A note on generalized bent criteria for Boolean functions

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***Abstract*—In this paper, we consider the spectra of Boolean functions with respect to the action of unitary transforms obtained by taking tensor products of the Hadamard, denoted by H, and the nega–Hadamard, denoted by N, kernels. The set of all such transforms is denoted by $\{H,N\}^n$. A Boolean function is said to be bent$_4$ if its spectrum with respect to at least one unitary transform in $\{H,N\}^n$ is flat. We prove that the maximum possible algebraic degree of a bent$_4$ function on n variables is $\lceil \frac{n}{2} \rceil$, and hence solve an open problem posed by Riera and Parker [cf. IEEE-IT: 52(2)(2006) 4142–4159]. We obtain a relationship between bent and bent$_4$ functions which is a generalization of the relationship between bent and negabent Boolean functions proved by Parker and Pott [cf. LNCS: 4893(2007) 9–23].**

Keywords: Walsh–Hadamard transform, nega–Hadamard transform, bent function, bent$_4$ function, algebraic degree.

I. Introduction

Let us denote the set of integers, real numbers and complex numbers by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively and let the ring of integers modulo r be denoted by $\mathbb{Z}_r$. The vector space $\mathbb{Z}_2^n$ is the space of all n-tuples $\mathbf{x} = (x_n,\dots,x_1)$ of elements from $\mathbb{Z}_2$ with the standard operations. By '$+$' we denote the addition over $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, whereas '$\oplus$' denotes the addition over $\mathbb{Z}_2^n$ for all $n \geq 1$. Addition modulo q is denoted by '$+$' and it is understood from the context. If $\mathbf{x} = (x_n,\dots,x_1)$ and $\mathbf{y} = (y_n,\dots,y_1)$ are in $\mathbb{Z}_2^n$, we define the scalar (or inner) product by $\mathbf{x}\cdot\mathbf{y} = x_ny_n \oplus\cdots\oplus x_2y_2 \oplus x_1y_1$. The cardinality of a set S is denoted by $|S|$. If $z = a + b\imath \in \mathbb{C}$, then $|z| = \sqrt{a^2+b^2}$ denotes the absolute value of z, and $\overline{z} = a - b\imath$ denotes the complex conjugate of z, where $\imath^2 = -1$, and $a, b \in \mathbb{R}$.

We call any function from $\mathbb{Z}_2^n$ to $\mathbb{Z}_2$ a *Boolean function* on n variables and denoted the set of all Boolean functions by $\mathcal{B}_n$. In general any function from $\mathbb{Z}_2^n$ to $\mathbb{Z}_q$ ($q \geq 2$ a positive integer) is said to be a *generalized Boolean function* on n variables [7], the set of all such functions being denoted by $\mathcal{GB}_n^q$. Clearly $\mathcal{GB}_n^2 = \mathcal{B}_n$. For any $f \in \mathcal{B}_n$, the algebraic normal form (ANF) is

$$f(x_n,\dots,x_1) = \bigoplus_{\mathbf{a}=(a_n,\dots,a_1)\in\mathbb{Z}_2^n} \mu_{\mathbf{a}}\Big(\prod_{i=1}^{n} x_i^{a_i}\Big) \qquad (1)$$

where $\mu_{\mathbf{a}} \in \mathbb{Z}_2$, for all $\mathbf{a} \in \mathbb{Z}_2^n$. For any $\mathbf{a} \in \mathbb{Z}_2^n$, $wt(\mathbf{a}) := \sum_{i=1}^n a_i$ is the Hamming weight. The algebraic degree of f, $\deg(f) = \max\{wt(\mathbf{a}) : \mathbf{a} \in \mathbb{Z}_2^n, \mu_{\mathbf{a}} \neq 0\}$.

Now, let $q \geq 2$ be an integer, and let $\zeta = e^{2\pi\imath/q}$ be the complex q-primitive root of unity. The (generalized) *Walsh–Hadamard transform* of $f \in \mathcal{GB}_n^q$ at any point $\mathbf{u} \in \mathbb{Z}_2^n$ is the complex valued function

$$\mathcal{H}_f(\mathbf{u}) = 2^{-\frac{n}{2}} \sum_{\mathbf{x}\in\mathbb{Z}_2^n} \zeta^{f(\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}}. \qquad (2)$$

The inverse of the Walsh–Hadamard transform is given by

$$\zeta^{f(\mathbf{y})} = 2^{-\frac{n}{2}} \sum_{\mathbf{u}\in\mathbb{Z}_2^n} \mathcal{H}_f(\mathbf{u})(-1)^{\mathbf{u}\cdot\mathbf{y}}. \qquad (3)$$

If $q=2$, we obtain the (normalized) *Walsh–Hadamard transform* of $f \in \mathcal{B}_n$. A function $f \in \mathcal{GB}_n^q$ is a *generalized bent* function if $|\mathcal{H}_f(\mathbf{u})| = 1$ for all $\mathbf{u} \in \mathbb{Z}_2^n$. When $q=2$, then f is said to be bent (bent functions exist for n even, only).

The *nega–Hadamard transform* of $f \in \mathcal{B}_n$ at any vector $\mathbf{u} \in \mathbb{Z}_2^n$ is the complex valued function

$$\mathcal{N}_f(\mathbf{u}) = 2^{-\frac{n}{2}} \sum_{\mathbf{x}\in\mathbb{Z}_2^n} (-1)^{f(\mathbf{x})\oplus\mathbf{u}\cdot\mathbf{x}} \imath^{wt(\mathbf{x})}. \quad (4)$$

A function $f \in \mathcal{B}_n$ is said to be *negabent* if and only if $|\mathcal{N}_f(\mathbf{u})| = 1$ for all $\mathbf{u} \in \mathbb{Z}_2^n$. If $f \in \mathcal{B}_n$, then the inverse of the nega–Hadamard transform $\mathcal{N}_f$ is

$$(-1)^{f(\mathbf{y})} = 2^{-\frac{n}{2}} \imath^{-wt(\mathbf{y})} \sum_{\mathbf{u}\in\mathbb{Z}_2^n} \mathcal{N}_f(\mathbf{u})(-1)^{\mathbf{y}\cdot\mathbf{u}}, \quad (5)$$

for all $\mathbf{y} \in \mathbb{Z}_2^n$. We recall the following result.

Proposition 1: [9, Lemma 1] We have

$$\sum_{\mathbf{x}\in\mathbb{Z}_2^n} (-1)^{\mathbf{u}\cdot\mathbf{x}} \imath^{wt(\mathbf{x})} = 2^{\frac{n}{2}} \omega^n \imath^{-wt(\mathbf{u})}, \quad (6)$$

where $\omega = (1+\imath)/\sqrt{2}$ is a primitive 8th root of unity.

The Hadamard kernel, the nega–Hadamard kernel and the identity transform on $\mathbb{Z}_2^2$, denoted by H, N and I, respectively, are as follows.

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \imath \\ 1 & -\imath \end{pmatrix}$$

and

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The set of 2^n different unitary transforms that are obtained by performing tensor products H and N, n times in any possible sequence is denoted by $\{H,N\}^n$. If $\mathbf{R}_H$ and $\mathbf{R}_N$ partition $\{1,\ldots,n\}$ then the unitary transformation, U of dimension $2^n \times 2^n$, corresponding to this partition is

$$U = \prod_{j\in\mathbf{R}_H} H_j \prod_{j\in\mathbf{R}_N} N_j \quad (7)$$

where

$$K_j = I \otimes I \otimes \ldots \otimes I \otimes K \otimes I \otimes \ldots \otimes I$$

with K in the jth position, $K \in \{H,N\}$ and "$\otimes$" indicating the tensor product of matrices. Let $i_{\mathbf{x}} \in \{0,1,\ldots,2^n-1\}$ denote a row or column number of the unitary matrix U. We write

$$i_{\mathbf{x}} = x_n 2^{n-1} + x_{n-1} 2^{n-2} + \cdots + x_2 2 + x_1,$$

where $\mathbf{x} = (x_n,\ldots,x_1) \in \mathbb{Z}_2^n$. Given a Boolean function $f \in \mathcal{B}_n$, we consider the $2^n \times 1$ column vector $(-1)^{\mathbf{f}}$, whose $i_{\mathbf{u}}$th row contains $(-1)^{f(\mathbf{u})}$, for all $\mathbf{u} \in \mathbb{Z}_2^n$. The spectrum of f with respect to $U \in \{H,N\}^n$ is the vector $U(-1)^{\mathbf{f}}$. If $\mathbf{R}_H = \{1,\ldots,n\}$ then we write that the corresponding matrix $U \in \{H\}^n$ and the $i_{\mathbf{u}}$th row element of $U(-1)^{\mathbf{f}}$ is $\mathcal{H}_f(\mathbf{u})$. If $\mathbf{R}_N = \{1,\ldots,n\}$ then we write that the corresponding matrix $U \in \{N\}^n$ and the $i_{\mathbf{u}}$th row element of $U(-1)^{\mathbf{f}}$ is $\mathcal{N}_f(\mathbf{u})$. In the former case, $U(-1)^{\mathbf{f}}$ is said to be the Walsh–Hadamard spectrum of f, while in the latter case it is the nega–Hadamard spectrum of f. The spectrum of a function f with respect to a unitary transformation U is said to be flat if and only if the absolute value of each entry of $U(-1)^{\mathbf{f}}$ is 1.

Definition 2: A function $f \in \mathcal{B}_n$ is said to be bent if and only if its Walsh–Hadamard spectrum is flat, negabent if and only if its nega–Hadamard spectrum is flat and bent$_4$ if there exists at least one $U \in \{H,N\}^n$ such that $U(-1)^{\mathbf{f}}$ is flat.

In this paper, we consider the spectra of Boolean functions with respect to the action of unitary transforms in $\{H,N\}^n$. We prove that the maximum possible algebraic degree of a bent$_4$ function on n variables is $\lceil \frac{n}{2} \rceil$, and hence solve an open problem posed by Riera and Parker [4]. We obtain a relationship between bent and bent$_4$ functions which is a generalization of the relationship between bent and negabent Boolean functions proved by Parker and Pott [3]. We also refer to the recent Su, Pott and Tang [13] for related results.

II. Bent properties with respect to $\{H,N\}^n$

Let $s_r(\mathbf{x})$ be the homogeneous symmetric function of algebraic degree r, whose ANF is

$$s_r(\mathbf{x}) = \bigoplus_{1\le i_1<\ldots<i_r\le n} x_{i_1}\ldots x_{i_r}. \quad (8)$$

The intersection of two vectors $\mathbf{c} = (c_n,\ldots,c_1), \mathbf{x} = (x_n,\ldots,x_1) \in \mathbb{Z}_2^n$ is defined as

$$\mathbf{c} * \mathbf{x} = (c_n x_n, \ldots, c_1 x_1).$$

Following this notation we define the function $s_r(\mathbf{c}*\mathbf{x})$ as

$$s_r(\mathbf{c}*\mathbf{x}) = \bigoplus_{1\le i_1<\ldots<i_r\le n} (c_{i_1}x_{i_1})\ldots(c_{i_r}x_{i_r}). \quad (9)$$

Suppose, the function $g \in \mathcal{GB}_n^4$ defined as $g(\mathbf{x}) = wt(\mathbf{x}) \mod 4$, for all $x \in \mathbb{Z}_2^n$. In the following proposition and its corollary we obtain a connection between g and s_2 which plays a crucial role in developing connections between different bent criteria. It is to be noted that the result of Propostion 3 is mentioned earlier by Su, Pott and Tang [13] and a proof by induction is suggested. We provide an alternative proof.

Proposition 3: If $g \in \mathcal{GB}_n^4$ is defined by $g(\mathbf{x}) = wt(\mathbf{x}) \mod 4$ for all $\mathbf{x} \in \mathbb{Z}_2^n$, then

$$g(\mathbf{x}) = \mathbf{1} \cdot \mathbf{x} + 2s_2(\mathbf{x}) = wt(\mathbf{x}) \mod 4, \qquad (10)$$

for all $\mathbf{x} \in \mathbb{Z}_2^4$.

Proof: By Proposition 1, we have

$$2^{-\frac{n}{2}} \sum_{\mathbf{x}\in\mathbb{Z}_2^n} (-1)^{\mathbf{u}\cdot\mathbf{x}} \imath^{wt(\mathbf{x})} = \omega^n \imath^{-wt(\mathbf{u})}. \qquad (11)$$

Therefore, $g(\mathbf{x}) = wt(\mathbf{x}) \mod 4$ is a generalized bent on $\mathbb{Z}_4$, which we refer to as $\mathbb{Z}_4$-bent. According to [12, Corollary 15] and [7], there exist $a, b \in \mathcal{B}_n$ such that b and $a+b$ are bent functions and $g(\mathbf{x}) = a(\mathbf{x})+2b(\mathbf{x}) = wt(\mathbf{x}) \mod 4$, for all $\mathbf{x} \in \mathbb{Z}_2^n$. From this we have

$$2b(\mathbf{x}) \equiv wt(\mathbf{x}) - a(\mathbf{x}) \pmod 4,$$

i.e.,

$$2|(wt(\mathbf{x}) - a(\mathbf{x})),$$

i.e.,

$$a(\mathbf{x}) = \mathbf{1} \cdot \mathbf{x}$$

where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}_2^n$, for all $\mathbf{x} \in \mathbb{Z}_2^n$. Therefore,

$$g(\mathbf{x}) = \mathbf{1}\cdot\mathbf{x}+2b(\mathbf{x}) = wt(\mathbf{x}) \mod 4, \text{ for all } \mathbf{x} \in \mathbb{Z}_2^n,$$

i.e.,

$$b(\mathbf{x}) = \frac{-\mathbf{1}\cdot\mathbf{x} + wt(\mathbf{x})}{2} \mod 2, \text{ for all } \mathbf{x} \in \mathbb{Z}_2^n.$$

Since $b \in \mathcal{B}_n$ is a symmetric bent function and $b(\mathbf{0}) = 0$ we have $b(\mathbf{x}) = s_2(\mathbf{x})$ or $s_2(\mathbf{x}) \oplus s_1(\mathbf{x})$. Since $b(0\ldots01) = 0$, we have $b(\mathbf{x}) = s_2(\mathbf{x})$. Therefore

$$g(\mathbf{x}) = \mathbf{1}\cdot\mathbf{x}+2s_2(\mathbf{x}) = wt(\mathbf{x}) \mod 4, \text{for all } \mathbf{x} \in \mathbb{Z}_2^n.$$

■

The following corollary generalizes (10) which is useful in finding a general expression of entries of any matrix $U \in \{H, N\}^n$.

Corollary 4: Let $\mathbf{x}, \mathbf{c} \in \mathbb{Z}_2^n$. Then

$$\mathbf{c}\cdot\mathbf{x} + 2s_2(\mathbf{c} * \mathbf{x}) = wt(\mathbf{c} * \mathbf{x}) \mod 4, \qquad (12)$$

for all $\mathbf{x} \in \mathbb{Z}_2^n$.

Proof: In Proposition 3 it is proved that

$$\mathbf{1}\cdot\mathbf{x} + 2s_2(\mathbf{x}) = wt(\mathbf{x}) \mod 4, \text{ for all } \mathbf{x} \in \mathbb{Z}_2^n,$$

i.e.,

$$\begin{aligned}&(1, \ldots, 1)\cdot(x_n, \ldots, x_1) + 2s_2(x_n, \ldots, x_1)\\ &= wt(x_n, \ldots, x_1) \mod 4, \text{ for all } \mathbf{x} \in \mathbb{Z}_2^n.\end{aligned}$$

Replacing x_i by $c_i x_i$ we get

$$\begin{aligned}&(1, \ldots, 1)\cdot(c_n x_n, \ldots, c_1 x_1) + 2s_2(c_n x_n, \ldots, c_1 x_1)\\ &\quad = wt(c_n x_n, \ldots, c_1 x_1) \mod 4, \text{ for all } \mathbf{x} \in \mathbb{Z}_2^n,\end{aligned}$$

i.e.,

$$\begin{aligned}&(c_n x_n \oplus \ldots \oplus c_1 x_1) + 2s_2(c_n x_n, \ldots, c_1 x_1)\\ &= wt(c_n x_n, \ldots, c_1 x_1) \mod 4, \text{ for all } \mathbf{x} \in \mathbb{Z}_2^n.\end{aligned}$$

Therefore,

$$\mathbf{c}\cdot\mathbf{x}+2s_2(\mathbf{c}*\mathbf{x}) = wt(\mathbf{c}*\mathbf{x}) \mod 4, \text{ for all } \mathbf{x} \in \mathbb{Z}_2^n.$$

■

Riera and Parker [4, Lemma 7] have obtained a general expression for the entries of any matrix $U \in \{H, N\}^n$. We obtain an alternative description below which we use to connect the spectrum $U(-1)^{\mathbf{f}}$ of any $f \in \mathcal{B}_n$ to the Walsh–Hadamard spectra of some associated functions.

Theorem 5: If $U = \prod_{j\in\mathbf{R}_H} H_j \prod_{j\in\mathbf{R}_N} N_j$, is a unitary matrix constructed as in (7), corresponding to the partition $\mathbf{R}_H, \mathbf{R}_N$ of $\{1, \ldots, n\}$ where $n \geq 2$, then for any $\mathbf{u}, \mathbf{x} \in \mathbb{Z}_2^n$ the element in the $i_{\mathbf{u}}$th row and $i_{\mathbf{x}}$th column of $2^{\frac{n}{2}} U$ is

$$(-1)^{\mathbf{u}\cdot\mathbf{x}\oplus s_2(\mathbf{c}*\mathbf{x})} \imath^{\mathbf{c}\cdot\mathbf{x}},$$

where $\mathbf{c} = (c_n, \ldots, c_1) \in \mathbb{Z}_2^n$ is such that $c_i = 0$ if $i \in \mathbf{R}_H$ and $c_i = 1$ if $i \in \mathbf{R}_N$.

Proof: We prove by induction. Let $n = 2$. If $\mathbf{c} = (0, 0)$ then clearly $U = H \otimes H$, and if $\mathbf{c} = (1, 1)$ then $U = N \otimes N$. We explicitly compute U when $\mathbf{c} = (0, 1)$ and $\mathbf{c} = (1, 0)$ and find that U is equal to

$$H \otimes N = \frac{1}{2}\begin{pmatrix} 1 & \imath & 1 & \imath \\ 1 & -\imath & 1 & -\imath \\ 1 & \imath & -1 & -\imath \\ 1 & -\imath & -1 & \imath \end{pmatrix},$$

and

$$N \otimes H = \frac{1}{2}\begin{pmatrix} 1 & 1 & \imath & \imath \\ 1 & -1 & \imath & -\imath \\ 1 & 1 & -\imath & -\imath \\ 1 & -1 & -\imath & \imath \end{pmatrix},$$

respectively. By Corollary 4

$$(-1)^{\mathbf{u}\cdot\mathbf{x}\oplus s_2(\mathbf{c}*\mathbf{x})} \imath^{\mathbf{c}\cdot\mathbf{x}} = (-1)^{\mathbf{u}\cdot\mathbf{x}} \imath^{wt(\mathbf{c}*\mathbf{x})}.$$

Suppose the result is true for n. Let $\mathbf{u}, \mathbf{x}, \mathbf{c} \in \mathbb{Z}_2^n$, and $\mathbf{u}' = (u_{n+1}, \mathbf{u}), \mathbf{x}' = (x_{n+1}, \mathbf{x}), \mathbf{c}' = (c_{n+1}, \mathbf{c}) \in \mathbb{Z}_2^{n+1}$. Let $U \in \{H, N\}^n$ be the unitary transformation induced by the partition corresponding to $\mathbf{c} \in \mathbb{Z}_2^n$. The transformation corresponding to the partition induced by $\mathbf{c}' = (0, \mathbf{c}) \in \mathbb{Z}_2^{n+1}$ is $H \otimes U$. By taking the tensor product of H and U we obtain

$$2^{\frac{n+1}{2}}(H \otimes U) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where

$$
\begin{aligned}
A_{11} &= \left((-1)^{\mathbf{u}\cdot\mathbf{x}}\imath^{wt(\mathbf{c}*\mathbf{x})}\right)_{2^n\times 2^n}\\
&= \left((-1)^{(0,\mathbf{u})\cdot(0,\mathbf{x})}\imath^{wt((0,\mathbf{c})*(0,\mathbf{x}))}\right)_{2^n\times 2^n},
\end{aligned}
$$

$$
\begin{aligned}
A_{12} &= \left((-1)^{\mathbf{u}\cdot\mathbf{x}}\imath^{wt(\mathbf{c}*\mathbf{x})}\right)_{2^n\times 2^n}\\
&= \left((-1)^{(0,\mathbf{u})\cdot(1,\mathbf{x})}\imath^{wt((0,\mathbf{c})*(1,\mathbf{x}))}\right)_{2^n\times 2^n},
\end{aligned}
$$

$$
\begin{aligned}
A_{21} &= \left((-1)^{\mathbf{u}\cdot\mathbf{x}}\imath^{wt(\mathbf{c}*\mathbf{x})}\right)_{2^n\times 2^n}\\
&= \left((-1)^{(1,\mathbf{u})\cdot(0,\mathbf{x})}\imath^{wt((0,\mathbf{c})*(0,\mathbf{x}))}\right)_{2^n\times 2^n}
\end{aligned}
$$

and

$$
\begin{aligned}
A_{22} &= \left((-1)(-1)^{\mathbf{u}\cdot\mathbf{x}}\imath^{wt(\mathbf{c}*\mathbf{x})}\right)_{2^n\times 2^n}\\
&= \left((-1)^{(1,\mathbf{u})\cdot(1,\mathbf{x})}\imath^{wt((0,\mathbf{c})*(1,\mathbf{x}))}\right)_{2^n\times 2^n}.
\end{aligned}
$$

Therefore,

$$2^{\frac{n+1}{2}}(H\otimes U) = \left((-1)^{\mathbf{u}'\cdot\mathbf{x}'}\imath^{wt(\mathbf{c}'*\mathbf{x}')}\right)_{2^{n+1}\times 2^{n+1}}.$$

The transform corresponding to the partition induced by $\mathbf{c}' = (1,\mathbf{c}) \in \mathbb{Z}_2^{n+1}$ is $N\otimes U$. By taking the tensor product of H and U we obtain

$$2^{\frac{n+1}{2}}(N\otimes U) = \begin{pmatrix} B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}$$

where

$$
\begin{aligned}
B_{11} &= \left((-1)^{\mathbf{u}\cdot\mathbf{x}}\imath^{wt(\mathbf{c}*\mathbf{x})}\right)_{2^n\times 2^n}\\
&= \left((-1)^{(0,\mathbf{u})\cdot(0,\mathbf{x})}\imath^{wt((1,\mathbf{c})*(0,\mathbf{x}))}\right)_{2^n\times 2^n},
\end{aligned}
$$

$$
\begin{aligned}
B_{12} &= \left(\imath(-1)^{\mathbf{u}\cdot\mathbf{x}}\imath^{wt(\mathbf{c}*\mathbf{x})}\right)_{2^n\times 2^n}\\
&= \left((-1)^{(0,\mathbf{u})\cdot(1,\mathbf{x})}\imath^{wt((1,\mathbf{c})*(1,\mathbf{x}))}\right)_{2^n\times 2^n},
\end{aligned}
$$

$$
\begin{aligned}
B_{21} &= \left((-1)^{\mathbf{u}\cdot\mathbf{x}}\imath^{wt(\mathbf{c}*\mathbf{x})}\right)_{2^n\times 2^n}\\
&= \left((-1)^{(1,\mathbf{u})\cdot(0,\mathbf{x})}\imath^{wt((1,\mathbf{c})*(0,\mathbf{x}))}\right)_{2^n\times 2^n}
\end{aligned}
$$

and

$$
\begin{aligned}
B_{22} &= \left((-\imath)(-1)^{\mathbf{u}\cdot\mathbf{x}}\imath^{wt(\mathbf{c}*\mathbf{x})}\right)_{2^n\times 2^n}\\
&= \left((-1)^{(1,\mathbf{u})\cdot(1,\mathbf{x})}\imath^{wt((1,\mathbf{c})*(1,\mathbf{x}))}\right)_{2^n\times 2^n}.
\end{aligned}
$$

Therefore,

$$2^{\frac{n+1}{2}}(N\otimes U) = \left((-1)^{\mathbf{u}'\cdot\mathbf{x}'}\imath^{wt(\mathbf{c}'*\mathbf{x}')}\right)_{2^{n+1}\times 2^{n+1}}.$$

This proves the result. ■

Using Theorem 5 we can state that given any $U \in \{H,N\}^n$ there exists $\mathbf{c}\in\mathbb{Z}_2^n$ such that for any $f\in\mathcal{B}_n$ the $i_{\mathbf{u}}$th row of the column vector $U(-1)^{\mathbf{f}}$ is

$$
\begin{aligned}
\mathcal{U}_f^{\mathbf{c}}(\mathbf{u}) &= 2^{-\frac{n}{2}}\sum_{\mathbf{x}\in\mathbb{Z}_2^n}(-1)^{f(\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})}\imath^{\mathbf{c}\cdot\mathbf{x}}(-1)^{\mathbf{u}\cdot\mathbf{x}}\\
&= 2^{-\frac{n}{2}}\sum_{\mathbf{x}\in\mathbf{c}^\perp}(-1)^{f(\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}}\\
&\quad + \imath 2^{-\frac{n}{2}}\sum_{\mathbf{x}\notin\mathbf{c}^\perp}(-1)^{f(\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}}.
\end{aligned}
\tag{13}
$$

Therefore, $\mathcal{U}_f^{\mathbf{c}}(\mathbf{u})$ is related to the Walsh–Hadamard transform of restrictions $f(\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})$ to the subspace $\mathbf{c}^\perp$ and its coset. From another perspective this transformation provides a measure of the distance of the function f to the functions of the form $s_2(\mathbf{c}*\mathbf{x})\oplus\mathbf{u}\cdot\mathbf{x}$. Thus, if $|\mathcal{U}_f^{\mathbf{c}}(\mathbf{u})|$ has high value for a choice of $\mathbf{u},\mathbf{c}\in\mathbb{Z}_2^n$ then f has low Hamming distance from the function of the form $s_2(\mathbf{c}*\mathbf{x})\oplus\mathbf{u}\cdot\mathbf{x}$. This means that the function may be approximated efficiently by the function $s_2(\mathbf{c}*\mathbf{x})\oplus\mathbf{u}\cdot\mathbf{x}$. This may have some cryptographic significance for the spectra of f with respect to the transformations $U\in\{H,N\}^n$.

Riera and Parker [4, p. 4125] posed the following open problem:

What is the maximum algebraic degree of a bent$_4$ Boolean function of n variables?

The following theorem provides the solution to this problem.

Theorem 6: The maximum algebraic degree of a bent$_4$ Boolean function on n variables is $\lceil\frac{n}{2}\rceil$.

Proof: Using Theorem 5 we can state that given any $U\in\{H,N\}^n$ there exists $\mathbf{c}\in\mathbb{Z}_2^n$ such that for any $f\in\mathcal{B}_n$ the $i_{\mathbf{u}}$th row of the column vector $U(-1)^{\mathbf{f}}$ is

$$
\begin{aligned}
2^{\frac{n}{2}}\mathcal{U}_f^{\mathbf{c}}(\mathbf{u}) &= \sum_{\mathbf{x}\in\mathbb{Z}_2^n}(-1)^{f(\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})}\imath^{\mathbf{c}\cdot\mathbf{x}}(-1)^{\mathbf{u}\cdot\mathbf{x}}\\
&= \sum_{\mathbf{x}\in\mathbf{c}^\perp}(-1)^{f(\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}}\\
&\quad + \imath\sum_{\mathbf{x}\notin\mathbf{c}^\perp}(-1)^{f(\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}}.
\end{aligned}
\tag{14}
$$

Let us suppose that f is bent$_4$ with respect to the chosen transformation U. Therefore, we have $|\mathcal{U}_f^{\mathbf{c}}(\mathbf{u})| = 1$, for all $\mathbf{u}\in\mathbb{Z}_2^n$. By (14)

$$
\begin{aligned}
2^n &= (\sum_{\mathbf{x}\in\mathbf{c}^\perp}(-1)^{f(\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}})^2\\
&\quad + (\sum_{\mathbf{x}\notin\mathbf{c}^\perp}(-1)^{f(\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}})^2.
\end{aligned}
\tag{15}
$$

By Jacobi's two-square theorem we know that 2^n has a unique representation (disregarding the sign and order) as a sum of two squares, namely $2^n = (2^{\frac{n}{2}})^2 + 0$, if n is even, and $2^n = (2^{\frac{n-1}{2}})^2 + (2^{\frac{n-1}{2}})^2$, if n is odd. Let $g_{\mathbf{c}}(\mathbf{x}) = s_2(\mathbf{c} * \mathbf{x})$, for all $\mathbf{x} \in \mathbb{Z}_2^n$.

$$\begin{aligned} |\mathcal{H}_{f\oplus g_{\mathbf{c}}}(\mathbf{u})| &= |2^{-\frac{n}{2}} \sum_{\mathbf{x}\in\mathbb{Z}_2^n} (-1)^{f(\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}}| \\ &= |2^{-\frac{n}{2}} \sum_{\mathbf{x}\in\mathbf{c}^\perp} (-1)^{f(\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}} \\ &\quad + 2^{-\frac{n}{2}} \sum_{\mathbf{x}\notin\mathbf{c}^\perp} (-1)^{f(\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}}| \\ &= 1, \end{aligned} \tag{16}$$

for all $\mathbf{u} \in \mathbb{Z}_2^n$. Therefore, $f \oplus g_{\mathbf{c}}$ is a bent function and its algebraic degree is bounded above by $\frac{n}{2}$. The algebraic degree of $g_{\mathbf{c}}$ is upper-bounded by 2, so the upper bound of the algebraic degree of a bent$_4$ Boolean function f is $\frac{n}{2}$, when n is even.

In case n is odd by a similar argument we get $|\mathcal{H}_{f\oplus g_{\mathbf{c}}}(\mathbf{u})| \in \{0, \sqrt{2}\}$, that is $f \oplus g_{\mathbf{c}}$ is semibent, and therefore the algebraic degree of f is bounded above by $\frac{n+1}{2}$. ■

III. Connecting bent and bent$_4$ functions

The following lemma is well known.

Lemma 7: Let $n = 2k$, $f \in \mathcal{B}_n$ a bent function, V be an $(n-1)$-dimensional subspace of $\mathbb{Z}_2^n$, $\mathbf{a} \in \mathbb{Z}_2^n \setminus V$ such that $\mathbb{Z}_2^n = V \cup (\mathbf{a} \oplus V)$. Then the restrictions of f to V and $\mathbf{a} \oplus V$, denoted $f|_V$ and $f|_{\mathbf{a}\oplus V}$ respectively, are semibent functions and $\mathcal{H}_{f|_V}(\mathbf{u})\mathcal{H}_{f|_{\mathbf{a}\oplus V}}(\mathbf{u}) = 0$ for all $\mathbf{u} \in \mathbb{F}_2^n$.

Proof: Since the dimension of V is $n-1$, the dimension of the orthogonal subspace $V^\perp$ is 1. Let $V^\perp = \{\mathbf{0}, \mathbf{b}\}$. Since $\mathbf{a} \notin V$, $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$. For all $\mathbf{u} \in \mathbb{Z}_2^n$ we have the following

$$\begin{aligned} 2^{\frac{n}{2}}\mathcal{H}(\mathbf{u}) &= \sum_{\mathbf{x}\in V} (-1)^{f(\mathbf{x})\oplus\mathbf{u}\cdot\mathbf{x}} \\ &\quad + (-1)^{\mathbf{u}\cdot\mathbf{x}} \sum_{\mathbf{x}\in V} (-1)^{f(\mathbf{x}+\mathbf{a})\oplus\mathbf{u}\cdot\mathbf{x}} \\ &\in \{-2^{\frac{n}{2}}, 2^{\frac{n}{2}}\} \end{aligned} \tag{17}$$

$$\begin{aligned} 2^{\frac{n}{2}}\mathcal{H}(\mathbf{u}\oplus\mathbf{b}) &= \sum_{\mathbf{x}\in V} (-1)^{f(\mathbf{x})\oplus\mathbf{u}\cdot\mathbf{x}} \\ &\quad - (-1)^{\mathbf{u}\cdot\mathbf{x}} \sum_{\mathbf{x}\in V} (-1)^{f(\mathbf{x}+\mathbf{a})\oplus\mathbf{u}\cdot\mathbf{x}} \\ &\in \{-2^{\frac{n}{2}}, 2^{\frac{n}{2}}\}. \end{aligned} \tag{18}$$

By adding (17) and (18) we obtain $\sum_{\mathbf{x}\in V} (-1)^{f(\mathbf{x})\oplus\mathbf{u}\cdot\mathbf{x}} \in \{-2^{\frac{n}{2}}, 0, 2^{\frac{n}{2}}\}$, and by subtracting (18) from (17) we obtain $\sum_{\mathbf{x}\in V} (-1)^{f(\mathbf{a}\oplus\mathbf{x})\oplus\mathbf{u}\cdot\mathbf{x}} \in \{-2^{\frac{n}{2}}, 0, 2^{\frac{n}{2}}\}$. This proves that both f and $f|_{\mathbf{a}\oplus V}$ are semibent functions. Further since the sums in (17) and (18) are both in $\{-2^{\frac{n}{2}}, 2^{\frac{n}{2}}\}$ for each $\mathbf{u} \in \mathbb{Z}_2^n$, the product of the Walsh–Hadamard transforms of the restrictions of f to V and $\mathbf{a} \oplus V$ at $\mathbf{u}$ is zero, that is $\mathcal{H}_{f|_V}(\mathbf{u})\mathcal{H}_{f|_{\mathbf{a}\oplus V}}(\mathbf{u}) = 0$, in other words, the Walsh–Hadamard spectra of f_V and $f|_{\mathbf{a}\oplus V}$ are disjoint. ■

This leads us to a generalization of [3, Theorem 12] due to Parker and Pott. Recall that for any $\mathbf{c} \in \mathbb{Z}_2^n$ define $g_{\mathbf{c}}(\mathbf{x}) = s_2(\mathbf{c} * \mathbf{x})$, for all $\mathbf{x} \in \mathbb{Z}_2^n$.

Theorem 8: Let $f \in \mathcal{B}_n$ where n is even. Then the following are true.

1) If f is bent, then $f \oplus g_{\mathbf{c}}$ is bent$_4$.
2) If f is bent$_4$, i.e., there exists $\mathbf{c} \in \mathbb{Z}_2^n$ such that $|\mathcal{U}_f^{\mathbf{c}}(\mathbf{u})| = 1$ for all $\mathbf{u} \in \mathbb{Z}_2^n$, then $f \oplus g_{\mathbf{c}}$ is bent.

Proof: Suppose f is a bent function. If $\mathbf{c} = \mathbf{0}$ there is nothing to prove. If $\mathbf{c} \neq \mathbf{0}$, then

$$\begin{aligned} 2^{\frac{n}{2}}\mathcal{U}_{f\oplus g_{\mathbf{c}}}^{\mathbf{c}}(\mathbf{u}) &= \sum_{\mathbf{x}\in\mathbb{Z}_2^n} (-1)^{f(\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})}\imath^{wt(\mathbf{c}*\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}} \\ &= \sum_{\mathbf{x}\in\mathbb{Z}_2^n} (-1)^{f(\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})\oplus s_2(\mathbf{c}*\mathbf{x})\oplus\mathbf{u}\cdot\mathbf{x}}\imath^{\mathbf{c}\cdot\mathbf{x}} \\ &= \sum_{\mathbf{x}\in\mathbb{Z}_2^n} (-1)^{f(\mathbf{x})\oplus\mathbf{u}\cdot\mathbf{x}}\imath^{\mathbf{c}\cdot\mathbf{x}} \\ &= \sum_{\mathbf{x}\in\mathbf{c}^\perp} (-1)^{f(\mathbf{x})\oplus\mathbf{u}\cdot\mathbf{x}} \\ &\quad + \imath \sum_{\mathbf{x}\notin\mathbf{c}^\perp} (-1)^{f(\mathbf{x})\oplus\mathbf{u}\cdot\mathbf{x}}. \end{aligned} \tag{19}$$

Since f is a bent function and $\mathbf{c}^\perp$ is a subspace of codimension 1, by Lemma 7 the restrictions of f on $\mathbf{c}^\perp$ and its remaining coset are semibent and their Walsh-Hadamard spectra are disjoint. Therefore, the right hand side of the above equation belongs to the set $\{\pm 2^{\frac{n}{2}}, \pm 2^{\frac{n}{2}}\imath\}$ for all $\mathbf{u} \in \mathbb{Z}_2^n$. So $f \oplus g_{\mathbf{c}}$ is a bent$_4$ function.

In the second part we assume f to be a bent$_4$ function such that there exists $\mathbf{c} \in \mathbb{Z}_2^n$ for which $|\mathcal{U}_f^{\mathbf{c}}(\mathbf{u})| = 1$ for all $\mathbf{u} \in \mathbb{Z}_2^n$,

$$\begin{aligned} \mathcal{U}_f^{\mathbf{c}}(\mathbf{u}) &= 2^{-\frac{n}{2}} \sum_{\mathbf{x}\in\mathbb{Z}_2^n} (-1)^{f(\mathbf{x})}\imath^{wt(\mathbf{c}*\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}} \\ &= 2^{-\frac{n}{2}} \sum_{\mathbf{x}\in\mathbb{Z}_2^n} \imath^{wt(\mathbf{c}*\mathbf{x})+2f(\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}}. \end{aligned} \tag{20}$$

Thus, the function $h(\mathbf{x}) = wt(\mathbf{c} * \mathbf{x}) + 2f(\mathbf{x}) \mod 4$, is a $\mathbb{Z}_4$-bent function which implies the existence of Boolean functions $a, b \in \mathcal{B}_n$ such that b, $a+b$ are bents [12, Corollary 15], with

$$h(\mathbf{x}) = a(\mathbf{x}) + 2b(\mathbf{x}) = wt(\mathbf{c} * \mathbf{x}) + 2f(\mathbf{x}) \mod 4. \tag{21}$$

Therefore, $2|(a(\mathbf{x})-wt(\mathbf{c}*\mathbf{x}))$, which implies $a(\mathbf{x}) = \mathbf{c}\cdot\mathbf{x}$. By Corollary 4 and (20) we have

$$b(\mathbf{x}) = f(\mathbf{x}) \oplus s_2(\mathbf{c}*\mathbf{x}).$$

Since $b \in \mathcal{B}_n$ is a bent function $f \oplus g_{\mathbf{c}}$ is a bent function. Thus, we have proved that if f is bent$_4$ function then $f \oplus g_{\mathbf{c}}$ is a bent function for some $\mathbf{c} \in \mathbb{Z}_2^n$. ■

REFERENCES

[1] T. W. Cusick, P. Stănică, Cryptographic Boolean functions and applications, Elsevier–Academic Press, 2009.

[2] F. J. MacWilliams, N. J. A. Sloane, The theory of error–correcting codes, North-Holland, Amsterdam, 1977.

[3] M. G. Parker, A. Pott, *On Boolean functions which are bent and negabent*. In: S.W. Golomb, G. Gong, T. Helleseth, H.-Y. Song (eds.), SSC 2007, LNCS 4893 (2007), Springer, Heidelberg, 9–23.

[4] C. Riera, M. G. Parker, *One and two-variable interlace polynomials: A spectral interpretation*, Proc. of WCC 2005, LNCS 3969 (2006), Springer, Heidelberg, 397–411.

[5] C. Riera, M. G. Parker, *Generalized bent criteria for Boolean functions*, IEEE Trans. Inform. Theory 52:9 (2006), 4142–4159.

[6] O. S. Rothaus, *On bent functions*, J. Comb. Theory – Ser. A 20 (1976), 300–305.

[7] P. Solé, N. Tokareva, *Connections between Quaternary and Binary Bent Functions*, http://eprint.iacr.org/2009/544.pdf; see also, Prikl. Diskr. Mat. 1 (2009), 16–18.

[8] K-U. Schmidt, *Quaternary Constant-Amplitude Codes for Multicode CDMA*, IEEE International Symposium on Information Theory, ISIT'2007 (Nice, France, June 24–29, 2007), 2781–2785; available at http://arxiv.org/abs/cs.IT/0611162.

[9] K. U. Schmidt, M. G. Parker, A. Pott, *Negabent functions in the Maiorana–McFarland class*. In: S.W. Golomb, M.G. Parker, A. Pott, A. Winterhof (eds.), SETA 2008, LNCS 5203 (2008), Springer, Heidelberg, 390–402.

[10] P. Stănică, S. Gangopadhyay, A. Chaturvedi, A. K. Gangopadhyay, S. Maitra, *Nega–Hadamard transform, bent and negabent functions*, Proc. of SETA 2010, LNCS 6338 (2010), 359–372.

[11] P. Stănică, S. Gangopadhyay, A. Chaturvedi, A. K. Gangopadhyay, S. Maitra, *Investigations on bent and negabent functions via the nega–Hadamard transform*, IEEE Trans. Inform. Theory 58:6 (2012), 4064–4072.

[12] P. Stănică, T. Martinsen, S. Gangopadhyay, B. K. Singh, *Bent and generalized bent Boolean functions*, Des. Codes Cryptogr. DOI 10.1007/s10623-012-9622-5.

[13] W. Su, A. Pott, X. Tang, *Characterization of negabent functions and construction of bent–negabent functions with maximum algebraic degree*, arXiv: 1205.6568v1 [cs.IT], 30 May 2012.