

# A Measure of Security for Ideal Functions

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## Abstract

In this work we present a modification of a well-established measure of dependence appropriate for the analysis of stopping times for adversarial processes on cryptographic primitives. We apply this measure to construct generic criteria for the ideal behavior of fixed functions in both the random oracle and ideal permutation setting. More significantly, we provide a nontrivial extension of the notion of hash function indistinguishability, transporting the theory from the status of providing security arguments for protocols utilizing ideal primitives into the more realistic setting of protocol assurance with fixed functions. The methodology this measure introduces to indistinguishability analysis connects the security of a hash function with an indistinguishable mode to the security of the underlying compression function in a quantitative way; thus, we prove that dependence results on cryptographic primitives provide a direct means of determining the practical resistance or vulnerability of protocols employing such primitives.

## 1 Introduction

Many real world phenomena can be studied by associating them with a sequence of identically distributed discrete random variables, corresponding to measurements, which take values in some finite set. If the random variables are independent, the behavior of such processes is well known, both in its long term, or asymptotic behavior and in its behavior over an intermediate finite number of measurements. The situation for dependent processes, however, is less clear. Although the limit theory for dependent random sequences has also been extensively developed, the behavior of dependent processes in the “semi-long term” has received less focus.

Hash analysis provides an excellent example of a process for which this “semi-long term” behavior is of the greatest significance. Specifically, an adaptive adversary trying to discover a weakness in a hash function will choose a strategy which depends on the information obtained through interaction with the function. The experiment will continue until some predetermined information is gleaned from this process. Since practical hash functions have a finite output, the stochastic process defined by the interactions of the adversary with the hash function has a stopping time.

The stopping time of such a dependent process is of critical importance in cryptography. An attack on a hash function can take many forms, as one may construe as an attack any process which gathers enough information to distinguish the fixed function from an ideal function with the same codomain. For any meaningful attack, the measure with which we can judge the resistance of such a function is determined by the stopping time of the adaptive adversarial process.

The purpose of this work is to provide a mathematical framework upon which we may study the stopping times of optimal adaptive adversarial strategies. More specifically, we introduce measures of dependence relevant to the study of stopping times for dependent stochastic processes and apply these measures, determining the properties required of a hash primitive to achieve various security criteria.

The manuscript is organized as follows. Section 2 introduces two classical measures of dependence employed in limit theory to determine the asymptotic behavior of strongly mixing random sequences. In Section 3 we present the appropriate terminology required for the cryptographic application. In the following sections, Sections 4 and 5, we derive analogues of the measure of dependence appropriate to the study of stopping times, and apply the resulting theory to fixed random functions and fixed permutations, respectively. Finally, in Section 6 we further employ the dependence condition extending the powerful indifferenciability framework to provide assurance in a more practical setting.

## 2 Mixing Coefficients

Strong mixing conditions have been used greatly, both in the context of random sequences and in the broader context of random fields, in order to study phenomena for which observations which are close to each other in time or location may show considerable influence on one another, while observations which are far apart in time or location are almost independent. Many results in the study of strong mixing conditions for random sequences have been in the literature in connection with various fields of study: Rosenblatt [22] and Zhurbenko [23], with respect to the estimation of spectral density; Phillips [20], for the use of ARMA models in the study of economics; Dabrowski, McDonald, and Rösler [8], in connection with the flow of electrical signals in the heart or nerve membrane; Halversam and Wise [12], related with the detection of a signal in the presence of noise; Philipp [19], in connection with some random processes arising from number theory; Kesten and Papanicolaou [14], with respect to the motion of a particle in a velocity field; and Davydov [9] and Meyn and Tweedie [16], for Markov chains.

In 1956, Rosenblatt [21] proposed one particularly useful type of dependence which he referred to as the “strong mixing condition” or the  $\alpha$ -mixing condition. Since then, a lot of progress has been made in this area (see [22], [20], [8], [12], [19], [14], [9], [16], [4]) in relation to the following mixing coefficients. Suppose  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability measure space, and  $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$  are two  $\sigma$ -fields. Define the following measures of dependence:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A)P(B)|,$$

and

$$\phi(\mathcal{A}, \mathcal{B}) := \sup_{A \in \mathcal{A}, B \in \mathcal{B}, P(A) > 0} |P(B|A) - P(B)|.$$

It is well known and elementary (see [3], Proposition 3.11(a)) that

$$0 \leq 2\alpha(\mathcal{A}, \mathcal{B}) \leq \phi(\mathcal{A}, \mathcal{B}) \leq 1.$$

It is obvious to notice that if the  $\sigma$ -fields  $\mathcal{A}$ ,  $\mathcal{B}$  are independent, then  $\alpha(\mathcal{A}, \mathcal{B}) = 0$  and  $\rho(\mathcal{A}, \mathcal{B}) = 0$ , and vice versa.

Suppose  $\xi := (\xi_k, k \in \mathbb{Z})$  is a (not necessarily stationary) sequence of random variables. For  $-\infty \leq J \leq L \leq \infty$ , define the  $\sigma$ -field

$$\mathcal{F}_J^L := \sigma(\xi_k, J \leq k \leq L),$$

the  $\sigma$ -field generated by the random variables  $(\xi_k, J \leq k \leq L)$ . It is understood that the index  $k$  is restricted to the integers. These notations will also be used for (not necessarily stationary) “one-sided” random sequences  $\xi := (\xi_1, \xi_2, \xi_3, \dots)$ , with the obvious modification that  $\sigma$ -field  $\mathcal{F}_J^L$  is defined only for  $1 \leq J \leq L \leq \infty$ . In this paper, for the “one-sided” random sequences  $\xi := (\xi_1, \xi_2, \xi_3, \dots, \xi_n)$  the  $\sigma$ -field  $\mathcal{F}_J^L$  is defined only for  $1 \leq J \leq L \leq n$ . For each integer  $n$ , define the following dependence coefficients:

$$\alpha(n) := \alpha(\xi, n) := \sup_{J \in \mathbb{Z}} \alpha(\mathcal{F}_{-\infty}^J, \mathcal{F}_{J+n}^\infty)$$

and

$$\phi(n) := \phi(\xi, n) := \sup_{J \in \mathbb{Z}} \phi(\mathcal{F}_{-\infty}^J, \mathcal{F}_{J+n}^\infty).$$

The random sequence  $\xi := (\xi_k, k \in \mathbb{Z})$  (whether stationary or not) is said to be “strongly mixing” (or  $\alpha$ -mixing), respectively “ $\phi$ -mixing” if  $\alpha(n) \rightarrow 0$  and  $\phi(n) \rightarrow 0$ , respectively, as  $n \rightarrow \infty$ . The strong mixing condition  $\alpha(n) \rightarrow 0$  was introduced by Rosenblatt [21], and the  $\phi$ -mixing condition  $\phi(n) \rightarrow 0$  was introduced by Ibragimov [13], and also studied by Cogburn [6]. While in this paper we are not interested in the asymptotic behavior of the random process, due to the fact that  $n$  is an arbitrary fixed integer, it is necessary to emphasize the challenge of showing the behavior for our random process when the number of steps is large and fixed, without looking at its limiting behavior.

### 3 Cryptographic Primitives and Modes

In cryptography, hash functions are designed in such a way to simulate a random function while committing to its output values. For a good hash function, it should be computationally infeasible to find a pre-image of a known hash value, alter a known input/output pair to find a second pre-image, or generate two inputs which hash to the same value. A simple reason for these design criteria is that any function satisfying these conditions behaves like a random oracle, a theoretical function often used to prove the security of complex protocols.

A finite random oracle,  $ro : A \rightarrow B$ , is a function chosen uniformly at random from among all functions from the finite set  $A$  to the finite set  $B$ . To collect information about any function  $F$  one may submit queries, inputs for which  $F$  provides outputs, and catalogue the results. If a query has never been made to the function previously, it is called a fresh query. Random oracles have the property that every fresh query produces an output which is uniformly distributed and independent of all previously catalogued information. We may

define a variable input length random oracle,  $RO : \mathbb{Z} \rightarrow B$ , with a similar property by selecting a sequence of uniformly distributed independent and identically distributed (i.i.d.)  $B$ -valued random variables,  $X_i$ , and defining  $RO(n) = X_n$ . In practice, the input to  $RO$  may be a bit string instead of an integer, in which case the composition of the necessary bijection is understood.

Random oracles are not appropriate for all theoretical applications. In particular, in some contexts it is important for a permutation to behave unpredictably. In such contexts, we employ ideal permutations. An ideal permutation,  $\pi : A \rightarrow A$ , is a permutation chosen uniformly at random from among all permutations of the finite set  $A$ .

One model for building a practical hash function is to chain together fixed finite functions which are designed to mimic the behavior of a finite random oracle or an ideal permutation. A hash mode is a method for creating a variable input length function from such functions of fixed input length. If a particular hash mode is proven secure when its components are assumed to be ideal primitives (either a finite random oracle or an ideal permutation, depending on the design), then the task of creating a secure hash function is reduced to the challenge of creating fixed functions which behave like the ideal functions.

These ideal functions provide a road-map for proving the security of a practical hash function. First, the hash mode is analyzed with the fixed primitives replaced by ideal primitives. If the resulting variable input length function is shown to be indistinguishable from a variable input length random oracle, the mode is considered secure. It then suffices to show that the actual fixed primitives employed in the hash design are indistinguishable from ideal primitives.

## 4 Fixed Random Functions

Consider a finite fixed function,  $F$ . An adversary, trying to reveal a weakness in  $F$  may attempt a collision attack (finding two inputs which map to the same output), a preimage attack (finding a preimage for some given fixed but arbitrary value of the codomain of  $F$ ), or a second-preimage attack (finding a second preimage for a known input/output pair). For such a fixed function, there may exist some symmetry which a diligent adversary can discover through interaction with the function and possibly exploit to effect one of the above attacks.

Consider in contrast the interaction of an adversary with a finite random oracle,  $ro$ . Clearly, there is no benefit to submitting a single query repeatedly to the oracle, thus we may assume that the adversary submits fresh queries. Consequently, we can define a sequence of uniformly distributed i.i.d. random variables,  $\xi_1, \xi_2, \dots$ , corresponding to each successive output of  $ro$ .

Let the random variable  $X$  denote the number of rounds of querying before the adversary is able to form a collision with outputs of  $ro$ . Then its probability density function is  $f(x) = x \frac{(n-1)!}{(n-x)!} n^{-x}$ , where  $x = 1, 2, \dots, n$ . Consequently, the expected value of  $X$  is given by

$$(4.1) \quad E(X) = \sum_{i=1}^n i^2 \frac{(n-1)!}{(n-i)!} n^{-i}.$$

A classical calculation shows the relative size of this quantity. Specifically, by Markov's inequality,  $E(X) \geq \alpha P(X \geq \alpha)$ , and we may find a lower bound for  $E(X)$  by finding  $\alpha$  satisfying  $P(X \geq \alpha) \geq \frac{1}{2}$ . Clearly, this is equivalent to finding  $P(X < \alpha) \leq \frac{1}{2}$ . Since for all  $i \leq n$  we have  $P(X > i - 1) > 0$ , we obtain, for all  $\alpha \leq n$ :

$$(4.2) \quad P(X < \alpha) \leq \sum_{i=1}^{\alpha-1} P(X = i | X > i - 1) = \sum_{i=1}^{\alpha-1} \frac{i}{n} = \frac{1}{n} \binom{\alpha}{2}.$$

Setting this quantity equal to  $\frac{1}{2}$ , we may derive that  $\alpha = \frac{\sqrt{4n+1}+1}{2}$ . This fact implies that  $E(X) \geq \frac{\sqrt{4n+1}+1}{2} P(X \geq \alpha) \geq \frac{\sqrt{4n+1}+1}{4}$ , which shows that the sum in (4.1) is  $\Omega(\sqrt{n})$ .

To get the other bound, notice that the expected number of rounds to get the first collision must be less than or equal to the number of rounds for the expected number of collisions to be at least one, since there can be at most one collision in any given round. Therefore, we can show that this number is order  $\sqrt{n}$ . Let  $I_{i,j}$  be the indicator that the  $i$ th and  $j$ th nodes visited coincide. Let  $Y$  be the number of collisions, that is

$$Y = \sum_{i=1}^{\sigma-1} \sum_{j=i+1}^{\sigma} I_{i,j},$$

where  $\sigma$  represents an arbitrary number of rounds. Therefore, considering the fact that  $P(I_{i,j} = 1) = \frac{1}{n}$ , we have:

$$(4.3) \quad E(Y) = \sum_{i=1}^{\sigma-1} \sum_{j=i+1}^{\sigma} \frac{1}{n} = \binom{\sigma}{2} \frac{1}{n}.$$

Thus, for  $E(Y) = 1$ , we need  $\sigma = \lceil \sqrt{2n + \frac{1}{4} + \frac{1}{2}} \rceil$ . Thus our sum is  $O(\sqrt{n})$ , and therefore,  $\Theta(\sqrt{n})$ .

The critical question in the context of random functions is, "What happens when we are not guaranteed independence among function output in each of the rounds?" For each  $i \in \{1, 2, \dots, n\}$ , let us define  $\xi_i$  to be a simple random variable taking the values  $\{1, 2, \dots, n\}$ , where each integer represents the output of a fixed function,  $F$ , in round  $i$ .

Clearly, whatever the adversary's criteria for a successful attack may be, he or she may only consider the attack successful by referring to information from the current round and past rounds. The adversary may only make judgements in round  $i$  based on information from the previous rounds. Therefore, to study the stopping time of the adversarial process, we are specifically interested in a measure of dependence involving past events and events in round  $i$ , a measure which consequently bounds the performance of an optimal adversary.

**Definition 4.1.** Consider the process  $\{\xi_k\}$  where  $k \in \{1, 2, \dots, n\}$  together with a collection of pairs of sigma fields,  $\mathcal{A}_i$  and  $\mathcal{B}_i$ , where  $\mathcal{A}_1$  is the trivial sigma field,  $\mathcal{A}_i = \mathcal{F}_1^{i-1}$  for  $2 \leq i \leq n$ , and  $\mathcal{B}_i = \mathcal{F}_i^i$  for  $1 \leq i \leq n$ . We define the dependence sequence of this process in the following way: for  $i \in \{1, 2, \dots, n\}$ ,

$$(4.4) \quad \phi_i = \phi(\mathcal{A}_i, \mathcal{B}_i) = \sup_{A \in \mathcal{A}_i, B \in \mathcal{B}_i} |P(B|A) - P(B)|, \quad P(A) > 0.$$

If the  $\xi_i$  are the outputs of a random function,  $F$ , with a finite codomain of size  $n$ , then we say that  $F$  is a  $\phi[p]$ -mixing random function provided that  $\phi_i \leq n^{-p}$  for all  $i < n$ .

By an easy calculation one may notice that for each  $i \in \{1, 2, \dots, n\}$ , if  $A \in \mathcal{F}_1^{i-1}$  and  $B \in \mathcal{F}_i^i$ , then

$$(4.5) \quad |P(A \cap B) - P(A)P(B)| \leq P(A) \cdot \phi_i.$$

Now we compute bounds on the probability of obtaining the first collision on round  $i$ . Clearly, larger probabilities of collision in early rounds result in a lower expected number of rounds to form a collision. With this in mind, we compute each probability assuming that each previous round attains the theoretical maximum probability of collision. For the first round we use the formal notation  $\phi_1$ , which simply represents zero, since there are no events occurring before the first round.

In this manner, we compute that the probability of getting a collision in the first round is  $\frac{1}{n} = \frac{1}{n} + \phi_1$ . Therefore, the probability of noncollision in the first round is  $\frac{n-1}{n} - \phi_1$ . Were the outcome in the second round uniformly distributed and independent of the outcome of the first round, the probability of a collision in the second round *and* a noncollision in the first round would be  $(\frac{n-1}{n} - \phi_1)\frac{2}{n}$ . The definition of  $\phi_2$  provides us with a possible error in this estimation; specifically, we may be as far off the correct value as  $(\frac{n-1}{n} - \phi_1)\phi_2$ . Therefore, the probability that the first collision occurs in round two is bounded by  $(\frac{n-1}{n} - \phi_1)(\frac{2}{n} + \phi_2)$ .

For each  $i \in \{1, 2, \dots, n\}$ , we may compute the probability of  $C_i$ , the event that the first collision occurs in round  $i$ , given that each collision event is assigned the maximal probability sequentially,

$$(4.6) \quad P(C_i) = \left(\frac{i}{n} + \phi_i\right) \prod_{1 \leq k < i} \left(\frac{n-k}{n} - \phi_k\right).$$

Consider this expression as a polynomial in  $\mathbb{R}[\phi_1] \dots [\phi_i]$ . For each  $i \in \{1, 2, \dots, n\}$ , we use  $\Psi_i^{(k)}$  to denote the sum of the total degree  $k$  terms in this bound. Then we obtain

$$(4.7) \quad \Psi_i^{(0)} = \frac{i}{n^i} \frac{(n-1)!}{(n-i)!}.$$

We denote the terms contained in  $\Psi_i^{(1)}$  by

$$(4.8) \quad \Psi_{i,j}^{(1)} = \begin{cases} \frac{\phi_i}{n^i} \frac{n!}{(n-i)!} & \text{if } j = i \\ -\frac{i\phi_j}{n^i} \frac{n!}{(n-i)!(n-j)} & \text{if } j < i, \end{cases}$$

where  $i$  represents the number of rounds and  $j$  represents the index of the variable  $\phi$  used. Thus, equation (4.6) becomes: for each  $i \in \{1, 2, \dots, n\}$ ,

$$(4.9) \quad P(C_i) = \Psi_i^{(0)} + \sum_{1 \leq j \leq i} \Psi_{i,j}^{(1)} + \text{higher degree terms.}$$

Now we can explicitly write down a lower bound formula on the expected number of rounds,  $X$ , required to form a collision relative to the  $\phi_i$ 's. Specifically, we have that

$$(4.10) \quad E(X) = \sum_{i=1}^n i\Psi_i^{(0)} + \sum_{i=1}^n i \sum_{j=1}^i \Psi_{i,j}^{(1)} + \text{sum of higher degree terms.}$$

From the birthday paradox we obtain  $\sum_{i=1}^n i\Psi_i^{(0)} = \Theta(\sqrt{n})$ . To evaluate the second summation in equation (4.10), we divide into two cases:

*Case 1:*  $j = i$ .

Under this condition, the second summation in (4.10) contributes positively to the expected value. Specifically, this sum is bounded below by

$$n \min_{l \leq n}(\phi_l) \sum_{i=1}^n \frac{i (n-1)!}{n^i (n-i)!}.$$

Notice that each term in the sum above is exactly  $\Psi_i^{(0)}$ , the probability of forming a collision among the  $n$  outputs at step  $i$ . Therefore this entire term simplifies to  $n \min_{l \leq n}(\phi_l)$ , which is nonnegative. On a side note, in conjunction with our subsequent calculations, this lower bound for the summand provides a natural upper bound on the dependence coefficients,  $\phi_i$ . Note that  $E(X) = O(\sqrt{n})$ , which implies that any fixed function has some dependence coefficient smaller than  $Cn^{-1/2}$  and *Case 1* is complete.

*Case 2:*  $j < i$ .

Under this condition, the second sum in equation (4.10) becomes:

$$(4.11) \quad - \sum_{i=2}^n \sum_{j=1}^{i-1} \frac{\phi_j i^2}{n^i} \frac{n!}{(n-i)!(n-j)} = - \sum_{j=1}^{n-1} \frac{n\phi_j}{n-j} \sum_{i=j+1}^n \frac{i^2 (n-1)!}{n^i (n-i)!}.$$

Note that the inner sum on the index  $i$  above is a partial sum with terms  $i\Psi_i^{(0)}$ , therefore it is bounded by  $C\sqrt{n}$ . Thus the right-hand side of (4.11) is lower bounded by

$$-Cn^{3/2} \sum_{j=1}^{n-1} \frac{\phi_j}{n-j} = -Cn^{3/2} \sum_{j=1}^{n-1} \frac{\phi_{n-j}}{j} \geq -Cn^{3/2} \max_i \phi_i \sum_{j=1}^{n-1} \frac{1}{j}.$$

Consequently, since the partial sum of the harmonic series above is bounded by  $\ln(n) + 1$ , we obtain that for  $n > 1$ ,

$$-Cn^{3/2} \max_i \phi_i \sum_{j=1}^{n-1} \frac{1}{j} \geq -Cn^{3/2}(\ln(n) + 1) \max_i \phi_i.$$

Therefore, the condition  $\phi_i < n^{-1-\epsilon}$  is sufficient to conclude that the second sum in (4.10) is  $o(n^{1/2})$  and hence, the lower bound of the expected value is  $\Omega(\sqrt{n})$ , provided that the higher degree terms are negligible for such values of  $\phi_i$ .

The following theorem will show the fact that the higher degree terms are indeed negligible if  $\phi_i < n^{-1-\epsilon}$  for all  $i \in \{1, 2, \dots, n\}$ .

**Theorem 4.1.** *A function,  $F$ , has ideal collision resistance provided it is  $\phi[1 + \epsilon]$ -mixing for some  $\epsilon > 0$ .*

*Proof.* From above, we have that  $E(X) = \Omega(\sqrt{n}) +$  sum of degree two and higher terms. We need to show that the sum of the terms with total degree greater than or equal to two

is  $o(\sqrt{n})$ . For this, since there are fewer than  $n$  possible total degrees, it suffices to show that the sum restricted to each total degree is  $o(n^{-1/2})$ . Consider the degree  $d \geq 2$  sum,  $\sum_{i=1}^n i\Psi_i^{(d)}$ . Allowing  $j_1 < j_2 < \dots < j_d \leq i$ , we denote the terms in  $\Psi_i^{(d)}$  by

$$(4.12) \quad \Psi_{i,j_1,\dots,j_d}^{(d)} = \begin{cases} (-1)^{d+1} \frac{\phi_i \phi_{j_1} \dots \phi_{j_{d-1}}}{n^{i-1}} \frac{n!}{(n-i)!(n-j_1)\dots(n-j_{d-1})} & \text{if } j_d = i \\ (-1)^d \frac{i \phi_{j_1} \dots \phi_{j_d}}{n^{i-1}} \frac{n!}{(n-i)!(n-j_1)\dots(n-j_d)} & \text{if } j_d < i. \end{cases}$$

We may bound the absolute values of sums of these terms with  $j_d = i$  and with  $j_d < i$  individually. As before, we consider the following two cases:

*Case 1:  $j_d = i$ .*

Under this condition, we obtain that

$$(4.13) \quad \sum_{i=d}^n \sum_{j_{d-1}=d-1}^{i-1} \dots \sum_{j_1=1}^{j_2-1} |i\Psi_{i,j_1,\dots,j_{d-1},i}| = \sum_{j_{d-1}=d-1}^{n-1} \dots \sum_{j_1=1}^{j_2-1} \sum_{i=j_{d-1}+1}^n |i\Psi_{i,j_1,\dots,j_{d-1},i}|.$$

By (4.12), replacing the  $\phi_k$ 's with their maximum values and multiplying and dividing by  $n$  in the right-hand side of (4.13), we obtain:

$$n \max_{l \leq n}(\phi_l^d) \sum_{j_{d-1}=d-1}^{n-1} \frac{1}{n-j_{d-1}} \dots \sum_{j_1=1}^{j_2-1} \frac{1}{n-j_1} \sum_{i=j_{d-1}+1}^n \frac{i}{n^i} \frac{n!}{(n-i)!}.$$

Since the innermost sum is the probability that it takes more than  $j_{d-1}$  rounds to have a collision in the outputs of a random oracle, our quantity above is bounded by

$$n \max_{l \leq n}(\phi_l^d) \sum_{j_{d-1}=d-1}^{n-1} \frac{1}{n-j_{d-1}} \dots \sum_{j_1=1}^{j_2-1} \frac{1}{n-j_1} \leq n \max_{l \leq n}(\phi_l^d) (\ln(n) + 1)^{d-1}.$$

The condition  $\phi_i < n^{-1-\epsilon}$  assures that the sum in (4.13) is  $o(n^{1-d})$ , which for  $d \geq 2$  means it is  $o(n^{-1})$  and in particular,  $o(n^{-1/2})$ . As a consequence, *Case 1* is complete.

*Case 2:  $j_d < i$ .*

In this second case, we have:

$$(4.14) \quad \sum_{i=d+1}^n \sum_{j_d=d}^{i-1} \dots \sum_{j_1=1}^{j_2-1} |i\Psi_{i,j_1,\dots,j_d}| = \sum_{j_d=d}^{n-1} \dots \sum_{j_1=1}^{j_2-1} \sum_{i=j_d+1}^n |i\Psi_{i,j_1,\dots,j_d}|.$$

We can rewrite the right-hand expression above as

$$\sum_{j_d=d}^{n-1} \dots \sum_{j_1=1}^{j_2-1} \frac{n \phi_{j_1} \dots \phi_{j_d}}{(n-j_1)\dots(n-j_d)} \sum_{i=j_d+1}^n \frac{i^2}{n^i} \frac{n!}{(n-i)!}.$$

Again, since the inner-most sum is a partial sum of the expected number of rounds required to form a collision in the outputs of a random oracle, we can bound it by  $Cn^{1/2}$ . Therefore,



we have

$$\begin{aligned} Cn^{3/2} \sum_{j_d=d}^{n-1} \frac{\phi_{j_d}}{n-j_d} \dots \sum_{j_1=1}^{j_2-1} \frac{\phi_{j_1}}{n-j_1} &\leq Cn^{3/2} \max_{l \leq n} (\phi_l^d) \left( \sum_{i=1}^{n-1} \frac{1}{i} \right)^d \\ &\leq Cn^{3/2} \max_{l \leq n} (\phi_l^d) (\ln(n) + 1)^d. \end{aligned}$$

Using the given bound,  $\phi_i < n^{-1-\epsilon}$ , we obtain a bound of  $o(n^{3/2-d})$  for the sum in (4.14), which for  $d \geq 2$  it implies the sum is  $o(n^{-1/2})$ . Therefore, Theorem 4.1 is complete.  $\square$

**Corollary 4.1.** *A fixed function,  $F$ , has ideal collision resistance if the dependence sequence satisfies  $\phi_i < n^{-1-\epsilon}$  for  $1 \leq i \leq C\sqrt{n}$  for some constant  $C$ .*

*Proof.* For any fixed function,  $F$ , let us denote

$$E(X|X \leq \sigma) := \sum_{i=0}^{\sigma} iP(X = i|X \leq \sigma).$$

Define now  $E_{X \leq \sigma}(X) = E(X|X \leq \sigma)P(X \leq \sigma)$ , which is exactly the  $\sigma$ th partial sum of  $E(X)$ . Therefore,  $\{E_{X \leq \sigma}\}$  forms a monotonically increasing sequence with index  $\sigma$ . Thus all ‘‘little-oh’’ bounds from Theorem 4.1 and before hold for the negative parts of  $\sum i\Psi_i^{(d)}$ ,  $d \geq 1$ . Hence, it suffices to show that for a random oracle,  $E_{X \leq C\sqrt{n}}(X) = \Omega(\sqrt{n})$ .

Fix  $\sigma$  to be the smallest integer such that  $P(X \leq \sigma) \geq \frac{1}{2}$ . From the discussion at the beginning of this section,  $\sigma = C\sqrt{n}$ . We need only to show that  $E(X|X \leq \sigma) = \Omega(\sqrt{n})$ .

Again, by Markov’s Inequality,  $E(X|X \leq \sigma) \geq \alpha P(X \geq \alpha|X \leq \sigma)$ . We find an  $\alpha$  such that  $P(X \geq \alpha|X \leq \sigma) \geq \frac{1}{2}$ ; this is equivalent to finding  $\alpha$  such that  $P(X < \alpha|X \leq \sigma) \leq \frac{1}{2}$ , which for  $\alpha < \sigma$  becomes  $P(X < \alpha) \leq \frac{P(X \leq \sigma)}{2}$ . Since  $P(X \leq \sigma) \geq \frac{1}{2}$ , it suffices to find an  $\alpha$  such that  $P(X < \alpha) = \frac{1}{4}$ .

As in this section’s prologue,  $P(X < \alpha) \leq \frac{1}{n} \binom{\alpha}{2}$ , and we find that  $\alpha$  must be roughly  $\sqrt{\frac{n}{2}}$  to satisfy our inequality. Therefore, the partial sums of  $E(X)$  for a random oracle are  $\Omega(\sqrt{n})$  provided that there are at least  $\Omega(\sqrt{n})$  summands. Thus, if  $\phi_i < n^{-1-\epsilon}$  for  $1 \leq i \leq C\sqrt{n}$ ,  $E(X) > E_{X \leq C\sqrt{n}}(X) = \Omega(\sqrt{n})$ , and the corollary is complete.  $\square$

## 5 Fixed Permutations

Clearly, collision resistance is not a concern for a permutation; however, we can analyze a random walk model similar to the random oracle process of the previous section to simulate the process of deriving a cycle in the permutation. Obviously, having an easily derived short cycle is a weakness for a permutation, since such a cycle may produce a free-start collision in a naive mode using the permutation. With this in mind, we compute the expected number of queries required to form a cycle in an ideal permutation.

Let  $F : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$  be an ideal permutation. Since the properties of an ideal permutation imply that each element in the domain has the same probability of belonging to a cycle of any specified length, we may analyze a random walk starting with the initial value  $IV$ , computing  $F(IV), F(F(IV)), \dots, F^i(IV) = IV$ .

To determine the probability that  $F^i(IV) = IV$ , we first count the number of permutations for which  $F^j(IV) \neq IV$  for all  $0 < j < i$ . There are  $\frac{(n-1)!}{(n-i)!}$  choices for a prefix of an  $i$ -cycle beginning with  $IV$ , the choice  $F^i(IV) = IV$  is forced, and there are  $(n-i)!$  ways of permuting the remaining  $n-i$  elements. Therefore, there are  $(n-1)!$  permutations in which  $F^i(IV) = IV$ . Thus, for all  $i$ ,  $P(F^i(IV) = IV) = \frac{1}{n}$ .

To find the expected number of queries,  $X$ , required to form a cycle, we calculate:

$$E(X) = \sum_{i=1}^n iP(X=i) = \sum_{i=1}^n \frac{i}{n} = \frac{n+1}{2}.$$

**Definition 5.1.** A permutation,  $F$ , is said to have the ideal cycle property if the number of queries,  $Q$ , required to discover a cycle,  $F^{(i)}$ , is  $\Theta(n)$ , the number of queries to perform the same task for an ideal permutation.

For an ideal permutation, the hypotheses of Theorem 4.1 are not satisfied. In particular, for each  $i \in \{1, 2, \dots, n\}$  consider calculating the dependence sequence,  $\{\phi_i\}$ , corresponding to the ideal permutation. Again,  $\phi_1 = 0$  since there are no events occurring before the first round. For the second dependence coefficient,

$$\phi_2 = \sup_{A \in \sigma(\xi_1), B \in \sigma(\xi_2)} |P(B|A) - P(B)|, P(A) > 0.$$

Consider the events  $A = [\xi_1 = IV]$  and  $B = [\xi_2 \neq IV]$ . Clearly,  $P(A) = \frac{1}{n}$ . By the above calculation,  $P(B) = \frac{n-1}{n}$ . Notice, however, that  $A \cap B = \emptyset$ , since if  $\xi_1 = IV$  then  $\xi_2 = IV$ . Therefore,  $\phi_2 \geq \frac{n-1}{n}$ . Although Theorem 4.1 is not applicable, we can, however, still apply the techniques from Section 4 and generalize the results to give us a meaningful result.

In contrast to a random oracle, the outputs of fresh queries to an ideal permutation do not form an i.i.d. sequence of random variables. In particular, the outputs of the permutation are definitively not independent. If we consider the dependence sequence of the previous section in a new light, however, we can generalize this technique to construct an useful measure of variation between the output distributions of a fixed permutation and an ideal permutation.

Note that if the  $i$ th input to a random oracle is fresh, then for all  $A \in \sigma(\xi_1, \dots, \xi_{i-1})$  and  $B \in \sigma(\xi_i)$ , we have that  $P(B|A) = P(B)$ . Therefore we can consider the dependence sequence of the previous section as a measure of variation between the distribution of outputs of the fixed function,  $F$ , and the random oracle  $RO$ . To make this formal, we need to define a sequence of relative dependence coefficients depending on two sequences of random variables.

**Definition 5.2.** Let  $F$  be a fixed function with a finite codomain of size  $n$  and let  $\pi$  be an ideal function. Define  $\xi_i$  to be the random sequence of outputs of  $F$  under fresh queries. Similarly, define  $\zeta_i$  to be the random sequence of outputs of  $\pi$  under fresh queries. Define a map  $\tau : \sigma(\xi_1, \dots, \xi_n) \rightarrow \sigma(\zeta_1, \dots, \zeta_n)$  by  $\tau[(\xi_1, \dots, \xi_n) \in A] = [(\zeta_1, \dots, \zeta_n) \in A]$  for any subset,  $A$ , of  $\{1, \dots, n\}^n$ . For each  $i \in \{1, 2, \dots, n\}$ , define the relative dependence sequence,

$$(5.1) \quad \hat{\phi}_i = \sup_{A_i \in \sigma(\xi_1, \dots, \xi_{i-1}), B_i \in \sigma(\xi_i)} |P(B_i|A_i) - P(\tau(B_i)|\tau(A_i))|,$$

with  $P(A_i)P(\tau(A_i)) > 0$ .  $F$  is said to be  $\hat{\phi}[p]$ -mixing if  $\hat{\phi}_i \leq n^{-p}$  for all  $i < n$ .

Note that if  $\xi_i$  and  $\zeta_i$  are identically distributed and in addition  $\zeta_i$  are also independent, then  $\hat{\phi}_i = \phi_i$ . This fact is the analytic expression of the intuition that the measure of dependence  $\phi$  is essentially a measure of variation between the given process and an idealized independent version of the process. As a consequence,  $\hat{\phi}_i = \phi_i$  if  $\tau$  maps to the sigma field generated by the outputs of a random oracle.

Define an  $i$ -singleton event by  $A_i = [\xi_1 = a_1, \dots, \xi_i = a_i] \in \sigma(\xi_1, \dots, \xi_i)$ , and an  $(i, k)$ -singleton event by  $B_{i,k} = [\xi_{k+1} = b_{k+1}, \dots, \xi_i = b_i] \in \sigma(\xi_{k+1}, \dots, \xi_i)$ . Clearly, in the case of an ideal permutation, we have that  $P(A_i) = \frac{(n-i)!}{n!}$  for all  $i$  and  $P(B_{i,k}) = \frac{(n-i+k)!}{n!}$  for all  $k < i$ . Using this relation, we derive the following permutation measuring lemma:

**Lemma 5.1.** *Fix  $1 \leq i \leq n$ . Let  $F$  be a fixed permutation. Assume that  $\xi_i$ , fresh query outputs for  $F$ , and  $\zeta_i$ , fresh query outputs for an ideal permutation,  $\pi$ , are identically distributed. Let  $A_i \in \mathcal{F}_1^{i-1}$  and  $B_i \in \mathcal{F}_i^i$ . Then  $|P(A_i \cap B_i) - \frac{n}{n+1-i} P(A_i) P(B_i)| \leq P(A_i) \hat{\phi}_i$ .*

*Proof.* If  $P(A_i) = 0$ , then the result is obviously valid. If  $P(A_i) > 0$ , then  $P(\tau(A_i)) > 0$ ; consequently,  $|P(B_i|A_i) - P(\tau(B_i)|\tau(A_i))| \leq \hat{\phi}_i$ . Note  $|P(A_i \cap B_i) - \frac{n}{n+1-i} P(A_i) P(B_i)| = |P(A_i \cap B_i) - P(A_i) \frac{n P(B_i)}{n+1-i}|$ .

Since  $B_i \in \sigma(\xi_i)$  and  $\xi_i$  and  $\zeta_i$  are identically distributed, we obtain the relation  $P(B_i) = P(\tau(B_i))$ . Now  $\tau(A_i)$  can be expressed as the disjoint union of  $a$   $(i-1)$ -singleton events, and  $\tau(B_i)$  can be expressed as a disjoint union of  $b$   $(i-1)$ -singleton events of the form  $[\xi_i = k]$ , each having probability  $\frac{1}{n}$ . Therefore we compute, using the above mentioned relation for  $(i-1)$ -singleton events, the quantities  $P(\tau(A_i)) = \frac{a(n+1-i)!}{n!}$ , and  $P(\tau(B_i)) = \frac{b}{n}$ .

Moreover, since the event  $\tau(A_i) \cap \tau(B_i)$  must be the disjoint union of  $a \cdot b$   $i$ -singleton events, it follows that  $P(\tau(A_i) \cap \tau(B_i)) = \frac{ab(n-i)!}{n!}$ . We therefore have that  $P(\tau(B_i)|\tau(A_i)) = \frac{b}{n+1-i} = \frac{n P(\tau(B_i))}{n+1-i} = \frac{n Pr(B_i)}{n+1-i}$ . We thus obtain,  $|P(A_i \cap B_i) - P(A_i) P(\tau(B_i)|\tau(A_i))| = P(A_i) |P(B_i|A_i) - P(\tau(B_i)|\tau(A_i))|$ , since  $P(A_i) > 0$ . This quantity is bounded by  $P(A_i) \hat{\phi}_i$ , and, consequently, the lemma holds.  $\square$

Using a technique similar to that of Section 4, we compute the probability of obtaining the first cycle on round  $i$ . Again, larger probabilities of cycle formation in early rounds result in a lower expected number of rounds to form a cycle; therefore, we compute each probability assuming that every previous round attains the theoretical maximum probability of cycle formation. As in the random oracle case,  $\hat{\phi}_1 = 0$ .

The probability of forming a cycle in the first round is  $\frac{1}{n} = \frac{1}{n} + \hat{\phi}_1$ ; consequently, the probability of noncollision in the first round is  $\frac{n-1}{n} - \hat{\phi}_1$ . By Lemma 5.1, the probability of getting a cycle in the second round *and* no cycle in the first round is bounded by  $P(A) \left( \hat{\phi}_2 + \frac{n-1}{n-1} \frac{1}{n} \right)$ , where  $A$  represents the event that there is no cycle formed in the first round. This quantity simplifies to  $\left( \frac{n-1}{n} - \hat{\phi}_1 \right) \cdot \left( \hat{\phi}_2 + \frac{1}{n-1} \right)$ .

Similarly, for each  $i \in \{1, 2, \dots, n\}$ , we derive the probability of  $C_i$ , the event that the first cycle formation occurs in round  $i$ , given that each cycle formation event is assigned the maximal probability sequentially,

$$(5.2) \quad P(C_i) = \left( \frac{1}{n+1-i} + \hat{\phi}_i \right) \prod_{1 \leq k < i} \left( \frac{n-k}{n+1-k} - \hat{\phi}_k \right).$$

Again, let us consider this expression as a polynomial in  $\mathbb{R}[\hat{\phi}_1] \dots [\hat{\phi}_i]$  with  $i \in \{1, 2, \dots, n\}$ . Let  $\Psi_i^{(k)}$  denote the sum of the total degree  $k$  terms in this bound. Then we obtain

$$\Psi_i^{(0)} = \frac{1}{n}.$$

We denote the terms in  $\Psi_i^{(1)}$  by

$$\Psi_{i,j}^{(1)} = \begin{cases} \frac{(n+1-i)\hat{\phi}_i}{n} & \text{if } i = j \\ -\frac{(n+1-j)\hat{\phi}_j}{n(n-j)} & \text{if } j < i. \end{cases}$$

Thus equation (5.2) becomes:

$$(5.3) \quad P(C_i) = \Psi_i^{(0)} + \sum_{1 \leq j \leq i} \Psi_{i,j}^{(1)} + \text{higher degree terms.}$$

We now derive an explicit lower bound formula for the expected number of rounds,  $X$ , required to form a cycle relative to the  $\hat{\phi}_i$ 's. We obtain

$$(5.4) \quad E(X) = \sum_{i=1}^n i\Psi_i^{(0)} + \sum_{i=1}^n i \sum_{j=1}^i \Psi_{i,j}^{(1)} + \text{sum of higher degree terms.}$$

From the analysis earlier in this section,  $\sum_{i=1}^n i\Psi_i^{(0)} = \frac{n+1}{2}$ . The following theorem provides conditions for which the sum of these terms constitutes the dominant term.

**Theorem 5.1.** *A permutation,  $F$ , has the ideal cycle property provided it is  $\hat{\phi}[1 + \epsilon]$ -mixing for some  $\epsilon > 0$ .*

*Proof.* First, we note that the terms in  $\Psi_i^{(d)}$ , the sum of the total degree  $d$  terms, satisfy:

$$\Psi_{i,j_1, \dots, j_d}^{(d)} = \begin{cases} (-1)^{d+1} \frac{(n+1-j_1) \dots (n+1-j_d) \hat{\phi}_{j_1} \dots \hat{\phi}_{j_d}}{n(n-j_1) \dots (n-j_{d-1})} & \text{if } j_d = i \\ (-1)^d \frac{(n+1-j_1) \dots (n+1-j_d) \hat{\phi}_{j_1} \dots \hat{\phi}_{j_d}}{n(n-j_1) \dots (n-j_d)} & \text{if } j_d < i. \end{cases}$$

By the calculations shown at the beginning of this section,  $\sum_{i=1}^n i\Psi_i^{(0)} = \Theta(n)$ . It therefore suffices to show that the absolute value of the sum of the negative terms,  $\sum_{i=d}^n i\Psi_i^{(d)}$  with  $d > 0$  is  $o(n)$ .

Now,  $\Psi_i^{(d)}$  is negative when either  $j_d = i$  and  $d > 0$  is even, or  $j_d < i$  and  $d$  is odd. In particular, since we have taken care of the  $d = 0$  case, we consider the absolute value of the sum  $\sum_{i=d}^n i\Psi_{i,j_1, \dots, j_d}^{(d)}$  when  $j_d < i$  and  $d \geq 1$  is odd, and when  $j_d = i$  and  $d \geq 2$  is even.

*Case 1:*  $j_d = i$  and  $d \geq 2$  is even.

Under these conditions, we have that

$$(5.5) \quad \sum_{i=d}^n i |\Psi_{i,j_1, \dots, j_{d-1}, i}^{(d)}| \leq \max_{l \leq n} \hat{\phi}_l^d \sum_{i=1}^n \frac{i(n+1-i)}{n} \left( \sum_{j=1}^{n-1} \frac{n+1-j}{n-j} \right)^{d-1}.$$

The first sum in the right-hand side of (5.5) evaluates to

$$\sum_{i=1}^n \frac{i(n+1-i)}{n} = \frac{1}{3} \binom{n+1}{2},$$

while the second sum in (5.5) simplifies to

$$(5.6) \quad \left( \sum_{j=1}^{n-1} \frac{n+1-j}{n-j} \right)^{d-1} = \left( n-1 + \sum_{k=1}^{n-1} \frac{1}{k} \right)^{d-1} \leq (n + \ln(n))^{d-1}.$$

Thus the right-hand sum in (5.5) is bounded by the quantity

$$\frac{1}{3} \max_{l \leq n} \hat{\phi}_l^d \binom{n+1}{2} (n + \ln(n))^{d-1}.$$

Since by hypothesis  $\max_{l \leq n} \hat{\phi}_l < n^{-1-\epsilon}$ , we have

$$\frac{1}{3} \max_{l \leq n} \hat{\phi}_l^d \binom{n+1}{2} (n + \ln(n))^{d-1} < \frac{1}{3} n^{-d-d\epsilon} \binom{n+1}{2} (n + \ln(n))^{d-1},$$

which is  $O(n^{1-d\epsilon})$ , and hence, *Case 1* is complete.

*Case 2:*  $j_d < i$  and  $d \geq 1$  is odd.

Under these conditions, we obtain that

$$\sum_{i=d}^n i |\Psi_{i,j_1,\dots,j_d}^{(d)}| \leq \max_{l \leq n} \hat{\phi}_l^d \sum_{i=1}^n \frac{i}{n} \left( \sum_{j=1}^{n-1} \frac{n+1-j}{n-j} \right)^d.$$

Using the bound  $\max_{l \leq n} \hat{\phi}_l < n^{-1-\epsilon}$ , this expression simplifies to

$$n^{-d-d\epsilon} \frac{n+1}{2} (n + \ln(n))^d = O(n^{1-d\epsilon}),$$

and *Case 2* is complete.

Taking the maximum constant,  $C$ , from each of the “big-oh” expressions in the  $n$  values of  $d$ , we obtain the following bound for the sum of the negative terms in (5.4):

$$(5.7) \quad C \sum_{d=1}^n n^{1-d\epsilon} \leq Cn \frac{1}{n^\epsilon - 1} = o(n).$$

Therefore the total sum in (5.4) is  $\Theta(n)$  and Theorem 5.1 is complete.  $\square$

## 6 Application to Indifferentiability

The indifferentiability framework for hash functions provides a security criterion for the replacement of an ideal function with a fixed function. Specifically, we have the following definition based on [7, 15].

**Definition 6.1.** An interactive Turing machine  $T$  with oracle access to an ideal primitive  $\mathcal{F}$  is said to be  $(t_{\mathcal{A}}, t_S, \sigma, \epsilon)$ -indifferentiable from an ideal function  $\mathcal{G}$  if there exists a simulator  $S$  such that, for any distinguisher  $\mathcal{A}$ , we have:

$$|P(\mathcal{A}^{T, \mathcal{F}} \Rightarrow 1) - P(\mathcal{A}^{\mathcal{G}, S} \Rightarrow 1)| \leq \epsilon.$$

The simulator  $S$  is an interactive Turing machine with oracle access to  $\mathcal{G}$  running in time at most  $t_S$ , the distinguisher  $\mathcal{A}$  runs in time at most  $t_{\mathcal{A}}$ , the number of queries  $\mathcal{A}$  is allowed is  $\sigma$ , and  $\epsilon$  is a negligible function of the security parameter of  $T$ .

The indifferentiability framework provides the most important theoretical means of verifying the security of a hash mode. In particular, an indifferentiable hash mode together with an ideal compression function is resistant to every generic attack, including any possible generic attack found in the future. As a result, the recent years have witnessed a great deal of work devoted to the indifferentiability security of the most significant hash modes. In [1, 5], an indifferentiability bound of  $\frac{n}{2}$  bits is derived for the hash function BLAKE; the same bound is also derived in [10, 11, 18, 2] for Skein, Gröstl, JH, and the SHA-3 winner Keccak, respectively.

As an application of the framework developed in the previous sections, we are able to extend any generic indifferentiability proof of hash mode  $H$  using an ideal primitive to the case of a  $\hat{\phi}[p]$ -mixing primitive. This advancement is due primarily to the observation put forth in [17] and [18] that any indifferentiability proof can be modeled with the use of only three games:  $\text{Game}(RO, S)$ , a game modelling interaction with a variable input length random oracle  $RO$  and simulator  $S$ ,  $\text{Game}(H, ip)$ , a game modelling interaction with the actual hash mode  $H$  and ideal primitive  $ip$ , and  $G1$ , a single hybrid game with an input/output distribution identical to that of  $\text{Game}(H, ip)$  and different from  $\text{Game}(RO, S)$  on an exhaustively specified set of events defined for each round  $i$ , denoted by  $BAD_i$ .

The methodology employed in [17] and [18] recasts the typical quest for a long sequence of games into a purely mathematical problem involving the events  $BAD_i$ , denoting the event that some occurrence in the  $i$ th round may allow an adversary to determine with which entity she is interacting, and  $GOOD_i$ , denoting  $\bigcap_{j < i} \neg BAD_j$ . Here  $\neg BAD_j$  denotes the complement of  $BAD_j$ . Specifically, any indifferentiability proof can in this way be reduced to checking that the catalogue of information retrieved by any indifferentiability adversary  $\mathcal{A}$  is identical for the two games  $G1$  and  $\text{Game}(RO, S)$  in round  $i$  if  $GOOD_i$  occurred followed by a simple computation of  $\neg GOOD_i$ .

A key concept from this analysis is the fact that given a fixed view, any adversary  $\mathcal{A}$ , being a probabilistic Turing machine, has a fixed distribution of state transitions. Consequently, even though given the same input/output distributions  $\mathcal{A}$  may in one instance output 1 while in another instance output 0, the probability of each output is unchanged from one experiment to the next.

**Lemma 6.1.** Let  $0 \leq a_1, \dots, a_k, b_1, \dots, b_k \leq 1$ . Then  $\left| \prod_{i=1}^k a_i - \prod_{i=1}^k b_i \right| \leq \sum_{i=1}^k |a_i - b_i|$ .

*Proof.* The result is clearly true if  $k = 1$ . Assume the result is true for all values less than

$k$ .

$$(6.1) \quad \left| \prod_{i=1}^k a_i - \prod_{i=1}^k b_i \right| = \left| \prod_{i=1}^k a_i - b_k \prod_{i=1}^{k-1} a_i + b_k \prod_{i=1}^{k-1} a_i - \prod_{i=1}^k b_i \right| \\ \leq \prod_{i=1}^{k-1} a_i |a_k - b_k| + b_k \left| \prod_{i=1}^{k-1} a_i - \prod_{i=1}^{k-1} b_i \right| \leq |a_k - b_k| + \left| \prod_{i=1}^{k-1} a_i - \prod_{i=1}^{k-1} b_i \right|.$$

Thus by induction,

$$\left| \prod_{i=1}^k a_i - \prod_{i=1}^k b_i \right| \leq \sum_{i=1}^k |a_i - b_i|.$$

□

**Theorem 6.1.** *Let  $\mathcal{A}$  be an indifferentiability adversary interacting with the hybrid game  $G(H, \mathcal{F}_p)$ , which utilizes a  $\hat{\phi}[p]$ -mixing primitive  $\mathcal{F}_p$ , and  $\text{Game}(RO, S)$  while limited to  $\sigma$  queries. If when interacting with  $\text{Game}(RO, S)$  and the hybrid game  $G(H, ip)$ , which utilizes an ideal primitive  $ip$ , and  $\mathcal{A}$  is limited by  $\sigma$  queries we have*

$$(6.2) \quad \left| P(\mathcal{A}^{G(H, ip)} \Rightarrow 1) - P(\mathcal{A}^{RO, S} \Rightarrow 1) \right| \leq \epsilon(\sigma),$$

then the following inequality holds as well:

$$(6.3) \quad \left| P(\mathcal{A}^{G(H, \mathcal{F}_p)} \Rightarrow 1) - P(\mathcal{A}^{RO, S} \Rightarrow 1) \right| \leq \frac{2\sigma}{2^{2pn}} + \epsilon(\sigma).$$

*Proof.* The left-hand side of (6.3) can be expanded in the following form:

$$(6.4) \quad \left| P(\mathcal{A}^{G(H, \mathcal{F}_p)} \Rightarrow 1) - P(\mathcal{A}^{G(H, ip)} \Rightarrow 1) + P(\mathcal{A}^{G(H, ip)} \Rightarrow 1) - P(\mathcal{A}^{RO, S} \Rightarrow 1) \right|.$$

By (6.2), the equation (6.4) is bounded by

$$\left| P(\mathcal{A}^{G(H, \mathcal{F}_p)} \Rightarrow 1) - P(\mathcal{A}^{G(H, ip)} \Rightarrow 1) \right| + \epsilon(\sigma).$$

Now let  $G_i$  denote the event  $GOOD_i$  for all  $i$ . Note that we can split the left summand above into:

$$(6.5) \quad \left| P(\mathcal{A}^{G(H, \mathcal{F}_p)} \Rightarrow 1 | \tau^{-1}(G_\sigma)) P(\tau^{-1}(G_\sigma)) - P(\mathcal{A}^{G(H, ip)} \Rightarrow 1 | G_\sigma) P(G_\sigma) \right. \\ \left. + P(\mathcal{A}^{G(H, \mathcal{F}_p)} \Rightarrow 1 | \neg \tau^{-1}(G_\sigma)) P(\neg \tau^{-1}(G_\sigma)) - P(\mathcal{A}^{G(H, ip)} \Rightarrow 1 | \neg G_\sigma) P(\neg G_\sigma) \right|.$$

By Lemma 6.1 and the triangle inequality, it suffices to obtain bounds on the following three quantities:

$$\left| P(\mathcal{A}^{G(H, \mathcal{F}_p)} \Rightarrow 1 | \tau^{-1}(G_\sigma)) - P(\mathcal{A}^{G(H, ip)} \Rightarrow 1 | G_\sigma) \right|, \\ \left| P(\mathcal{A}^{G(H, \mathcal{F}_p)} \Rightarrow 1 | \neg \tau^{-1}(G_\sigma)) - P(\mathcal{A}^{G(H, ip)} \Rightarrow 1 | \neg G_\sigma) \right|,$$

and

$$|P(\tau^{-1}(G_\sigma)) - P(G_\sigma)|.$$

By the definition of  $\tau$ , the view of the adversary in the game  $G(H, \mathcal{F}_p)$ ,  $V_{1,\sigma}$ , and the view of the adversary in  $G(H, ip)$ ,  $V_{2,\sigma}$ , given either of the event pairs  $(\tau^{-1}(G_\sigma), G_\sigma)$  or  $(\neg\tau^{-1}(G_\sigma), \neg G_\sigma)$ , are identically distributed. Therefore, since the possible states of  $\mathcal{A}$  interacting with  $G(H, \mathcal{F}_p)$  and  $G(H, ip)$  are identically distributed, the corresponding output distributions are identical. Thus we have:

$$\left| P(\mathcal{A}^{G(H, \mathcal{F}_p)} \Rightarrow 1 | \tau^{-1}(G_\sigma)) - P(\mathcal{A}^{G(H, ip)} \Rightarrow 1 | G_\sigma) \right| = 0$$

and

$$\left| P(\mathcal{A}^{G(H, \mathcal{F}_p)} \Rightarrow 1 | \neg\tau^{-1}(G_\sigma)) - P(\mathcal{A}^{G(H, ip)} \Rightarrow 1 | \neg G_\sigma) \right| = 0.$$

To analyze the last of the three quantities, notice that  $G_i$  has the special property that  $G_i = \cap_{k \leq i} G_k$ . Therefore,

$$P(G_\sigma) = P(G_1) \prod_{i=2}^{\sigma} P(G_i | G_{i-1}),$$

and  $\tau^{-1}(G_\sigma)$  satisfies a similar relation. Thus by Lemma 6.1,  $|P(\tau^{-1}(G_\sigma)) - P(G_\sigma)|$  is bounded by

$$\sum_{i=1}^{\sigma} |P(\tau^{-1}(G_i) | \tau^{-1}(G_{i-1})) - P(G_i | G_{i-1})| \leq \sum_{i=1}^{\sigma} \hat{\phi}_i.$$

Therefore, (6.5) is bounded by  $2 \sum_{i=1}^{\sigma} \hat{\phi}_i$ , which is bounded by  $\sigma 2^{-2pm+1}$ .  $\square$

## 7 Conclusion

The measures of dependence derived in Sections 4 and 5 provide a measure for fixed functions, determining such a function's proximity to an ideal primitive in some sense. This framework supports the direct analysis of cryptographic primitives as exemplified in Sections 4 and 5 as well as providing a means of porting protocol security arguments involving ideal primitives into a more realistic setting.

Perhaps the most interesting application for these dependence coefficients has been the stopping time analysis for an indifferenciability adversary provided in Section 6. An analysis of the dependence sequence for the primitives of the SHA-3 finalists and the winner of the SHA-3 competition, Keccak, may solidify the indifferenciability arguments already present in the literature. Although the direct calculation of the dependence sequence of such fixed functions is itself a challenging task, such functions are often constructed from block ciphers and other iterative protocols which can be further broken down and analyzed.

On the other hand, the determination of lower bounds for the dependence coefficients of these hash primitives is a more immediately approachable problem. The discovery of particularly large dependence coefficients, while not constituting an invalidation of indifferenciability results, may call the practical resistance to distinguishing attacks of protocols employing such primitives into question. Furthermore, this methodology can be utilized for the construction of new heuristic arguments which may influence the construction of primitive random functions.



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