prove the computational soundness in the presence of active adversaries.

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