Fully Homomorphic Message Authenticators

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Abstract

We define and construct *fully homomorphic message authenticators*. In such a scheme, anybody can perform arbitrary computations over authenticated data and produce a *short* tag that authenticates the result of the computation. The user verifies this tag with her private key to ensure that the claimed result is indeed the correct output of the specified computation over previously authenticated data, *without needing to know the underlying data itself*. For example, a user can outsource the storage of large amounts of authenticated data to a remote server, and the server can later non-interactively certify the outputs of various computations over this data with only a short tag. Our construction uses fully homomorphic encryption in a novel way.

1 Introduction

The rise of the *cloud computing* paradigm requires that users can securely outsource their data to a remote service provider, and have it reliably perform computations over the data. The recent groundbreaking development of *fully homomorphic encryption* [22] allows us to maintain *confidentiality/privacy* of outsourced data, while enabling arbitrary computation over it. In this work, we look at the analogous (but orthogonal) question of providing *integrity/authenticity* for computations over outsourced data. In particular, if the remote server claims that the execution of some program $\mathcal{P}$ over the user's outsourced data results in an output y, how can the user be sure that this is indeed the case?

Toward this goal, we define and instantiate a new primitive, called a *fully homomorphic message authenticator*. This primitive can be seen as a *symmetric-key* version of fully homomorphic *signatures*, which were defined by Boneh and Freeman [9], but whose construction remains an open problem. We will return to survey the related work on (partially) homomorphic signatures and authenticators, as well as related work on *delegating memory and computation*, in Section 1.3. First, we describe our notion of fully homomorphic message authenticators, which will be the focus of this work.

1.1 What are Homomorphic Message Authenticators?

In a homomorphic message-authenticator scheme, the user can authenticate some arbitrary data D using her secret key. Later anybody can homomorphically execute some arbitrary program $\mathcal{P}$ over the authenticated data to produce a short authentication tag ψ, which authenticates the value $y = \mathcal{P}(D)$ as the output of $\mathcal{P}$. It is important that ψ does *not* simply authenticate y out of context; it only authenticates y as the output of the specified program $\mathcal{P}$.[1] In particular, the user can verify the triple $(y, \mathcal{P}, \psi)$ with her secret key to ensure that y is indeed the output of the program $\mathcal{P}$ evaluated on the previously authenticated data D, which she does *not* need to know during verification. The tag ψ should be *succinct*, meaning that its size is independent of the size of the input data D or the complexity of $\mathcal{P}$.

[1] Otherwise this notion of fully homomorphic authenticators would be vacuous; given any authenticated data D, the attacker could just homomorphically compute an authentication tag for *any* possible value y.

Labeled Data and Programs. The above high-level description only considers a scenario where the user authenticates a single value D in one shot. However, if the user wants to authenticate multiple items (say, many different files) on-the-fly, we need to establish some syntax for specifying *which data-items* the program $\mathcal{P}$ should be evaluated on. For this purpose, we rely on the notion of *labeled* data and programs.

Whenever the user wants to authenticate some data D, she chooses some *label* τ for it, and the authentication algorithm authenticates the data D with respect to the label τ. For example, the label τ could be a file name. For the greatest level of granularity, we will assume that the user authenticates individual *bits* of data separately. Each bit b is authenticated with respect to its own label τ via a secretly-keyed authentication algorithm $\sigma \leftarrow \mathsf{Auth}_{sk}(b, \tau)$. For example, to authenticate a long file named "salaries" containing the data-bits $D = (b_1, \ldots, b_t)$, the user can create separate labels $\tau_i =$("salaries", i) to denote the i^{th} bit of the file, and then authenticate each bit b_i of the file under its own label τ_i.

Correspondingly, we consider *labeled programs* $\mathcal{P}$, where each input bit of the program has an associated label τ_i indicating which data it should be evaluated on. For example, a labeled-program $\mathcal{P}$ meant to compute the *median* of the "salaries" data would have its input bits labeled by $\tau_i =$("salaries", i).

Homomorphic authenticators allow us to authenticate the output of a labeled program, given authentication tags for correspondingly labeled input data. In particular, given a labeled program $\mathcal{P}$ whose input-labels are $\tau_1, \ldots, \tau_t$, along with authentication tags σ_i that authenticate some data-bits b_i with respect to these labels τ_i, anybody can homomorphically compute a short authentication tag $\psi = \mathsf{Eval}(\mathcal{P}, \sigma_1, \ldots, \sigma_t)$ that authenticates the output $y = \mathcal{P}(b_1, \ldots, b_t)$ for the program $\mathcal{P}$. The secretly-keyed verification algorithm $\mathsf{Ver}_{sk}(y, \mathcal{P}, \psi)$ verifies that y is indeed the output of the labeled program $\mathcal{P}$ on previously authenticated labeled input-data, without needing to know the original data itself.

Composition. We also desire homomorphic authenticators that are *composable* so that we can incrementally combine authenticated outputs of partial computations to derive an authenticated output of a larger computation. In particular, if the tags $\psi_1, \ldots, \psi_t$ authenticate some bits $b_1, \ldots, b_t$ as the outputs of some labeled programs $\mathcal{P}_1, \ldots, \mathcal{P}_t$ respectively, then $\psi^* = \mathsf{Eval}(\mathcal{P}^*, \psi_1, \ldots, \psi_t)$ should authenticate the bit $b^* = \mathcal{P}^*(b_1, \ldots, b_t)$ as the output of the *composed program*, which first evaluates $\mathcal{P}_1, \ldots, \mathcal{P}_t$ on the appropriate authenticated data, and then runs $\mathcal{P}^*$ on the outputs.

Succinct Tags vs. Efficient Verification. The main requirement that makes our definition of homomorphic authenticators interesting is that the tags should be *succinct*. Otherwise, there is a trivial solution where we can authenticate the output of a computation $\mathcal{P}$ by simply providing all of its input bits and their authentication tags. The succinctness requirement ensures that we can certify the output of a computation $\mathcal{P}$ over authenticated data with much smaller *communication* than that of simply transmitting the input data. Therefore, this primitive is especially useful when verifying computations that read a lot of input data but have a short output.

However, we note that the verification algorithm in our schemes may require large *computational effort* from the user, proportional to the complexity of the computation $\mathcal{P}$ being verified. Therefore, although homomorphic authenticators allow us to save on *communication*, they do not necessarily save on the *computational complexity* of verifying computations over outsourced data. In Section 5, we explore how to use techniques from *delegating computation* to also get efficient verification.

1.2 Overview of Our Construction

Our construction of fully homomorphic authenticators relies on the use of fully homomorphic encryption (FHE). We let n denote the *security parameter* and $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$. During the key generation phase of our construction, the user Alice chooses a *secret* random subset $S \subseteq [n]$ of size $|S| = n/2$. She also chooses a public/secret key for an FHE scheme and a pseudo-random function (PRF) $f_K(\cdot)$.

To authenticate a bit b under a label τ, Alice creates n ciphertexts $c_1, \ldots, c_n$. For $i \in [n] \setminus S$, she chooses the ciphertexts $c_i \leftarrow \mathsf{Enc}_{pk}(b)$ as random encryptions of the bit b being authenticated. For

$i \in S$, she computes $c_i = \mathsf{Enc}_{pk}(0; f_K((i,\tau)))$ as *pseudorandom encryptions of 0*, where the random coins are derived using the PRF. Notice that for $i \in S$, these pseudorandom ciphertexts c_i can be easily re-computed from Alice's secret key and the label τ alone, *without knowing the data* bit b. Alice outputs the *authentication tag* $\sigma = (c_1, \ldots, c_n)$, consisting of the n ciphertexts.

Given some program $\mathcal{P}$ with t inputs and authentication tags $\{\sigma_j = (c_{1,j}, \ldots, c_{n,j})\}_{j\in[t]}$, we can homomorphically derive an authentication tag $\psi = (c_1^*, \ldots, c_n^*)$ for the output as follows. We derive each ciphertext c_i^* by homomorphically evaluating the program $\mathcal{P}$ over the t ciphertexts that lie in position i within the tags $\{\sigma_j\}_{j\in[t]}$. In particular, we set $c_i^* = \mathsf{Eval}(\mathcal{P}, c_{i,1}, \ldots, c_{i,t})$, where Eval is the homomorphic evaluation of the FHE scheme. We assume (without loss of generality) that the evaluation procedure for the FHE scheme is deterministic so that the results are reproducible.

Alice can verify the triple $(y, \mathcal{P}, \psi)$, where the tag $\psi = (\hat{c}_1, \ldots, \hat{c}_n)$ is supposed to certify that y is the output of the labeled program $\mathcal{P}$. Let $\tau_1, \ldots, \tau_t$ be the input labels of $\mathcal{P}$. For the indices $i \in S$, Alice can re-compute the pseudo-random input ciphertexts $\{c_{i,j} = \mathsf{Enc}_{pk}(0; f_K((i,\tau_j)))\}_{j\in[t]}$ using the PRF, without knowing the actual input bits. She then computes $c_i^* = \mathsf{Eval}(\mathcal{P}, c_{i,1}, \ldots, c_{i,t})$ and checks that the ciphertexts in the tag ψ were computed correctly with $\hat{c}_i \stackrel{?}{=} c_i^*$ for indices $i \in S$. [2] If this is the case, and all of the other ciphertexts $\hat{c}_i$ for $i \in [n] \setminus S$ decrypt to the claimed bit y, then Alice accepts.[3]

Intuitively, the only way that an attacker can lie about the output of some program $\mathcal{P}$ is by producing a tag $\psi = (\hat{c}_1, \ldots, \hat{c}_t)$ where the ciphertexts $\hat{c}_i$ for $i \in S$ are computed correctly but for $i \in [n] \setminus S$ they are all modified so as to encrypt the wrong bit. But this is impossible since the security of the FHE should ensure that the attacker cannot distinguish encryptions of 0 from those of the authenticated bits, and hence cannot learn anything about the set S. We point out the novel use of FHE in our construction. It is used to hide the difference between data-independent pseudorandom ciphertexts $c_i : i \in S$, which allow Alice to check that the computation was performed correctly, and data-containing ciphertexts $c_i : i \in [n] \setminus S$, which allow Alice to check that the output bit y is the correct one. This is reminiscent of the use of FHE in recent schemes for *delegating computation* [19, 14, 2] (see Section 1.3 on related work).

Note that the authentication tags ψ in our scheme always consist of n (= security parameter) ciphertexts, no matter how many inputs the program $\mathcal{P}$ takes and what its complexity is. Therefore, we satisfy the succinctness requirement.

Security & Verification Queries. We show that our construction is secure in the setting where the attacker can make arbitrarily many *authentication queries* for arbitrary labels, but cannot make *verification queries* to test whether a maliciously produced tag verifies correctly. In practice, this means that the user needs to abort and completely stop using the scheme whenever she gets the first tag that doesn't verify correctly. It is easy to allow for some fixed a-priori bounded number of verification queries q, just by increasing the number of ciphertexts contained in an authentication tag from n to $n + q$.

The difficulty of allowing arbitrarily many verification queries also comes up in essentially all prior schemes for delegating computation in the "pre-processing" model [19, 14, 2] (see Section 1.3 on related work), and remains an important open problem in both areas. In Section 4, we explore this question further in the context of our scheme, and show how to resolve it given an FHE scheme that satisfies an additional *randomness homomorphism* property. This property is independently interesting and raises an interesting open problem in the area of FHE schemes.

Fast Verification. One of the limitations of our solution above is that the verification algorithm is *no more efficient* than running the computation $\mathcal{P}$. Therefore, although it saves on the *communication complexity* of verifying computations over outsourced data, it does not save on user's computational complexity. In Section 5, we explore the option of using schemes for *delegating computation* to also offload the computational cost of the verification procedure to the remote server.

[2] Note that, in this step, Alice has to perform work comparable to that of computing $\mathcal{P}$.

[3] The above description of the scheme is slightly simplified and inaccurate. See Section 3 for the actual full construction.

1.3 Related work

Homomorphic Signatures and MACs. Many prior works consider the question of homomorphic message authentication (private verification) and signatures (public verification) for restricted homomorphisms, and almost exclusively for *linear functions*. Perhaps the first work to propose this problem is that of Johnson et al. [27]. Since then, many works have considered this notion in the context of *network coding*, yielding a long line of positive results [1, 8, 21, 9, 5, 10, 13, 18]. Another line of works considered this notion in the context of *proofs of data possession and retrievability* [3, 29, 16, 4].

The only work that considers a larger class of homomorphisms *beyond linear functions* is that of Boneh and Freeman [9], who show how to get homomorphic signatures for *bounded (constant) degree polynomials*. In that work, they also present a general definition along the same lines as the definition we use in this work, and pose the question of constructing *fully* homomorphic signatures for *arbitrary* functions.[4] Although the question of fully homomorphic *publicly verifiable* signatures under standard assumptions still remains open, our work provides the first positive result for the case of private verification.

Succinct Arguments of Knowledge. One method that would allow us to construct fully homomorphic (publicly verifiable) signatures is to rely on CS-Proofs [28] or, more generally, any *succinct non-interactive argument of knowledge* (SNARK) [6]. This primitive allows us to create a short "argument" π for any NP statement, to prove "knowledge" of the corresponding witness. The length of π is independent of the statement/witness size.

Using SNARKs, we can authenticate the output y of a labeled program $\mathcal{P}$, by creating a short argument π that proves the knowledge of some "labeled input data D along with valid signatures authenticating D under the appropriate labels, such that $\mathcal{P}(D) = y$". Since this is an argument of *knowledge*, a forged signature for the output of some program $\mathcal{P}$ would allow us to extract out a forged signature for the underlying input data, breaking the security of signatures.

Unfortunately, constructing succinct non-interactive arguments is known to require the use of *non-standard* assumptions [23]. Current constructions either rely on the *random-oracle model* [28] or on various "knowledge" assumptions (see, e.g., [26, 6, 7, 20]). Moreover, a solution using SNARKs does *not* seem to support arbitrary composition (only constant-depth) [30, 7].

Delegating Computation. Several prior works consider the problem of *delegating computation* to a remote server while maintaining the ability to efficiently verify the result [25, 19, 14, 2]. In this scenario, the server needs to convince the user that $\mathcal{P}(x) = y$, where the user knows the program $\mathcal{P}$, the input x and the output y, but does not want to do the *work* of computing $\mathcal{P}(x)$. In contrast, in our scenario the verifier only knows $\mathcal{P}, y$, but does *not* know the previously authenticated inputs that $\mathcal{P}$ should have been executed on. On the other hand, we are not trying to minimize work, just communication.

Despite these differences, some of the results on delegating computation in the "pre-processing" model [19, 14, 2], can also be (re-)interpreted for our setting. In this model, the user "pre-processes" a circuit C and stores some value σ on the server. Later, the user can ask the server to compute $C(x)$ for various inputs x, and the server uses σ to derive a short and efficiently verifiable proof ψ that certifies correctness of the computation. One caveat is that, in all these schemes, the user first needs to send some challenge $c = \mathit{chall}(x)$ to the server, and the proof ψ is computed as a response to c.

We can apply these results to our setting of outsourced data as follows. Consider outsourcing the "universal circuit" $C_D(\cdot)$ that has the data D "hard-coded", gets as input a program $\mathcal{P}$, and outputs $C_D(\mathcal{P}) = \mathcal{P}(D)$. Then, we can think of the pre-processing of C_D as creating an authentication tag σ for the data D. Later, the user can take a program $\mathcal{P}$, create a challenge $c = \mathit{chall}(\mathcal{P})$, and get back a short tag ψ that authenticates $y = C_D(\mathcal{P}) = \mathcal{P}(D)$.[5] One advantage of this approach is that the tag ψ can be

[4] See [9, 18] for an explanation of how this definition generalizes that of prior works on network coding.

[5] Here, we assume that $\mathcal{P}$ has a short uniform description so that reading/trasnmitting $\mathcal{P}$ is much more efficient than evaluating $\mathcal{P}$.

verified efficiently, with less work than that of computing $\mathcal{P}(D)$. However, there are several important disadvantages of this approach as compared to our notion of homomorphic authenticators:

1. *Interaction:* Homomorphic authenticators allow anybody to evaluate a chosen program $\mathcal{P}$ over authenticated data and *non-interactively* authenticate the output. The above delegation-based schemes require a *round of interaction*; the user first creates a challenge $chall(\mathcal{P})$ for the program $\mathcal{P}$, and only then can the server authenticate the output $\mathcal{P}(D)$ with respect to this challenge.

2. *Single Use:* Homomorphic authenticators allow the user to authenticate labeled data *"on the fly"* in a piecemeal way and verify computations over all of it using a fixed secret key. The above delegation-based schemes require that the user outsources all of the data D in *one shot*, and stores some small secret state associated with the data to verify computations over it in the future.

3. *Bounded Size:* The above delegation-based schemes require that the circuit-size of the computations $\mathcal{P}$ is *a-priori bounded* by some fixed polynomial chosen during authentication. Furthermore, the complexity of authentication is proportional to this polynomial. Our fully homomorphic authenticators have no such restriction.

4. *No composition:* The above delegation-based schemes does not support the *composition* of several partial authenticated computations.

Memory Delegation. The work of Chung et al. [15] on *memory delegation* explicitly considers the problem of outsourcing a large amount of data while maintaining the ability to efficiently verify later computations over it. The two main *advantages* of memory delegation over our work are that: (I) the verification is efficient, and (II) the data can be updated by the user in the future. However, the solutions to memory delegation suffer from some of the same *disadvantages* as the above delegation-based schemes. In particular, current memory delegation schemes are *interactive*, and the only general solution for arbitrary polynomial-time programs requires 4 rounds of interaction during verification. Furthermore, memory delegation schemes require that the user is *stateful*, whereas homomorphic authenticators only require a fixed secret key.

2 Definitions

2.1 Homomorphic Authenticators

Labeled Programs. We begin by defining the concept of a labeled program, where the labels denote *which* data the program should be evaluated on. Formally, a *labeled-program* $\mathcal{P} = (f, \tau_1, \ldots, \tau_k)$ consists of a circuit $f : \{0,1\}^k \to \{0,1\}$ along with a distinct *input label* $\tau_i \in \{0,1\}^*$ for each input wire $i \in [k]$. [6]

Given some labeled programs $\mathcal{P}_1, \ldots, \mathcal{P}_t$ and a circuit $g : \{0,1\}^t \to \{0,1\}$, we can define the *composed program*, denoted by $\mathcal{P}^* = g(\mathcal{P}_1, \ldots, \mathcal{P}_t)$, which corresponds to evaluating g on the outputs of $\mathcal{P}_1, \ldots, \mathcal{P}_t$. The labeled inputs of the composed program $\mathcal{P}^*$ are just all the *distinct* labeled inputs of $\mathcal{P}_1, \ldots, \mathcal{P}_t$, meaning that we collect all the input wires with the same label and convert them into a single input wire.

We define the *identity program with label* τ as $\mathcal{I}_\tau := (g_{id}, \tau)$ where g_{id} is the *canonical identity circuit* and $\tau \in \{0,1\}^*$ is some label. Notice that any program $\mathcal{P} = (f, \tau_1, \ldots, \tau_k)$ can be written as a composition of identity programs $\mathcal{P} = f(\mathcal{I}_{\tau_1}, \ldots, \mathcal{I}_{\tau_k})$.

[6] Although the above description of $\mathcal{P}$ is long (proportional to its input size and complexity), in many scenarios it is possible that $\mathcal{P}$ may also have an alternative succinct description. For example, $\mathcal{P}$ may compute the *median* value in a large file called "salaries" and its input labels are simply $\tau_i = (\text{"salaries"}, i)$ for each bit i. Therefore, although we are concerned with succinctness, we will ignore the cost of communicating the program $\mathcal{P}$ from future consideration.

Syntax. A *homomorphic authenticator scheme* consists of the probabilistic-polynomial time algorithms (KeyGen, Auth, Ver, Eval) with the following syntax:

- $\mathsf{KeyGen}(1^n) \to (evk, sk)$: Outputs the secret key sk and an evaluation key evk.
- $\mathsf{Auth}_{sk}(b, \tau) \to \sigma$: Creates a *tag* σ that authenticates the bit $b \in \{0,1\}$ under the label $\tau \in \{0,1\}^*$. (Equivalently, we say that σ authenticates b as the output of the identity program $\mathcal{I}_\tau$.)
- $\mathsf{Eval}_{evk}(f, \boldsymbol{\sigma}) \to \psi$: The deterministic evaluation procedure takes a vector of tags $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_k)$ and a circuit $f : \{0,1\}^k \to \{0,1\}$. It outputs a tag ψ. If each σ_i authenticates a bit b_i as the output of some labeled-program $\mathcal{P}_i$ (possibly the identity program), then ψ should authenticate $b^* = f(b_1, \ldots, b_k)$ as the output of the composed program $\mathcal{P}^* = f(\mathcal{P}_1, \ldots, \mathcal{P}_k)$.
- $\mathsf{Ver}_{sk}(e, \mathcal{P}, \psi) \to \{\mathtt{accept}, \mathtt{reject}\}$: The deterministic verification procedure uses the tag ψ to check that $e \in \{0,1\}$ is the output of the program $\mathcal{P}$ on previously authenticated labeled data.

We require that the scheme satisfies the following properties, defined below: *authentication correctness*, *evaluation correctness*, *succinctness* and *authenticator security.*

Authentication Correctness. We require that for any $b \in \{0,1\}$ and any label $\tau \in \{0,1\}^*$, we have:

$$\Pr\left[\mathsf{Ver}_{sk}(b, \mathcal{I}_\tau, \sigma) = \mathtt{accept} \;\middle|\; (evk, sk) \leftarrow \mathsf{KeyGen}(1^n), \sigma \leftarrow \mathsf{Auth}_{sk}(b, \tau)\right] = 1$$

where $\mathcal{I}_\tau$ is the identity program with label τ. In other words, the tag $\sigma = \mathsf{Auth}_{sk}(b, \tau)$ correctly authenticates b under the label τ, which is equivalent to saying that it authenticates b as the output of the identity program $\mathcal{I}_\tau$.

Evaluation Correctness. Fix any (evk, sk) in the support of $\mathsf{KeyGen}(1^n)$. Fix any circuit $g : \{0,1\}^t \to \{0,1\}$ and any set of program/bit/tag triples $\{ (\mathcal{P}_i, b_i, \psi_i) \}_{i=1}^t$ such that $\mathsf{Ver}_{sk}(b_i, \mathcal{P}_i, \psi_i) = \mathtt{accept}$. Set:

$$b^* := g(b_1, \ldots, b_t),\ \mathcal{P}^* := g(\mathcal{P}_1, \ldots, \mathcal{P}_t),\ \psi^* := \mathsf{Eval}_{evk}(g, (\psi_1, \ldots, \psi_t)).$$

Then we require $\mathsf{Ver}_{sk}(b^*, \mathcal{P}^*, \psi^*) = \mathtt{accept}$.

In words, assume that each tag ψ_i certifies that the output of the labeled program $\mathcal{P}_i$ is b_i. Then the tag ψ^* certifies that b^* is the output of the composed program $\mathcal{P}^*$. If all of the programs $\mathcal{P}_i = \mathcal{I}_{\tau_i}$ are identity programs, then the above says that as long as the tags ψ_i authenticate bits b_i under the labels τ_i, the tag ψ^* authenticates b^* as the output of $\mathcal{P}^* = (g, \tau_1, \ldots, \tau_t)$. Therefore, the above definition captures the basic correctness of computing over freshly authenticated data, as well as the *composability* of computing over the authenticated outputs of prior computations.

Succinctness. We require that the tag-size is always bounded by some fixed polynomial in the security parameter, and is independent of the size of the evaluated circuit or the number of inputs it takes. That is, there exists some polynomial $p(\cdot)$ such that, for any (evk, sk) in the support of $\mathsf{KeyGen}(1^n)$, the output-size of $\mathsf{Auth}_{sk}(\cdot, \cdot)$ and of $\mathsf{Eval}_{evk}(\cdot, \cdot)$ is bounded by $p(n)$ for any choice of their input.

Authenticator Security. Consider the following game $\mathsf{ForgeGame}^{\mathcal{A}}(1^n)$ between an attacker $\mathcal{A}(1^n)$ and a challenger:

1. The challenger chooses $(evk, sk) \leftarrow \mathsf{KeyGen}(1^n)$ and gives evk to $\mathcal{A}$. It initializes $T := \emptyset$.
2. The attacker $\mathcal{A}$ can adaptively submit arbitrarily many *authentication queries* of the form (b, τ) to the challenger. On each such query, if there is some $(\tau, \cdot) \in T$ (i.e. the label τ is not fresh) then the challenger ignores it. Else it updates $T := T \cup \{(\tau, b)\}$, associating the label τ with the authenticated bit b, and replies with $\sigma \leftarrow \mathsf{Auth}_{sk}(b, \tau)$.

3. Finally, the attacker outputs some *forgery* $(e^*, \mathcal{P}^* = (f^*, \tau_1^*, \ldots, \tau_k^*), \psi^*)$. The output of the game is 1 iff $\mathsf{Ver}_{sk}(e^*, \mathcal{P}^*, \psi^*) = \texttt{accept}$ *and* one of the following two conditions holds:

   - *Type I Forgery:* There is some $i \in [k]$ such that the label $(\tau_i^*, \cdot)$ does not appear in T. (*i.e., No bit was ever authenticated under the label τ_i^* involved in the forgery.*)
   - *Type II Forgery:* The set T contains tuples $(\tau_1^*, b_1), \ldots, (\tau_k^*, b_k)$, for some bits $b_1, \ldots, b_k \in \{0,1\}$ such that $f^*(b_1, \ldots, b_k) \neq e^*$. (*i.e., The labeled program $\mathcal{P}^*$ does not output e^* when executed on previously authenticated labeled data $b_1, \ldots, b_k$.*)

We say that a homomorphic authenticator scheme is *secure (without verification queries)* if, for any probabilistic polynomial-time $\mathcal{A}$, we have $\Pr[\mathsf{ForgeGame}^{\mathcal{A}}(1^n) = 1] \leq \mathsf{negl}(\lambda)$. We can also define a stronger variant, called *security with verification queries*, where we we insist that the above probability holds for a modified version of the game, in which the attacker can also adaptively make arbitrarily many verification queries of the form $(e, \mathcal{P}, \psi)$, and the challenger replies with $\mathsf{Ver}_{sk}(e, \mathcal{P}, \psi)$.

2.2 Homomorphic Encryption

A *fully homomorphic (public-key) encryption* (FHE) scheme is a quadruple of PPT algorithms $\mathsf{HE} = (\mathsf{HE.KeyGen}, \mathsf{HE.Enc}, \mathsf{HE.Dec}, \mathsf{HE.Eval})$ defined as follows.

- $\mathsf{HE.KeyGen}(1^n) \to (pk, evk, sk)$: Outputs a public encryption key pk, a public evaluation key evk and a secret decryption key sk.
- $\mathsf{HE.Enc}_{pk}(b) \to c$: Encrypts a bit $b \in \{0,1\}$ under public key pk. Outputs ciphertext c.
- $\mathsf{HE.Dec}_{sk}(c) \to b$: Decrypts ciphertext c using sk to a plaintext bit $b \in \{0,1\}$.
- $\mathsf{HE.Eval}_{evk}(g, c_1, \ldots, c_t) \to c^*$: The *deterministic evaluation algorithm* takes the evaluation key evk, a boolean circuit $g : \{0,1\}^t \to \{0,1\}$, and a set of t ciphertexts $c_1, \ldots, c_t$. It outputs the result ciphertext c^*.

An FHE should also satisfy the following properties.

Encryption Correctness. For all $b \in \{0,1\}$ we have:

$$\Pr\left[\mathsf{HE.Dec}_{sk}(\mathsf{HE.Enc}_{pk}(b)) = b \mid (pk, evk, sk) \leftarrow \mathsf{HE.KeyGen}(1^n)\right] = 1$$

Evaluation Correctness. For any (pk, evk, sk) in the support of $\mathsf{HE.KeyGen}(1^n)$, any ciphertexts $c_1, \ldots, c_t$ such that $\mathsf{HE.Dec}_{sk}(c_i) = b_i \in \{0,1\}$, and any circuit $g : \{0,1\}^t \to \{0,1\}$ we have

$$\mathsf{HE.Dec}_{sk}(\mathsf{HE.Eval}_{evk}(g, c_1, \ldots, c_t)) = g(b_1, \ldots, b_t).$$

Succinctness. We require that the ciphertext-size is always bounded by some fixed polynomial in the security parameter, and is independent of the size of the evaluated circuit or the number of inputs it takes. That is, there exists some polynomial $p(\cdot)$ such that, for any (pk, evk, sk) in the support of $\mathsf{HE.KeyGen}(1^n)$, the output-size of $\mathsf{HE.Enc}_{pk}(\cdot)$ and of $\mathsf{Eval}_{evk}(\cdot)$ is bounded by $p(n)$, for any choice of their inputs.

Semantic Security. Lastly, an FHE should satisfy the standard notion of *semantic security* for public-key encryption, where we consider the evaluation key evk as a part of the public key. That is, for any PPT attacker $\mathcal{A}$ we have:

$$\left| \Pr\left[\mathcal{A}(1^n, pk, evk, c_0) = 1\right] - \Pr\left[\mathcal{A}(1^n, pk, evk, c_1) = 1\right] \right| \leq \mathsf{negl}(n)$$

where the probability is over $(pk, evk, sk) \leftarrow \mathsf{KeyGen}(1^n), \{c_b \leftarrow \mathsf{HE.Enc}_{pk}(b)\}_{b \in \{0,1\}}$, and the coins of $\mathcal{A}$.

Canonical FHE. We can take any FHE scheme and make it *canonical*, meaning that the $\mathsf{HE.Eval}$ procedure just evaluates the circuit recursively, level-by-level and gate-by-gate. In particular, for any circuit $g : \{0,1\}^k \to \{0,1\}$ taking input bits $b_1, \ldots, b_k$, if the top gate of g is $h : \{0,1\}^t \to \{0,1\}$ and the inputs to h are computed by sub-circuits $f_1(b_{i_{1,1}}, \ldots b_{i_{1,k_1}}) \ldots, f_t(b_{i_{t,1}}, \ldots b_{i_{t,k_t}})$ then

$$\mathsf{HE.Eval}_{evk}(g, c_1, \ldots, c_k) = \mathsf{HE.Eval}_{evk}(h, c_1^*, \ldots, c_t^*)$$

where $c_j^* = \mathsf{HE.Eval}_{evk}(f_j, c_{i_{j,1}}, \ldots, c_{i_{j,k_j}})$ for $j \in [t]$. We also assume that, if g_{id} is the canonical identity circuit on one input, then $\mathsf{HE.Eval}_{evk}(g_{id}, c) = c$.

See, e.g., the works of [22, 31, 12, 11] for constructions of fully homomorphic encryption.

3 Constructing Homomorphic Authenticators

We now describe our construction of homomorphic authenticators. Although it closely follows our high-level description in the introduction, there are some differences. Most notably, the simple scheme in the introduction does not appropriately protect against *type I* forgeries. For example, it is possible that for the function g_0 which always outputs 0, the FHE scheme always outputs $\mathsf{HE.Eval}_{evk}(g_0, c) = C_0$ where C_0 is some fixed and known ciphertext encrypting 0. In that case, the attacker can always authenticate the output of the labeled-program $\mathcal{P}_0 = (g_0, \tau)$ for any label τ, even if he never saw any previously authenticated data under the label τ. This already classifies as a type I forgery. To fix this, we add an extra component to our tags that ensures that the attacker must have seen authentication tags for all of the underlying input labels. We describe this component below.

Hash Tree of a Circuit. If $g : \{0,1\}^k \to \{0,1\}$ is a circuit and $H : \{0,1\}^* \to \{0,1\}^m$ is some hash function, we define the *hash-tree of* g, denoted g^H, as a *Merkle-Tree* that has the same structure as the circuit g, but replaces all internal gates with the hash function H. More precisely, the hash-tree $g^H : (\{0,1\}^*)^k \to \{0,1\}^m$ is a function which takes as input strings $\nu_i \in \{0,1\}^*$ for each input wire of g. For every wire w in the circuit g, we define the *value of* $g^H(\nu_1, \ldots, \nu_k)$ *at* w inductively as:

- $val(w) = H(\nu_i)$ if w is the ith input wire of g.
- $val(w) = H(val(w_1), \ldots, val(w_t))$ if w is the output wire of some gate with input wires $w_1, \ldots, w_t$.

We define the output of the function $g^H(\nu_1, \ldots, \nu_k)$ to be its its value at the output wire of g.

Construction. Let $\mathsf{HE} = (\mathsf{HE.KeyGen}, \mathsf{HE.Enc}, \mathsf{HE.Dec}, \mathsf{HE.Eval})$ be a canonical fully homomorphic encryption scheme, where the encryption algorithm uses $r = r(n) = \omega(\log(n))$ random bits. Let $\{f_K : \{0,1\}^* \to \{0,1\}^{r(n)}\}_{K \in \{0,1\}^n}$ be a (variable-input-length) *pseudo-random function* PRF family. Let $\mathcal{H}$ be a family of (variable-length) *collision-resistant hash functions (CRHF)* $H : \{0,1\}^* \to \{0,1\}^{m(n)}$. We define the authenticator scheme $\Pi = (\mathsf{KeyGen}, \mathsf{Auth}, \mathsf{Eval}, \mathsf{Ver})$ as follows:

$\mathsf{KeyGen}(1^n)$**:** Choose a PRF key $K \leftarrow \{0,1\}^n$ and a CRHF $H \leftarrow \mathcal{H}$. Choose an encryption key $(pk, evk', sk') \leftarrow \mathsf{HE.KeyGen}(1^n)$. Select a subset $S \subseteq [n]$ by choosing whether to add each index $i \in [n]$ to the set S independently with probability $\frac{1}{2}$. Output $evk = (evk', H), sk = (pk, evk', H, sk', K, S)$.

$\mathsf{Auth}_{sk}(b, \tau)$**:** Given $b \in \{0,1\}$ and $\tau \in \{0,1\}^*$ do the following:

1. Choose random coins $rand_1, \ldots, rand_n$ by setting $rand_i = f_k((\tau, i))$. Set $\nu := f_K(\tau)$.
2. Create n ciphertexts $c_1, \ldots, c_n$ as follows. For $i \in [n] \setminus S$, choose $c_i = \mathsf{HE.Enc}_{pk}(b; rand_i)$ as encryptions of the bit b. For $i \in S$, choose $c_i = \mathsf{HE.Enc}_{pk}(0; rand_i)$ as encryption of 0.
3. Output $\sigma = (c_1, \ldots, c_n, \nu)$.

$\mathsf{Eval}_{evk}(g, \boldsymbol{\sigma})$**:** Given $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_t)$, parse each $\sigma_j = (c_{1,j}, \ldots, c_{n,j}, \nu_j)$.

- For each $i \in [n]$, compute $c_i^* = \mathsf{HE.Eval}_{evk'}(g, c_{i,1}, \ldots, c_{i,t})$.
- Compute $\nu^* = g^H(\nu_1, \ldots, \nu_t)$ to be the output of the hash-tree of g evaluated at $\nu_1, \ldots \nu_t$.

Output $\psi = (c_1^*, \ldots, c_n^*, \nu^*)$.

$\mathsf{Ver}_{sk}(e, \mathcal{P}, \psi)$**:** Parse $\mathcal{P} = (g, \tau_1, \ldots, \tau_t)$ and $\psi = (c_1^*, \ldots, c_n^*, \nu^*)$.

1. Compute $\nu_1 := f_K(\tau_1), \ldots, \nu_t = f_K(\tau_t)$ and $\nu' := g^H(\nu_1, \ldots, \nu_t)$. If $\nu' \neq \nu^*$, output `reject`.
2. For $i \in S$, $j \in [t]$, compute $rand_{i,j} := f_K((\tau_j, i))$ and set $c_{i,j} := \mathsf{HE.Enc}_{pk}(0; rand_{i,j})$. For each $i \in S$, evaluate $c_i' := \mathsf{HE.Eval}_{evk'}(g, c_{i,1}, \ldots, c_{i,t})$ and if $c_i' \neq c_i^*$ output `reject`.
3. For each $i \in [n] \setminus S$, decrypt $e_i := \mathsf{HE.Dec}_{sk'}(c_i^*)$ and if $e \neq e_i$ output `reject`.

If the above doesn't reject, output `accept`.

Theorem 3.1. *If $\{f_K\}$ is a PRF family, $\mathcal{H}$ is a CRHF family and* HE *is a semantically secure canonical FHE, then the homomorphic authenticator scheme* Π *is secure without verification queries.*

Proof. It is easy to verify that the *authentication correctness* of Π just follows from the encryption correctness of HE, and the *evaluation correctness* of Π follows from that of HE, along with the fact that HE is *canonical*.

We now prove the security of Π (without verification queries). Let $\mathcal{A}$ be some PPT attacker and let $\mu(n) = \Pr[\mathsf{ForgeGame}^{\mathcal{A}}(1^n) = 1]$. We use a series of hybrid games modifying ForgeGame to prove that $\mu(n)$ must be negligible.

Game_1**:** We modify ForgeGame so as to replace the PRF outputs with truly random consistent values. That is, the challenger replaces all calls to f_k needed to answer authentication queries and to check the winning condition $\mathsf{Ver}_{sk}(e^*, \mathcal{P}^*, \psi^*) \stackrel{?}{=} \mathtt{accept}$, with calls to a completely random function $F : \{0,1\}^* \to \{0,1\}^{r(n)}$, whose outputs it chooses efficiently "on the fly".

By the pseudo-randomness of $\{f_k\}$, we must have $\Pr[\mathsf{Game}_1^{\mathcal{A}}(n) = 1] \geq \mu(n) - \mathsf{negl}(n)$.

Game_2**:** We now define Game_2 by modifying the winning condition, so that the attacker only wins (the game outputs 1) if the attacker outputs a valid *type (II) forgery* (and the game outputs 0 on a type I forgery). Let E be the event that attacker wins with a type I forgery in Game_1. Then we claim that, under the collision-resistance of $\mathcal{H}$, we have $\Pr[E] = \mathsf{negl}(n)$ and therefore:

$$\Pr[\mathsf{Game}_2^{\mathcal{A}}(n) = 1] \geq \Pr[\mathsf{Game}_1^{\mathcal{A}}(n) = 1] - \Pr[E] \geq \mu(n) - \mathsf{negl}(n)$$

Assume otherwise that $\Pr[E]$ is non-negligible. Recall that the event E only occurs when the attacker submits a forgery

$$e^*, \mathcal{P}^* = (g, \tau_1^*, \ldots, \tau_t^*), \psi^* = (c_1^*, \ldots, c_n^*, \nu^*)$$

such that the attacker never asked any authentication query containing one of the labels τ_j^* for some $j \in [t]$, and verification accepts. During the computation of $\mathsf{Ver}_{sk}(e^*, \mathcal{P}^*, \psi^*)$, when checking that verification accepts in Game_1, the challenger chooses the value $\nu_j = F(\tau_j^*) \leftarrow \{0,1\}^{r(n)}$ freshly at random, since the label τ_j^* was never queried before. If we rewind and re-sample $\nu_j' \leftarrow \{0,1\}^{r(n)}$ freshly an independently at random again, then the probability that verification accepts *both* times, which we denote by the event E^2, is at least $\Pr[E^2] \geq \Pr[E]^2$. Let C be the event that E^2 occurs *and* the values $\nu_j \neq \nu_j'$ are distinct, so that $\Pr[C] \geq \Pr[E]^2 - 2^{-r(n)} = \Pr[E]^2 - \mathsf{negl}(n)$ is *non-negligible*. When the event C occurs then we must have $\nu^* = g^H(\nu_1, \ldots, \nu_j, \ldots, \nu_t) = g^H(\nu_1, \ldots, \nu_j', \ldots, \nu_t)$

which immediately gives us some collision on H at some level of the hash tree g^H. Therefore, $\mathcal{A}$ can be used to efficiently find collisions on $H \xleftarrow{\$} \mathcal{H}$ with non-negligible probability, which gives us a contradiction.

Game_3: In Game_3 we modify the winning condition yet again. When answering authentication queries, the challenger now also remembers the tag σ that it uses, storing (τ, b, σ) in T. If the attacker outputs a type II forgery

$$e^*, \mathcal{P}^* = (g, \tau_1^*, \ldots, \tau_t^*), \psi^* = (c_1^*, \ldots, c_n^*, \nu^*),$$

we modify how the challenger checks $\mathsf{Ver}_{sk}(e^*, \mathcal{P}^*, \psi^*) \stackrel{?}{=} \texttt{accept}$. Recall that for a type II forgery, the tags τ_i^* were previously used in authentication queries, so that T must contain some tuples

$$((\tau_1^*, b_1, \sigma_1 = (c_{1,1}, \ldots, c_{n,1}, \nu_1)), \ldots, (\tau_t^*, b_t, \sigma_t = (c_{1,t}, \ldots, c_{n,t}, \nu_t)).$$

Let $\hat{c}_i := \mathsf{HE.Eval}_{evk}(g, c_{i,1}, \ldots, c_{i,t})$ for $i \in [n]$ be the *"honest ciphertexts"* that would be included in an honestly generated tag ψ for the program $\mathcal{P}^*$. In Game_3, we replace steps (2), (3) of the verification procedure as follows:

2'. For each $i \in S$: if $\hat{c}_i \neq c_i^*$ then output `reject`.

3'. For each $i \in [n] \setminus S$: if $\hat{c}_i = c_i^*$ then output `reject`.

Notice that step (2') is actually the same as the original step (2) used in Game_2, since in both cases we just check the forgery ciphertexts c_i^* against the honest ciphertexts $c_i' = \hat{c}_i$. The only difference is that previously we re-computed c_i' from scratch by re-encrypting $c_{i,j}$, and now we compute $\hat{c}_i$ using the stored ciphertexts $c_{i,j}$ in T (but the values are equivalent).

Step (3'), in Game_3 is different from the original step (3) in Game_2. In the original step (3), we decrypted the forgery ciphertexts for $i \in [n] \setminus S$ and checked that they decrypt to the claimed output $e^* \stackrel{?}{=} \mathsf{Dec}_{sk'}(c_i^*)$. Let $e = g(b_1, \ldots, b_t)$ be the correct output of g on previously authenticated data. In an accepting type II forgery, we must have $e^* \neq e$ but the decryption of the "honest ciphertexts" will satisfy $\mathsf{HE.Dec}_{sk'}(\hat{c}_i) = e$. So it must be the case that $c_i^* \neq \hat{c}_i$ for all $i \in [n] \setminus S$ for any accepting type II forgery in Game_2. Therefore, any type II forgery that's accepting in Game_2 is also accepting in Game_3 and hence: $\Pr[\mathsf{Game}_3^{\mathcal{A}}(n) = 1] \geq \Pr[\mathsf{Game}_2^{\mathcal{A}}(n) = 1] \geq \mu(n) - \mathsf{negl}(n)$.

Game_4: We modify Game_3 so that, when answering *authentication queries*, the challenger computes all of the ciphertexts c_i (even for $i \in S$) as encryptions of the correct bit b in step (2) of the authentication procedure. In particular, the choice of S is ignored when answering authentication queries.

We claim that:

$$\Pr[\mathsf{Game}_4^{\mathcal{A}}(n) = 1] \geq \Pr[\mathsf{Game}_3^{\mathcal{A}}(n) = 1] - \mathsf{negl}(n) \geq \mu(n) - \mathsf{negl}(n). \tag{1}$$

This simply follows by the semantic security of the encryption scheme HE. Given challenge ciphertexts which either encrypt the attacker's bits b or 0, we can embed these into the authentication procedures for positions $i \in S$ and either simulate Game_3 or Game_4. We can efficiently determine if the output of the game is 1, since the decryption secret key sk' is never used in these games. Therefore, if the above didn't hold, the attacker $\mathcal{A}$ would break semantic security.

Negligible Advantage. We now claim that, information theoretically, $\Pr[\mathsf{Game}_4^{\mathcal{A}}(n) = 1] \leq 2^{-n}$. Together with equation (1), this shows that $\mu(n) \leq 2^{-n} + \mathsf{negl}(n) = \mathsf{negl}(n)$, as we wanted to show.

In Game_4, the choice of the set $S \subseteq [n]$ is not used at all when answering authentication queries and so we can think of the challenger as only picking the set S during verification. For any type

II forgery $e^*, \mathcal{P}^*, \psi^* = (c_1^*, \ldots, c_n^*, \nu^*)$, let $c'_1, \ldots, c'_n$ be the *"honest ciphertexts"* that would be included in an honestly generated tag ψ for the output of $\mathcal{P}^*$ (see description of Game_3). Let $S' := \{i \in [n] \ : \ c_i^* = c'_i\}$ be the indices on which the forged and honest ciphertexts match. The attacker only wins if steps (2'), (3') of verification pass, which only occurs if $S = S'$. But this only occurs with probability 2^{-n} over the random choice of S.

□

Fully Homomorphic Authenticator-Encryption. We can also extend homomorphic message authenticators to homomorphic *authenticator-encryption*. Given the secret key sk, it should be possible to *decrypt* the correct bit b from the tag authenticating it, but given only the evaluation key evk, the tags should not reveal any information about the authenticated bits.

We can allow decryption generically. Take any homomorphic authenticator scheme (KeyGen, Auth, Eval, Ver) and define $\mathsf{VerDec}(\mathcal{P}, \psi) \to \{0, 1, \texttt{reject}\}$ as follows: run $\mathsf{Ver}(e, \mathcal{P}, \psi)$ for both choices of $e \in \{0, 1\}$ and, if exactly one of the runs is accepting, return the corresponding e, else return `reject`.

We notice that our specific construction of homomorphic authenticators already provides *chosen-plaintext-attack (CPA)* security for the above authenticator-encryption scheme. Even if the attacker gets evk and access to the authentication oracle $\mathsf{Auth}_{sk}(\cdot, \cdot)$, if he later sees a tag $\sigma \leftarrow \mathsf{Auth}_{sk}(b, \tau)$ for a fresh label τ, he cannot distinguish between the cases $b = 0$ and $b = 1$.

4 Security with Verification Queries?

The scheme presented in Section 3 only provides security *without* verification queries, and it remains an interesting open problem to construct fully homomorphic authenticators that allow for the stronger notion of security *with* verification queries. In this section we make some observations on possible strategies to resolve this question.

An Efficient Attack. Firstly, we note that there is an efficient attack against our basic scheme from Section 3, in the setting of security *with* verification queries.

The attacker gets evk and makes a single authentication query to get a tag $\sigma \leftarrow \mathsf{Auth}_{sk}(1, \tau)$, authenticating the bit $b = 1$ under some arbitrary label τ. We parse $\sigma = (c_1, \ldots, c_n, \nu)$.

The attacker then makes several verification queries whose aim is to learn the secret set $S \subseteq [n]$. It does so as follows: for each $i \in [n]$ he computes c'_i by adding an encryption of 0 to c_i (or performing any homomorphic operation that changes the ciphertext while preserving the plaintext) and sets the modified tag σ_i to be the same as σ, but with c_i replaced by c'_i. Then, for each $i \in [n]$, the attacker makes a verification query $\mathsf{Ver}_{sk}(1, \mathcal{I}_\tau, \sigma_i)$ to test if the modified tag σ_i is valid. If the query rejects then the attacker guesses that $i \in S$ and otherwise guesses $i \notin S$. With overwhelming probability the attacker correctly recovers the entire set S.

Now the attacker can construct a type II forgery for the identity program $\mathcal{I}_\tau$, claiming that its value is 0 (recall, that we previously authenticated $b = 1$ under the label τ). To do so, the attacker takes the tag σ and, for $i \notin S$, replaces the ciphertexts c_i with fresh encryptions of 0. Let's call the resulting modified tag σ^*. Then it's easy to see that $(0, \mathcal{I}_\tau, \sigma^*)$ is a valid type II forgery, breaking the scheme.

Bounded Verification Queries. The above attack requires n verification queries to break the scheme. It is relatively easy to show that the scheme is secure against $O(\log(n))$ verification queries. In particular, the attacker only gets $O(\log(n))$ bits of information about the set S, which is not enough to break the scheme. Similarly, for any a-priori bound q, we can modify our scheme so that the tags contain $n + q$ ciphertexts, and then we get security against q verification queries.

4.1 Construction via Randomness-Homomorphic Encryption

We now show that it is possible to achieve the stronger notion of authenticator security *with* unbounded verification queries, given an FHE scheme that satisfies a new property, which we call *randomness homomorphism*. This notion seems independently interesting, but, unfortunately, is not satisfied by any of the known FHE schemes. Nevertheless, we hope that it yields a promising approach. We first define this property and then show how to modify our scheme from Section 3 to take advantage of it.

Definition 4.1 (Randomness Homomorphism)**.** *An FHE scheme* $(\mathsf{HE.KeyGen}, \mathsf{HE.Enc}, \mathsf{HE.Dec}, \mathsf{HE.Eval})$ *satisfies* randomness homomorphism *if there exists an efficient algorithm* $\mathbf{R}\mathsf{Eval}$ *such that the following holds. For any* $(pk, evk, sk) \leftarrow \mathsf{KeyGen}(1^n)$*, any circuit* $f : \{0,1\}^t \to \{0,1\}$ *and any ciphertexts* $c_1, \ldots, c_t$ *such that* $c_i = \mathsf{HE.Enc}_{pk}(b_i; rand_i)$ *encrypts the bits* b_i *under randomness* $rand_i$*, we have:*

$$\mathsf{HE.Eval}_{evk}(f, c_1, \ldots, c_t) = \mathsf{HE.Enc}_{pk}(f(b_1, \ldots, b_t); \mathbf{R}\mathsf{Eval}_{evk}(f, rand_1, \ldots, rand_t)).$$

In other words, "randomness homomorphism" ensures that the randomness in the evaluated ciphertext *only* depends on the randomness of the underlying ciphertexts and *not* on their encrypted bits. We note that the above definition of randomness homomorphism can be achieved for some *singly-homomorphic* encryption schemes such as ElGamal. However, we do not know if it can be achieved for any FHE scheme.

The Modified Authenticator Scheme. We now modify our authenticator scheme Π from Section 3, to construct a scheme Π_{Ver} that takes advantage of the "randomness homomorphism" property of the underlying FHE to get security with verification queries. We actually define Π_{Ver} by only modifying the verification procedure of Π (and in particular only step 3) which is now defined as follows:

$\mathsf{Ver}_{sk}(e, \mathcal{P}, \psi)$**:** Parse $\mathcal{P} = (g, \tau_1, \ldots, \tau_t)$ and $\psi = (c_1^*, \ldots, c_n^*, \nu^*)$.

1. Compute $\nu_1 := f_K(\tau_1), \ldots, \nu_t = f_K(\tau_t)$. If $\nu^* \neq g^H(\nu_1, \ldots, \nu_t)$ then output `reject`.
2. For $i \in [n]$, $j \in [t]$, compute $rand_{i,j} := f_K((\tau_j, i))$ and, for $i \in S$, set $c_{i,j} := \mathsf{HE.Enc}_{pk}(0; rand_{i,j})$. For each $i \in S$, evaluate $c_i' := \mathsf{HE.Eval}_{evk'}(g, c_{i,1}, \ldots, c_{i,t})$ and if $c_i' \neq c_i^*$, output `reject`.
3. For each $i \in [n] \setminus S$, evaluate $c_i' = \mathsf{HE.Enc}_{pk}(e; \mathbf{R}\mathsf{Eval}_{evk'}(g, rand_{i,1}, \ldots, rand_{i,t}))$ and if $c_i' \neq c_i^*$, output `reject`.

If the above doesn't reject, output `accept`.

With this modification, the construction Π_{Ver} satisfies security with verification queries. We state this theorem below and provide the proof in Appendix A.

Theorem 4.2. *If* $\{f_K\}$ *is a PRF family,* $\mathcal{H}$ *is a CRHF family and* HE *is a semantically secure canonical FHE with* randomness homomorphism*, then the above homomorphic authenticator scheme* Π_{Ver} *is secure* with *verification queries.*

5 Improving Verification Complexity

One of the main limitations of our homomorphic authenticator scheme from the previous sections is that the complexity of the verification algorithm is no better than that of executing the program $\mathcal{P}$. Therefore, although the scheme saves on the *communication complexity* of transmitting the input data to $\mathcal{P}$, it does not save on the *computation complexity* of executing $\mathcal{P}$. As we discussed in the introduction, works in the area of *delegating computation* obtain efficient verification whose complexity is independent (or at least much smaller than) the computation of $\mathcal{P}$, but require that the user knows the entire input x. We explore the idea of "marrying" these two techniques by *delegating* the computation required to *verify* the authentication tag in our homomorphic authenticator scheme.

Delegating Verification. Firstly, we notice that the verification procedure $\mathsf{Ver}_{sk}(e, \mathcal{P}, \psi)$ of our scheme as described in Section 3 has special structure. The only expensive computation (proportional to the complexity of the the program $\mathcal{P}$) is *independent* of the tag ψ. In particular, this computation uses the secret key sk and the program $\mathcal{P}$ to compute the output of the "hash-tree" ν' and the ciphertexts c'_i for $i \in S$ derived by evaluating $\mathcal{P}$ over the pseudorandom encryptions of 0. We call this computation $\texttt{Expensive}(\mathcal{P}, sk)$. Given the outputs of $\texttt{Expensive}(\mathcal{P}, sk)$, the rest of the verification procedure is efficient. Therefore, we can delegate the computation of $\texttt{Expensive}(\mathcal{P}, sk)$ prior to knowing the tag ψ that needs to be verified. One issue is that the computation does depend on the secret key sk, which needs to be kept private. We note that we can always (generically) keep the input of a delegated computation private by encrypting it under an FHE scheme. In our context, we can encrypt the value sk under an independently chosen FHE key, and publishing this ciphertext C_{sk} in the evaluation key. We can then delegate the computation $\texttt{Expensive}'(\mathcal{P}, C_{sk})$ which takes $\mathcal{P}$ as an input and homomorphically executed $\texttt{Expensive}(\mathcal{P}, sk)$.

We now explore the advantages of using the above approach with some concrete delegation schemes.

Using SNARGs. We can use succinct non-interactive arguments (SNARGs), which may not necessarily be arguments of knowledge (SNARKs). For example, if we instantiate the random-oracle CS-Proofs of Micali [28] with some cryptographic hash function, it may be more reasonable to assume that we get a SNARG, than it is to assume that we get a SNARK. Such SNARGs already allow for completely non-interactive delegation of computation (assuming the computation has a short uniform description). Therefore, using the above approach, we get homomorphic authenticators satisfying our original non-interactive syntax and security, but also allowing efficient verification for programs $\mathcal{P}$ having a short uniform description. During *evaluation*, we simply also have the server compute $\texttt{Expensive}'(\mathcal{P}, C_{sk})$ and provide a SNARG proof π that it was done correctly. Recall that, in the introduction, we described a significantly simpler solution to the problem of efficiently verifiable homomorphic authenticators (and signatures) using SNARKs directly. Therefore, the main advantage of the above technique is that now we only require SNARGs (a weaker primitive).

Using Delegation with Pre-Processing. Alternatively, we can use the delegation techniques in the "pre-processing" model [19, 14, 2] to outsource the computation of $\texttt{Expensive}'(\mathcal{P}, C_{sk})$ where $\mathcal{P}$ is given as an input. This scheme will have many of the same advantages and disadvantages as the approach of using delegation with "pre-processing" directly to outsource the data (see a description of this latter approach in Section 1.3). In particular, in both approaches, the verification will be efficient but the scheme will now require one round of interaction, where the user needs to create a challenge $\mathit{chall}(\mathcal{P})$ for the computation $\mathcal{P}$ that she wants to verify. The main advantages of our approach, combining delegation with homomorphic authenticators, over a direct delegation-based scheme is the following. When using delegation directly, the user needs to outsource all of the data in one shot and remember some short partial information about it; now the user can arbitrarily authenticate fresh labeled data "on the fly" and verify computations over all of it using a single independent secret key.

6 Conclusions

In this work we give the first solution to fully homomorphic message authenticators, allowing a user to verify computations over previously authenticated data. The authentication tag is short, independently of the size of the authenticated input to the computation. Our work leaves many interesting open questions. Perhaps the most ambitious one is to construct fully homomorphic signatures with *public verification*. Less ambitiously, construct fully homomorphic authenticators that allow an unbounded number of verification queries. We propose an avenue for doing so via an FHE with *randomness homomorphism*, and so an independently interesting open problem would be to construct the latter. Lastly, it would be interesting

to improve the efficiency of our construction. One pressing question is to make the verification complexity independent of the complexity of the program $\mathcal{P}$ while maintaining all of the advantages of our scheme (standard assumptions, no interaction). But a less ambitious, still interesting question is to just reduce the tag size from $O(n)$ ciphertexts to something smaller, say a single ciphertext.

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A Proof of Theorem 4.2

We restate and prove Theorem 4.2 below.

Theorem 4.2. *If $\{f_K\}$ is a PRF family, $\mathcal{H}$ is a CRHF family and* HE *is a semantically secure canonical FHE with* randomness homomorphism*, then the scheme Π_{Ver} is secure* with *verification queries.*

Proof. We let $\mathsf{ForgeGameVer}$ to denote the forgery game with verification queries (see Section 2.1). For ease of exposition, we say that a verification query $(e, \mathcal{P} = (g, \tau_1, \ldots, \tau_t), \psi)$ made during the course of the game is a *type I* query, if one of the labels τ_i is *fresh* and was never involved in any prior authentication query. Otherwise, we say that it is a *type II* query. This matches the terminology we used for the attacker's forgery.

Let $\mathcal{A}$ be some PPT attacker and let $\mu(n) = \Pr[\mathsf{ForgeGameVer}^{\mathcal{A}}(1^n) = 1]$. We use a series of hybrid games, to prove that $\mu(n)$ is negligible. The hybrid games $\mathsf{Game}_1, \ldots, \mathsf{Game}_4$ are very much analogous to those used in the proof of Theorem 3.1, and therefore we focus on the changes.

Game_1**:** Replace the PRF with a truly random function whose outputs are chosen "on-the-fly". This is indistinguishable by PRF security.

Game_2**:** Automatically output `reject` on all *type I* verification queries, including during the verification of the final forgery. The proof that Game_1 and Game_2 are indistinguishable relies on the collision-resistance of H and is essentially the same as the analogous step in the proof of Theorem 3.1.

Game_3**:** We now modify how we answer all *type II* verification queries, including the verification of the final forgery. When answering authentication queries, the challenger now also remembers the corresponding tag σ. For any type II verification query $e^*, \mathcal{P}^* = (g, \tau_1^*, \ldots, \tau_t^*), \psi^* = (c_1^*, \ldots, c_n^*, \nu^*)$, the challenger can now recall the correct bits b_j and tags $\sigma_j = (c_{1,j}, \ldots, c_{n,j}, \nu_j)$ associated with the input labels τ_j^* for $j \in [t]$. For $i \in [n]$, let $\hat{c}_i := \mathsf{HE.Eval}_{evk}(g, c_{i,1}, \ldots, c_{i,t})$ be the *honest tag ciphertexts* for the program $\mathcal{P}^*$, and let let $e = g(b_1, \ldots, b_t)$ be the *honest output* of $\mathcal{P}^*$.

In Game_3, we replace steps (2), (3) of the verification procedure as follows:

2'. For each $i \in S$: if $\hat{c}_i \neq c_i^*$ then output `reject`.

3'.a If $e = e^*$:

- For each $i \in [n] \setminus S$: if $\hat{c}_i \neq c_i^*$ then output `reject`.

3'.b If $e \neq e^*$.

- For each $i \in [n] \setminus S$: if $\hat{c}_i = c_i^*$ then output `reject`.
- If above step doesn't reject, then the game stops and outputs 1.

Notice that step $2'$ is actually equivalent to step 2 of the original verification. Also, when $e = e^*$, then step $3'.a$ is equivalent to step 3 of original verification because[7]

$$\hat{c}_i = \mathsf{HE.Eval}_{evk}(g, c_{i,1}, \ldots, c_{i,t}) = \mathsf{HE.Enc}_{pk}(e^*; \mathbf{R}\mathsf{Eval}_{evk'}(g, rand_{i,1}, \ldots, rand_{i,t})) = c'_i.$$

Therefore, the only difference occurs when $e^* \neq e$, in which case step $3'.b$ differs from step 3. Whenever $3'.b$ rejects, it means that there is some $i \in [n] \setminus S$ such that:

$$c_i^* = \hat{c}_i \neq \mathsf{HE.Enc}_{pk}(e^*; \mathbf{R}\mathsf{Eval}_{evk'}(g, rand_{i,1}, \ldots, rand_{i,t}))$$

which means that step 3 in the original verification would have rejected as well. Whenever $3'.b$ does not reject, the game automatically outputs 1. Therefore, whenever Game_2 and Game_3 differ, Game_3 just outputs 1 and hence: $\Pr[\mathsf{Game}_3^{\mathcal{A}}(n) = 1] \geq \Pr[\mathsf{Game}_2^{\mathcal{A}}(n) = 1] \geq \mu(n) - \mathsf{negl}(n)$.

Game_4: We modify Game_3 so that, when answering *authentication queries*, the challenger computes all of the ciphertexts c_i (even for $i \in S$) as encryptions of the correct bit b in step (2) of the authentication procedure. In particular, the choice of S is ignored when answering authentication queries. The indistinguishability follows from the semantic security of the encryption scheme.

Game_5: Assume the attacker makes $q = q(n)$ verification queries during the course of the game, where we include the final forgery as the last verification query. We now define q hybrid games $\mathsf{Game}_{4.i}$ where we change how the challenger responds to the first i verification queries. In particular, for the first i queries, if the query is *type II*, we replace step $3'.b$ of the verification procedure (see definition of Game_3) to just always output `reject` when $e^* \neq e$. We define $\mathsf{Game}_{4.0} = \mathsf{Game}_4$ and $\mathsf{Game}_{4.q} = \mathsf{Game}_5$. Notice that this means that Game_5 *never* outputs 1 since the verification of any type II verification query with $e^* \neq e$ (which must be true for the forgery) is rejecting.

We claim that for $i \in [q]$, we have $\Pr[\mathsf{Game}_{4.i}^{\mathcal{A}}(n) = 1] \geq \Pr[\mathsf{Game}_{4.(i-1)}^{\mathcal{A}}(n) = 1] - 2^{-n}$. We prove this via an information theoretic argument. Let E_i be the event that the ith verification query in $\mathsf{Game}_{4.(i-1)}$ is a type II query with $e^* \neq e$ and the output is `accept`. Since the two games are the same as long as E_i doesn't occur, it is sufficient to prove $\Pr[E_i] \leq 2^{-n}$. To see this, notice that the choice of S is not used anywhere in $\mathsf{Game}_{4.(i-1)}$ prior to verifying the ith query.[8] Therefore, we can think of the challenger as choosing S only at this time. For *any* choice of a type II verification query that the attacker can make for its ith query, let c_j^* be the ciphertexts contained in the queried tag and $\hat{c}_j$ be the honest ciphertext (as defined in description of Game_3) for $j \in [n]$. Let $S' := \{j \in [n] : c_j^* = \hat{c}_j\}$. Then, $\Pr[E_i] = \Pr[S' = S] \leq 2^{-n}$ over the random choice of S.

Finally, this means that $0 = \Pr[\mathsf{Game}_5^{\mathcal{A}}(n) = 1] \geq \Pr[\mathsf{Game}_4^{\mathcal{A}}(n) = 1] - q2^{-n} \geq \mu(n) - \mathsf{negl}(n)$ Therefore the attacker's original success probability in $\mathsf{ForgeGameVer}$ is at most $\mu(n) = \mathsf{negl}(n)$ as we wanted to show. □

[7] This is the only place where we use randomness homomorphism and this provides for the main difference in the scheme and the proof from that of Section 3. We could *not* say this for the verification procedure used in Section 3.

[8] In Game_3, we already defined steps 2' and 3'.a of verification to treat all indices $i \in [n]$ equally and step 3'.b always rejects. In Game_4, we changed authentication to treat all indices $i \in [n]$ equally.